Combinatorics
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Lecture 47
Colouring and Symmetries - Examples

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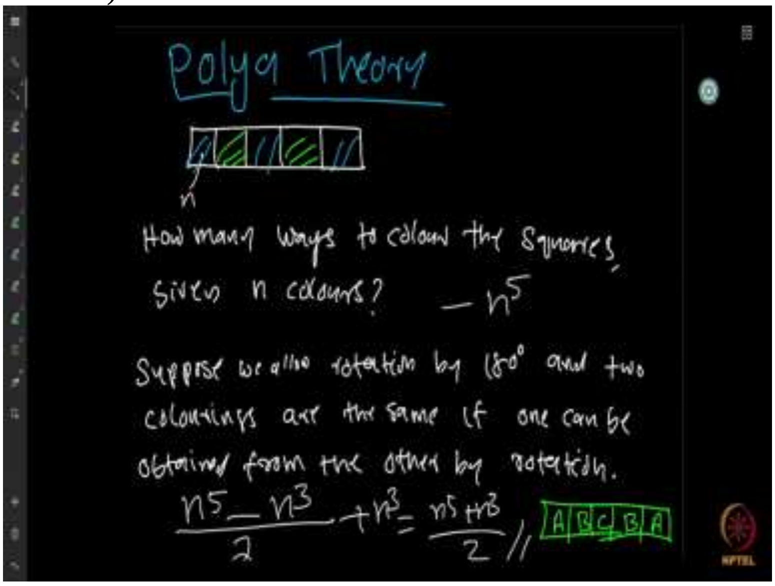

Now, we start a new topic called Polya theory. Let us start with one of the questions that we have visited many times so far in this course that is suppose I give you these 5 squares in a row and I give you n different colours and I ask you to colour these squares using your choice of colours.

So, you can choose for each cell you can decide which colour that you want to use and then you have to colour. Now, the first question that arises in combinatorics is how many different ways I can do this colouring. So, how many different ways are there to colour these 5 squares given n colours.

Of course, the answer is very easy for every square we know that there are n choices for the colours. And I am free to choose whichever colour that I want, so I have n choices for each of the 5 boxes so n^5 colourings in total. So, I get an n^5 distinct colourings of this squares.

Now, we can add some restrictions for example I say that, we allow rotation of this figure by 180 degrees like I can just rotate the entire thing by 180 degree. And then I say that if I can get 1 colouring a fixed colouring of this strip by rotating another colouring then I will say you know I consider them to be equivalent and I do not distinguish between them. So, if under rotation I say that the colouring that is given to this is the same then I can say that the colour colourings are equivalent. For example, colouring like this if I rotate, I will get the same colouring. Now, then I you know we can ask like how many different colourings are there now.

How many inequivalent colourings are there? So, well one can think of trying to solve this. We see that if I have n^5 colourings in total, if I reflect them, I will get a set of colourings. And, these colourings are equivalent for corresponding like if I take any colouring its reflection is also equivalent.

So, maybe I can try to divide by 2 but then there is a problem if I divide you know all the colourings by 2 will it work? Well, the problem is that like some colourings might be counted twice for this. For example, I might have a colouring where the reflection also gives the same colouring so in that case what happens?

For example, let us say that I have this strip and I colour the first one with colour A second with B third with C then I colour the fourth with B and the final one with A again. Now, if I reflect this there is no difference, I get the same colouring of the squares and therefore when I divide by 2 I am actually under counting.

So, I need to take care of this. I want to make sure that I do not overlook this kind of colourings. So, what I do is that I just observe that the only type of colourings which remains the same under the reflection are precisely or rotation by 180.

Are precisely the colourings where you know some three colours are used in the first three boxes and then the colour of the second cell is used on the fourth and the colour of the first cell is used around the fifth. So, any colouring like this if I rotate I will get the same colouring it is precisely the same.

So therefore, such colourings I want to read separately. So out of the total n^5 colourings this n cube colourings, because, the first three cells I can choose any of the n colours but then you know there is no choice because I just use the colour that I already have on the remaining sides.

So, out of this $n^5 - n^3$ colourings each of them under the rotation will give a different colouring but they are equivalent so therefore I can divide by 2. But, on the other hand I have this n^3 colourings which are the same under other rotation also so I have to add that so this will be, $\frac{n^5 - n^3}{2} + n^3 = \frac{n^5 + n^3}{2}$.

So, we get this to be the number of inequivalent colourings. The distinct colourings where two colourings are same under rotation by 180 degree. Now, let us try to see how much more we can go in this direction. So, the idea of Polya theory is to somehow generalize this.

So, what is the kind of thing that we can use to generalize something like this? So, one thing that we observe if you think about this is you know this rotation that we are talking about for example is a kind of symmetry or in fact it is a type of permutation of the elements of the set.

So basically, I am permuting I can think of this rotation as permuting the cells of this set of squares, or the squares, individual squares. For example I can say that now this rotation is basically like you know I am permuting this square going here and this square going here similarly this one going here and this one going here and this square is mapped to itself.

So, such permutations will give you the effect of rotation and of course you know the identity permutation will keep the same colouring. So, if you want to think of this you can think of this as basically you know these two permutations identity and the permutation that maps 1 to 5 and 5 to 1 then 2 to 4 and 4 to 2 and 3 to itself squares to itself as representing the rotation.

And then you see that these two basically form a 2-element group and you know it is going to be a subgroup of the symmetric group of all the permutations of the squares. Things like that. So, before going further with this let me just recall what is this group and all we are talking about. It is a very quick review because you know I expect all of you to know group theory.

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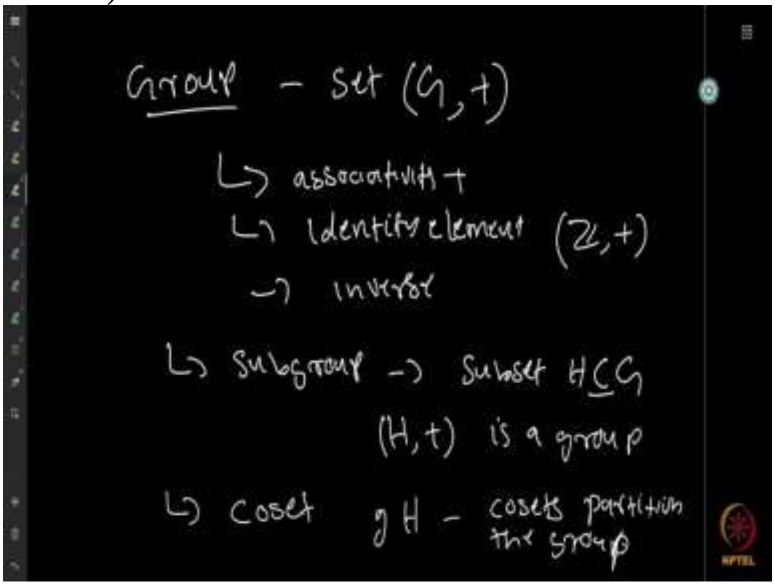

So, what is a group, group is basically a set let us say G together with a binary operation such that under the binary operation the set is closed. That if I add elements of the set I will get still another element of the set.

So, it is closed under this binary operation. Now, then we have some additional properties the group properties which are for example associativity of the addition, then existence of identity element, then existence of inverse element so these are the properties that we want I know for a group to have.

So, we have such structures which we call groups. Now, example of a group, for example is the set of all integers under addition. Now, if groups have subgroups what is a subgroup, subgroup of a group is basically a subset of a subset of the set G and such that (H,+) is a group by itself a group. So, if this is true for some subset then it is a it is called a subgroup. So, we follow the same binary operation, the same rules and you know this subset together with the binary operation is a group by itself then it is a subgroup of the group.

Then associated concept is what is that is that of coset. Cosets with respect to a subgroup I can say that like you know g is an element of G and H is as a subgroup then elements gH is are the left cosets and similarly you can define right cosets. And cosets partition the group so this is something that you can take it as a homework and prove if you have not done it before. So, the cosets basically from a partition of the group. Now, and you know with respect to which subgroup you are looking at you will see that the cosets have the same cardinality and then it follows that like you know the order of the subgroup. So, the number of elements in the subgroup divides the order of the group and things like that. So, these results are all very standard in any mention of group theory.

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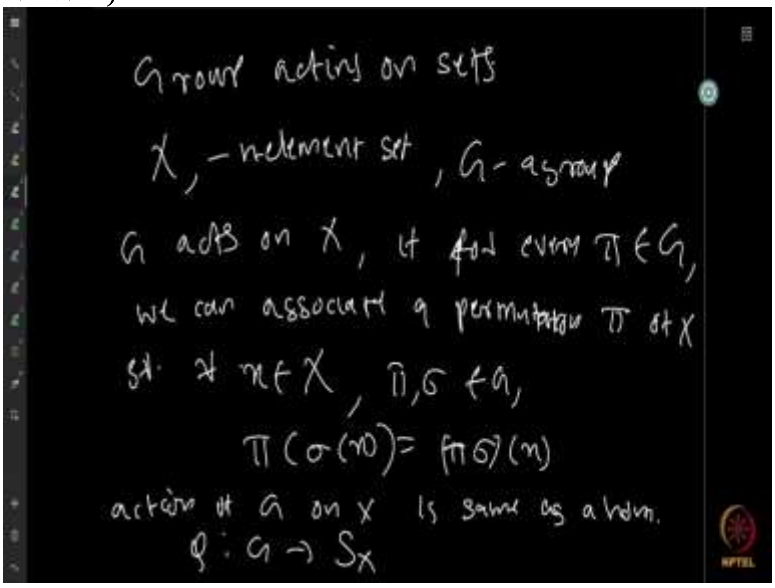

So now, these things that you should know then another related concept is the action of a group on a set. So, group acting on sets. So, let X is a set let us say that n element set, X is n element set and G is a group. Now, we say that the group G acts on the set X G acts on the set X if for every element of the group, if for every element let us say of the group G we can associate a permutation. We can associate a permutation and we will overload the symbol by saying that you know π is also a permutation.

So, we will use the same thing permutation of X. So, permutation of the elements of X such that for every element x in X and two elements let us say π and σ in G we have $\pi\big(\sigma(x)\big) = (\pi\sigma)(x)$. So, $\pi\sigma$ is basically the operation under the group so it is a binary operation in the group so $\pi\sigma$ is an element of the group.

So therefore, it has a corresponding associated permutation every element is associated permutation so $\pi\sigma$ is now an element which is also representing some permutation. So, $\pi\big(\sigma(x)\big) = (\pi\sigma)(x)$ so if this is true for every x and every π and σ then we say that the group G acts on X.

Now, if you think about this maybe you should try to give an argument why the action of the group G is the same as a homomorphism. So, action of the group G on the set X is equivalent to saying that there is a homomorphism say homomorphism let us say ϕ which maps from G to the symmetric group S_X.

So, S_X is the set of all permutations of X. So, this is the symmetric group S_X. Now this is something that you can work out the details and make sure that you understand what is a group what is coset what is action of a group on sets and things like that. So, these things you should be clear so once this is clear we can proceed.

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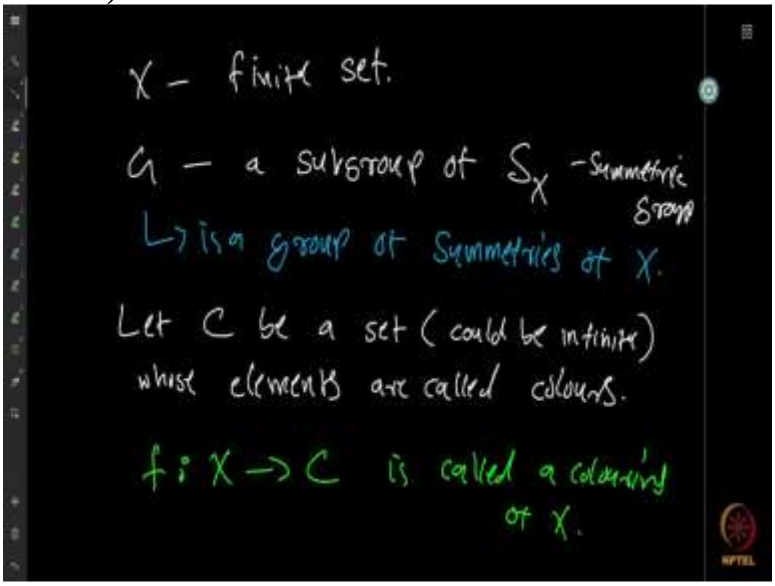

Now, so let us try to look these things in a more general way. So, let us start with the set X which is a finite set now we consider a group G, G as a subgroup of S_X which is the symmetric group. Now, this group is basically a group of symmetries of X.

Because, permutations can basically you know as a group can represent symmetries and that is basically the reason we call the group of permutations of the symmetric group itself. Now, we have this group of symmetries we have this group of symmetries and in fact see when we are talking about the definition when we are defining the groups acting on sets we were saying that we are basically associating permutations with this.

But, now one can see that you know whichever group that you find you start with you can associate a permutation then that permutations form a subgroup of the symmetric group and then you can as well start with the symmetric group as your group that you are looking at.

Now suppose, you know so you have the finite set X and you have a group G and then consider a set of colours C which could be an infinite set of colours. So, C could be finite or infinite and elements of C are called colours and the colouring of the set X is basically a function from X to C.

So, you are basically associating colours with elements. So, we have now a colouring of an arbitrary finite set X. Then we can of course ask several questions like what are the number of colourings then we can ask like you know if under the action of this symmetry symmetric group G can you say what how many inequality colourings are there. So, this is basically a generalization of what we are looking before.

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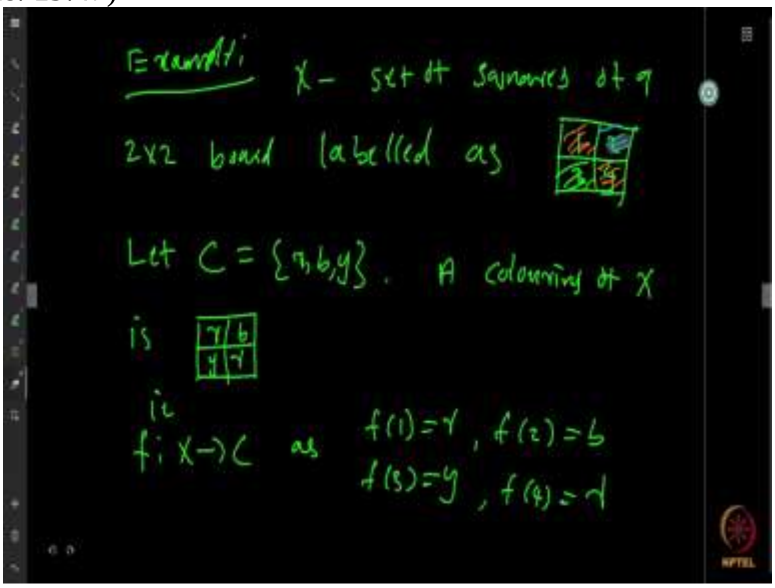

So, let us look at an example so X is the set of squares of a 2 by 2 chessboard or board labelled as let us say I think I have maybe I will redraw this. So I have these four squares 1, 2, 3, 4. Now, let us say that C is this finite set r, b, y it is a set of colours so red blue and yellow.

The colouring of X is using the colours from C you know it can be one of the examples is for example you know you give the first cell as r second as blue then the third is yellow and the fourth as red. So r, b, y, r. which means that f map from X to C is f(1) = r, f(2) = b, f(3) = y and f(4) = r. So, we have a colouring of finite set X.

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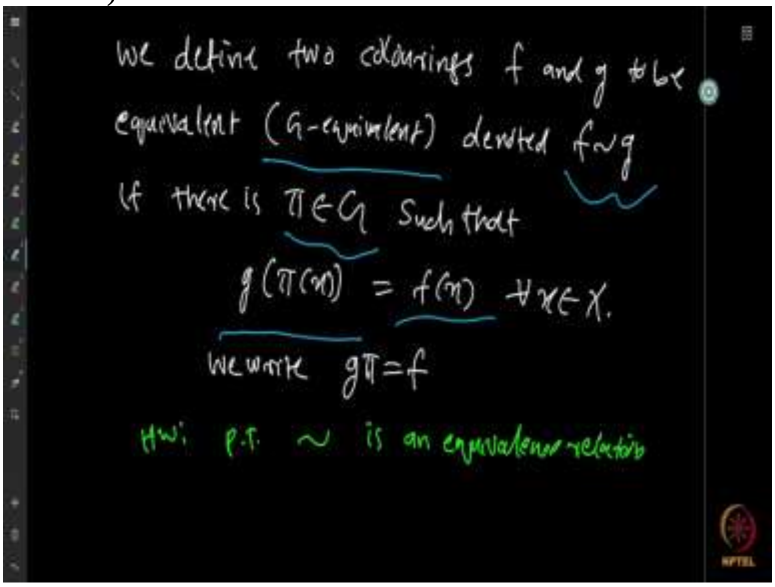

Now, given this we can ask some questions and for that let us let us define the following. So, we define two colourings f and g you know to be equivalent or g equivalent and we denote it by f $\sim$ g. So f $\sim$ g says that f and g are equivalent. If we can find some permutation G because G is a symmetric group so if you can find a permutation in G such that $g(\pi(x)) = f(x)$, for every x in X.

So, if I can do this then we say that f and g are equivalent. So, we can write for example $g\pi = f$. The composition of g and π is basically the colouring. Now, so colouring f. So, now prove that you know this relation that we defined the equivalence relation so G-equivalent is an equivalence relation. I mean so, so the reason we call this as if they are equivalent is because this happens to be an equivalence relation but you need to prove this. So, it is quite easy just prove that $\sim$ is an equivalence relation.

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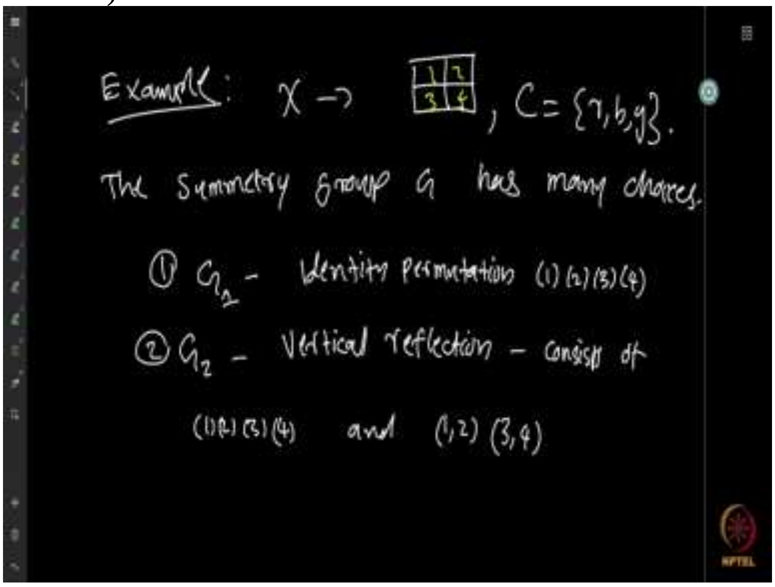

Continuing our example so we have the set X which is the set of these four squares and C is the colours r, b and y. Now, the symmetry group G so we said that G is basically you know it is a subgroup of the symmetric group on all these elements on X, S_X , and since our S_X in this case is basically S of 1, 2, 3, 4, there are four elements set X.

So, we can look at any of these subgroups of you know this S_4. So, for example maybe the symmetric group is precisely the identity permutation, identity by itself forms a group so therefore it is a subgroup of the of the symmetric group S_X.

So, G_1 let me say is the group with only one element which is identity permutation. So, what is identity permutation this one goes to itself 2 goes to itself 3 goes to itself and 4 goes to itself so all the elements goes to itself. Then G_2 so is basically the vertical reflections.

So, if you look at the vertical reflection what happens if I take a vertical mirror and reflect on this then you will see that, under the reflection 1 will be going to 2 and 3 will be going to 4. So, this is represented by this permutation (1, 2) (3, 4).

Of course, the identity is always equivalent the colouring is equivalent to itself. So, therefore identity is always there so therefore we have G_2 is consisting of these two permutations (1) (2) (3) (4) and (1, 2) (3, 4). So, these two permutations from another subgroup with just two elements.

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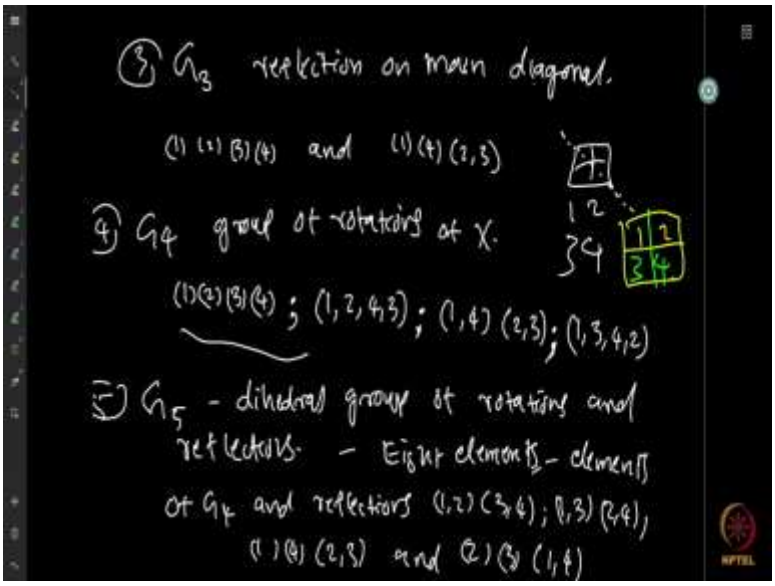

Then we can look at the reflection on the main diagonal. So, I can put a mirror on the diagonal and then try to find the reflection. So, (1) (2) (3) (4) again is there. Now, what happens under the reflection on the main diagonal of course you know this cell will be preserved as it is this cell will remain as it is but this will go here and this will go here.

So basically, this is represented by one going to itself 4 going to itself and 2 is going to 3 and 3 going to 2. So G_3 is the reflection on the main diagonal that is also a 2 element 2 element sub group. Now, you will see that you know as abstract groups if you have studied group theory you will see that as abstract groups G_2 and G_3 for example are isomorphic.

We have not defined isomorphism but in any group theory course you will see. So, they are isomorphic but we will see that you know their actions are different. So, you know as permutation groups they are basically different objects we will see this soon. Then we look at G_4 as the group of rotations of X.

So, what is basically the rotations? Rotations are basically like so this (1) (2) (3) (4) is there I just rotate the entire figure by 90 degrees. So, these are the rotations and what are the you know what are the elements of the permutation group which does this job. Of course, the identity is the rotation where we do not do anything then 1 going to 2, 2 going to 4 and 4 going to 3 and 3 going to 1. So 1 going to 2, 2 going to 4, 4 going to 3 and 3 going to 1 it is one rotation by 90 degree.

Then if I have two rotations then 1 will go to 4 after two rotations and 2 will go to 3 and 3 will go to 2 and similarly 4 will go to 1. So, I get this permutation and similarly if I rotate three times or rotate to the left then I will get 1 3 4 2. 1 going to 3, 3 going to 4, 4 going to 2 and 2 going to 1.

So, these are you know the rotations and this form is a subgroup which is called G4 in our case. Then we can talk about G5 which is the dihedral group of all rotations and reflections. So, we already saw that you know the rotation has all these four elements then we have the reflections, reflections on the main diagonal and the horizontal vertical etcetera.

So, these forms altogether eight elements. So, what are the elements? So, the reflections 1 going to 2, 3 going to 4 so 1 going to 2 and 3 going to 4. So, that is the reflection vertical deflection. Then 1 going to 3 and 2 going to 4 which is the horizontal reflection. Then 1 and 4 remains 2 and 3 are interchanged which is the reflection on the main diagonal and then 2 and 3 are remaining itself and 1 is going to 4 which is a reflection on this diagonal, diagonal including a 2 and 3. So, this is the dihedral group which consists of all rotations and reflections.

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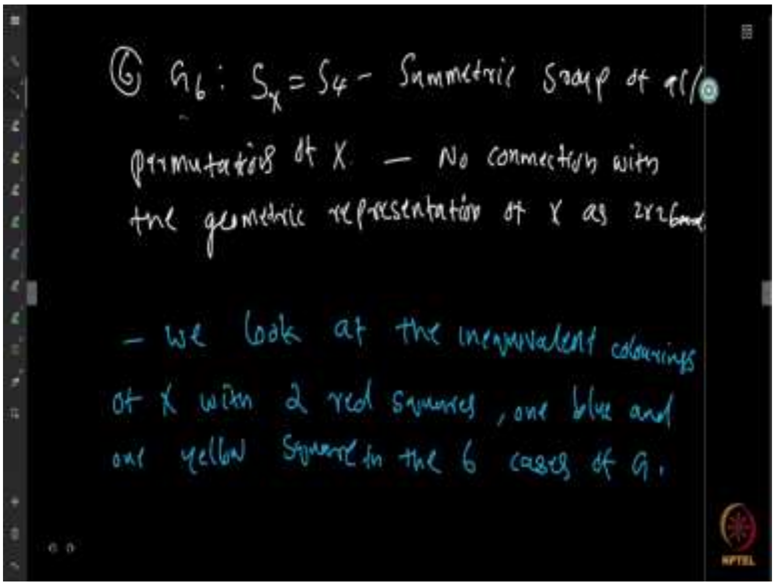

Then finally we can also look at the symmetry group of all permutations of X, G_6. So, S_X is S_4 itself. Now, so if you look at all possible permutations then we see that actually you know it loses the connection with the geometric representation of X as this 2 by 2 square.

Because, when we allow all possible permutations and they are all considered identical then you know there is precisely nothing to differentiate like you know whether there is a there is this figure you know where is this figure or not.

Whatever the shape of the figure it is, instead of 1, 2, 3, 4 as a 2 by 2 square you can have it 1, 2, 3, 4 as a sequence of 4 things or put in another fashion all of these are immaterial what is the shape it does not matter.

So, you will see that the you know the when you take the full symmetric group then the shape or geometric representation does not matter. So, that we lose the meaning but of course it is a valid permutation group. Then under all these six groups we can talk about the inequivalent colourings of X where we specify our set of colours.

Let us say we are saying that we always use two red colours like two squares will be coloured with red always and one will be coloured with blue and one will be coloured with yellow for example. So, if you specify the number of times each colour appears then we can talk about the action of the group.