Combinatorics
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Lecture 45
The Discharging Method – Part 2

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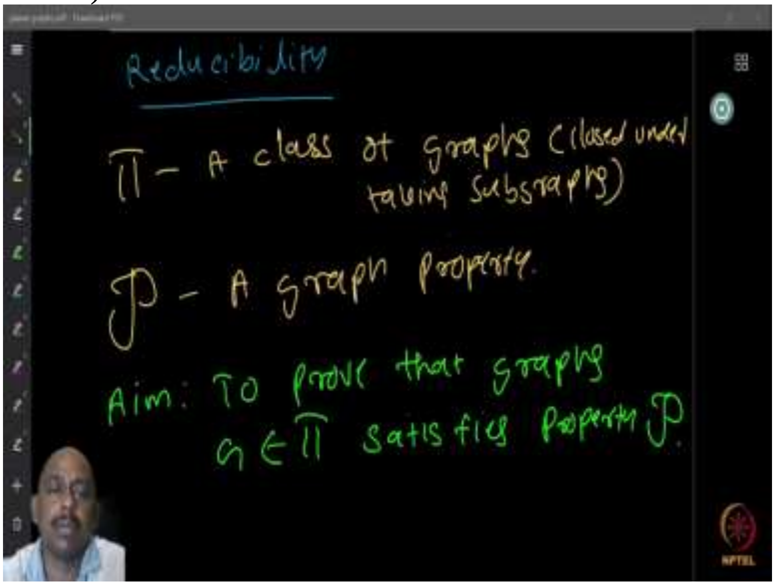

So, I continue to look at more involved examples of discharging. And to do this, let us look at some ideas behind the discharging method. One notion that I want to look at is that, the idea of reducibility. So, let Π be a class of graphs, closed under taking subgraphs. We want to look at graph which are hereditary. Hereditary means, if you take subgraphs, the property still remain. For example, planar graphs are having this property.

Because any subgraph of planar graph is also planar. But not all graphs have this property. For example, if I just say, non-planar, if I take subgraph, it could be planar. So, that does not belong to the class, so that kind of classes I cannot take so I take a class of graphs which are closed under taking subgraphs, so, Π be is such a class.

Then I have some graph property that want to establish, let us say property $\mathcal{P}$, it could be anything like what we have just told before or some other property like colourability or chromatic number or anything, some existence of some small neighbour. Or existence of a neighbourhood where some properties too… anything like this. Let $\mathcal{P}$ be some graph property. And our aim is to prove that graphs in Π, in the class that we are looking at, all satisfy the property $\mathcal{P}$. Every graph in this class has this particular property. Now this is what we want to prove, okay?

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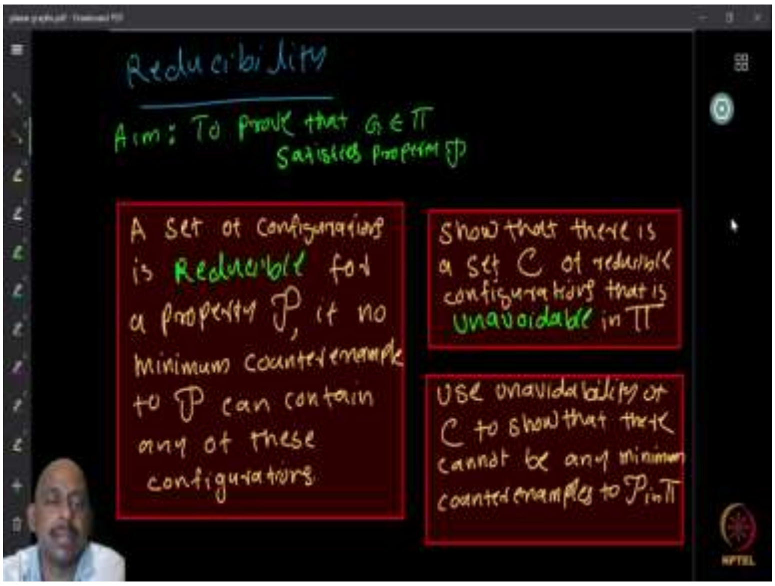

Now, if we are looking at such a question, then we say a set of configurations, configuration is basically a substructure of property, set of configurations is said to be reducible for a property $\mathcal{P}$, if no minimum counter example to the property can contain any of these configurations. So, I say, that a set of configurations is reducible for a given property if no minimum counter example to P can contain any of these configurations.

What is the meaning of this statement? Let us look at in a graph and a given property. If the graph does not satisfy this property, then we will say this graph is a counter example to the property.

Now, if you are looking at a class of graph, a class of graph we say, has a counter example, then we can look at all possible counter example. So, if you can find at least one graph which is a counter example to the property which does not satisfy the property, then we can talk about all possible such graphs which does not satisfy this.

So, look at all possible counter examples. Then we can define some kind of minimality, we can say that the graph with the smallest number of vertices and edges which does not satisfy the property because there are at least some graphs which violates the property, we can talk about one with the smallest size. So, for this kind of graphs we call minimum counter examples.

So, what we are saying is that, a sub-structure or a configuration is reducible if minimum counter examples cannot contain this configuration. So, what actually means is that, if you have a graph and this graph has this sub-structure, this reducible configuration and if the property is not true for this graph, then even after removing this particular sub-structure, the remaining graph will also be counter example. That the counter example cannot be minimum one, you can still find a smaller counter examples. Such cases are called minimum counter examples. Such configurations are called reducible configurations.

Now, suppose we can show that, you can find a set of reducible configurations that cannot be avoided in the class Π. So, I say that, I have a reducible configuration, a set of reducible configurations and this is unavoidable in the class Π which means every graph in the class Π must have at least one of these configurations. Without having one of this, no graph in this class can be there. If I can show this, then what does it says? It says that, every graph in this class Π contains one of these configurations.

Now, if any graph in this is a counter example to the property, then this cannot be minimum counter example because if there is a minimum counter example, we are saying that, then it cannot contain the reducible configurations. So, we are saying that, none of the graphs in this class can be minimum counter example. But if there is a counter example, there is a minimum counter example because you look all possible counter example, and find the one with the minimum size. So, therefore, if you show that reducible configuration is unavoidable, then we have the proof that, the property must be true for the class.

So, the idea of discharging is to show a set of reducible configurations is unavoidable. In that part alone, we will use the discharging method. So, the idea is to show some structural property like reducibility, reducible configuration. Now, reducibility of this configuration, that you have to do separately. So, once you show the configuration is reducible, you can try to use discharging method to show that it is unavoidable in the class. And that is the way this method is usually used.

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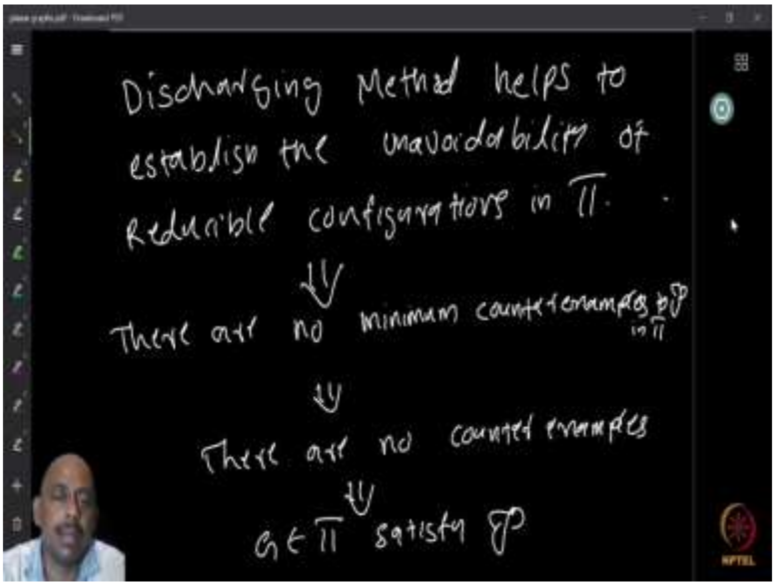

So, now let us look at more concrete example, so here was the trucks of the idea, discharging method helps to establish the unavoidability of reducible configuration in a class Π. If there are no minimum counter examples for a property $\mathcal{P}$ in Π, there are no counter examples for this property in Π. And therefore, the graphs in Π satisfy this property.

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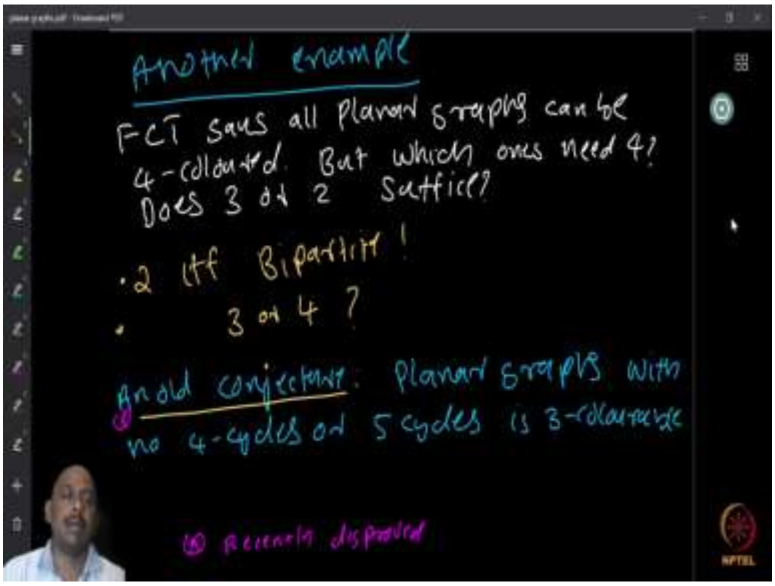

Now, we are going to look at very interesting result, this comes from the question that we were looking at before, like 4-colour theorem. We mentioned 4-colour theorem. So what was 4-colour theorem? The 4-colour theorem says that every planar graph can be vertex coloured such that no adjacent vertices get the same colour using at most 4-colours. Every planar graph admits a 4 colouring. You can colour with less than or equal to 4 colours. But now the question is that, if all planar graphs are 4 colours, does every such graphs require 4? Obviously, no, it says at most 4.

Now, the question is that, which graph actually require 4? Which graph will do with 3? Which graph will do with 2 etc. Now, the case 2 is very easy, right? We know that graph is 2 colourable if and only if it is bipartite. So, therefore, for planar graphs, it is 2 colourable if and only if it is bipartite. So, bipartite planar graphs are easy to characterise therefore, that question is resolved. Now, the question is when can you say the graph require 4 colours or 3 colours. Can you say that, this class of graph require only 3 colours or this graph requires 4 colours.

So, there were several attempts to resolve these kinds of questions, many of them are still unresolved and one of the interesting question was an old conjecture, so this conjecture was that, suppose we have planar graphs, now if you take a planar graph and you say that you do not allow any triangles in it, there is no triangle then you know that the triangle free planar graph has at most 2n minus 4 edges that we proved and this will easily show us that, the graph can be 3 coloured. So, planar graph without triangles can be 3 coloured which is very easy to show. I want you to come up with the proof if we want nice exercise, you can think about this. And so therefore, that is not interesting for triangle free-planar graph. So, we will allow planar graphs with triangles.

Now, the question is that, suppose we have planar graph but we do not allow 4 cycles or 5 cycles. Turns out that planar graphs without no 4 cycles only. We just forbid 4 cycles, then you can find examples where you need 4 colours. But then, somebody ask what if I do not allow 4 cycles and 5 cycles? So, the conjecture said that, if you have planar graph where there is no 4 cycles or 5 cycles then the graph is 3 colourable. Now, this conjecture was open for many years, only very recently in 2 years before or 3 years before, or may be 4 or 5 years before it was disproved. But it still gives many other interesting questions. What we are going to show is that, something related to this conjecture.

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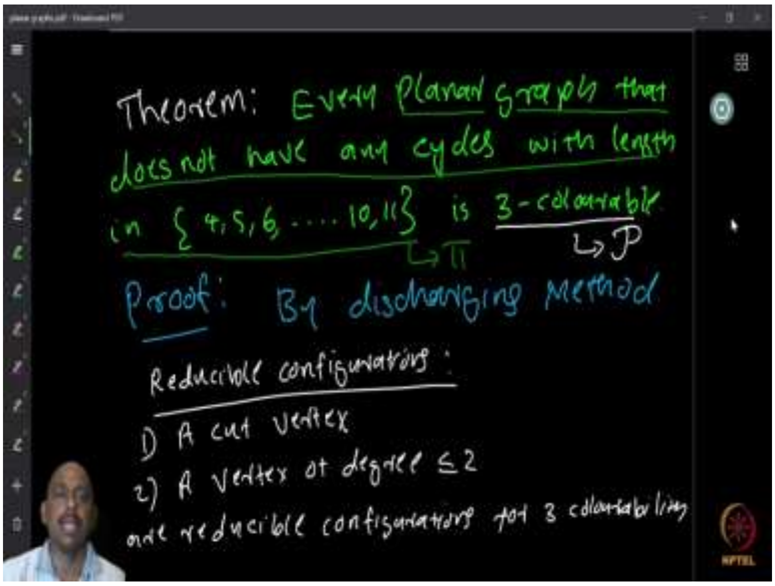

So, we are going to prove the following theorem that every planar graph that does not have cycles of length, 4, 5, 6, 7 etc, upto 11. There is no 4 cycle, 5 cycle, 6 cycle, 7 cycle etc, no 10 cycle, no 11 cycle. So, if all the cycles are not there, then the graph is 3 colourable. So, we have just made it weaker by adding more forbidden cycles. And we are going to prove it using discharging method.

Now, let me mention a little bit about this, so one person proved this very easily, if you do not allow 4 to 11 cycle, then we can say it is 3-colourable and then after some trials we could see that, it is easily improvable to 4 to 10. So, somebody proved that, if there are no cycles of length 4 to 10, then the graph is 3-colourable.

Now, the second part 4 to 10, can be good homework question after you do this lecture. And then somebody improved it further saying that, if there are no cycles of length 4 to 9, then the graph is 3-colourable, planar graphs. Again, after sometime somebody improved it, if there are no cycles of length 4 to 8, it is also 3-colourable. And finally, with much more difficulty, some body proved that if there are no cycle of length for 4 to 7, then the graph is 3-colourable.

Then after this somebody proved that, the original conjecture, that if, there are no 4 to 5 cycles, then the graph is 3-colourable is false, you need 4 colours, they come up with counter example saying that, these graphs have no 4 cycles or 5 cycles which are planar but still requires 4 colours. So, now the only question that remains open is that, if there are no 4 cycles, 5 cycles or 6 cycle, then can you say the graph is 3-colourable? I do not mean the only question? One natural question which is still open.

So, this is if you study this method and read up more interesting follow papers, you may be able to look at this technique again. All these proofs are using discharging method and again try to prove that, if there are no cycles of length 4, 5 or 6 then the graph is 3 colourable. If you can do that, it is a very nice result and you will be very happy with it. So, without further ado, let us try to prove, this very weak result, one of the weakest in the chain. That, if there are no cycles of length 4 to 11 the graph is 3-colourable

So, we start with the set of reducible configurations, I claim that for the 3 colourability of graphs, a cut vertex and a vertex of degree less than equal to 2 is a set of reducible configuration, which means that, if you are talking about 3 colourability of graphs, minimum counter examples to 3 colourability can not contain cut vertices and minimum counter example to 3 colourability cannot contain vertices of degree less than equal to 2.

And these are two very easy to prove statements and I want you to think about this and try to prove it yourself before going to the next phase. Stop and then wait for some time and try to work out this yourself, show why cut vertex and vertex of degree less than equal to 2 cannot be part of minimum counter example to 3 colourability.

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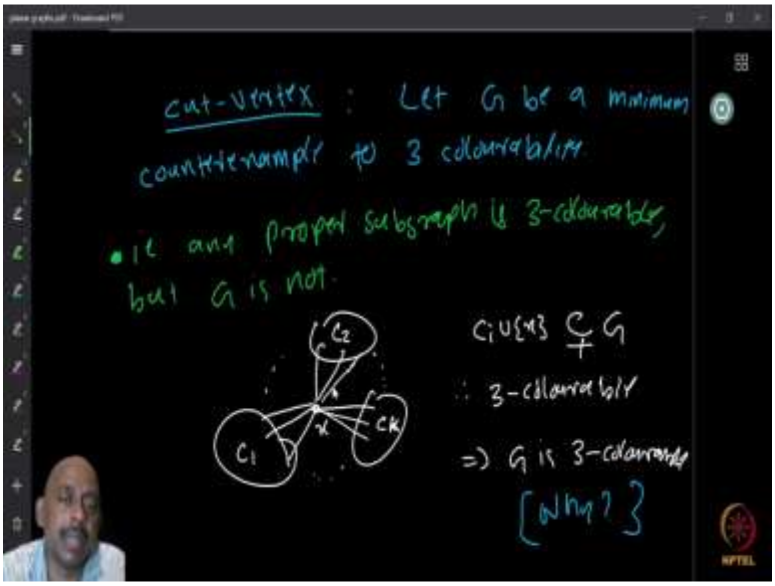

Let us look at why? Let us take a cut vertex, so suppose G is a minimum counter example to 3 colourability and the graph G has a cut vertex in it. Let x be a cut vertex in this graph. So, we have this cut vertex and by definition of cut vertex, if I remove it, I will get several components. So, let us say that, C_1 to C_k are our components after removing x. $C_1, C_2, C_3, \ldots, C_k$, so these are the components. C_1 to C_k are the components of this graph after removing x. Now, what I do is, I will look at the component C_1 together with the vertex x. So x has neighbours to C_1, so I look at all these neighbours including the component C_1, I look at this sub-graph.

Now, of course, the sub-graph is a smaller graph than G. So, if G was a minimum counter example, then to 3 colourability, then this graph, the component C_1 plus x, that sub-graph is definitely 3-colourable. Similarly, C_2 with x is also 3 colourable because it is smaller than G, similarly C_k with x is also 3-colourable. So, therefore, $C_i \cup x$ is all 3-colourable, strict sub-graphs of G, which are all 3-colourable. Now this tells that the graph G itself is 3 colourable.

Now, why is that? Can you think about this? If each component with x is 3-colourable then I claim that the graph itself together with x is also 3-colourable. Now, this is easy to see, because if I look at any colouring or C_1 with x, x will get some colour, let us say red, now C_2 with x also have a colouring, but x maybe getting a colour let us say blue there.

But now, it does not matter, what I do is that, I take the component C_2 and then I will change the name of the colours, all blue vertices, I will call green and all the green vertices, I will call blue. I will just rename the colour blue and green or exchange the colours. So, that the colouring of C_2 with x is still proper but x will get the colour green again.

Similarly, I can do it for each of the component. So, I will make sure that x gets the same colouring in all the colouring of the sub-graphs. So, therefore, altogether again it will be a proper colouring because each component gives a proper colouring with x getting green, I will get a proper colouring of the entire graph.

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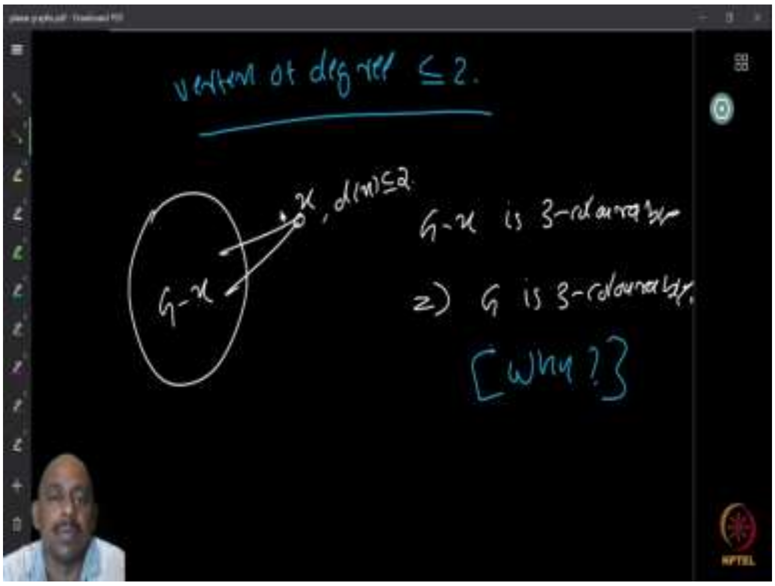

Now, vertex of degree less than equal to 2, cannot be part of a minimum counter example because if I look at G - x as a sub-graph, G - x can be 3 coloured, means that I can extend the colouring to include x also. Because if you look at the vertex x, it has atmost 2 neighbours and since I am using 3 colours, even if x is seeing 2 different colours in the neighbourhood, the third colour is not going to be used in the neighbourhood of x. So, I can give x with that colour. So that is also easy. So, therefore, vertices of degree less than equal to 2 or vertices of, which are cut vertices cannot be part of minimum counter example. So, therefore, they are reducible configurations for 3 colourability.

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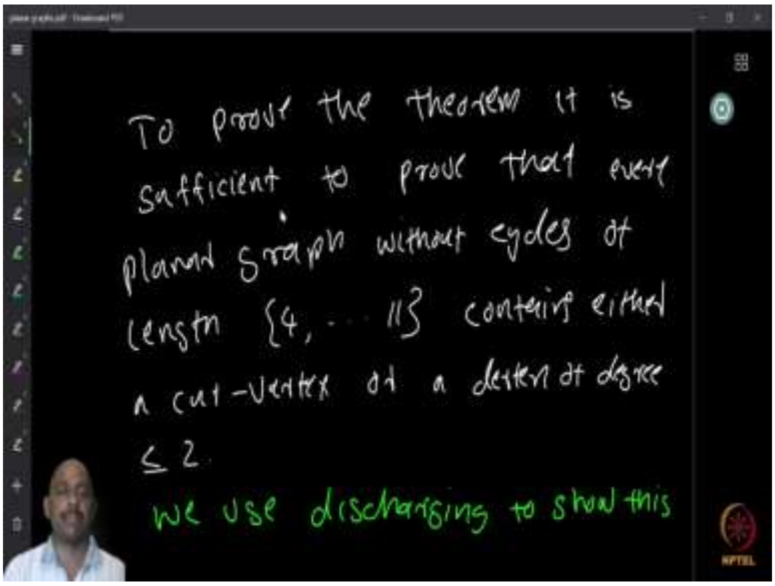

Now, we want to prove the theorem. For that, it is sufficient if you show that, every planar graph without cycles of length 4 to 11 contains either a cut vertex or a vertex of degree at most 2. Because as we just observed, the class of graphs without cycles of length 4 to 11, all sub-graphs also have this property. If there are no cycles of length 4 to 11 in a graph, if I delete something, you cannot create a cycle of length 4 to 11. Therefore, you will see that any sub-graph will have this property.

Now what we are saying is that, every such graph contains either a cut vertex or vertex of degree less than equal to 2, therefore none of the minimum counter example to the 3 colourability can be any of these graphs. But if there is a counter example in this class, I can always talk about this smallest counter example. And therefore, it cannot be there so that is the idea, so to prove that any such graph contains either a cut vertex or vertex of degree less than equal to 2, we are going to use the discharging method.

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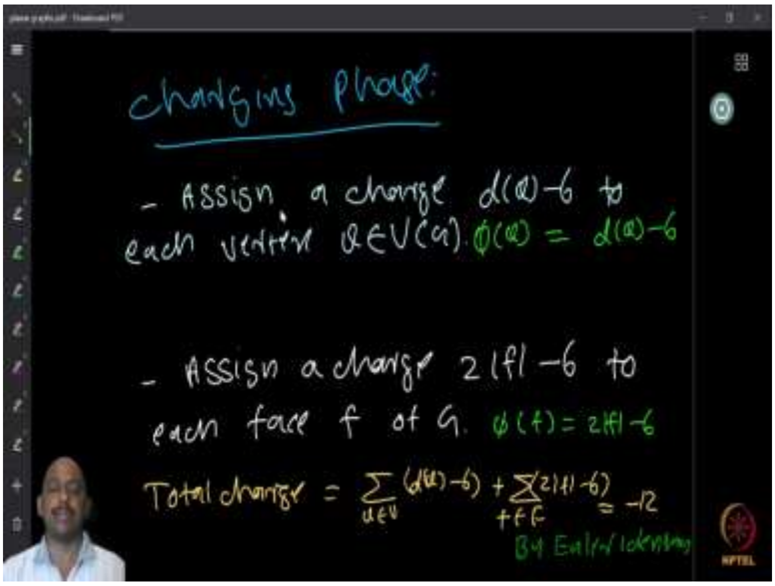

So, again we start with the charging phase. So what is our charging face? We are going to assign a charge, d(v) - 6 to each vertex of the graph.. This is the one thing that we did in the first example, the trivial example. So, the charge of v, $\phi(v) = d(v) - 6$. Then we assign a charge twice the length of the face minus 6 to each face of the graph G. Again, the charge of face, $\phi(f) = 2|f| - 6$. So,

total charge = $\sum_{v\in V} d(v) - 6v + \sum_{f\in F} 2|f| - 6 = -12,$ by Euler identity.

Now, why I say equal is that, because we assumed that there are no cut vertices, one can show that this is actually an equality.

Every face is actually incident to exactly 2 faces, and that will tell you that the degree sum will appear as an equality. So, you will get the total charge in the graph to be -12. This is true for any planar graph in this class that we are looking at. So any such planar graph will have total charge to be -12. Whatever the size of the graph, total will be always -12.

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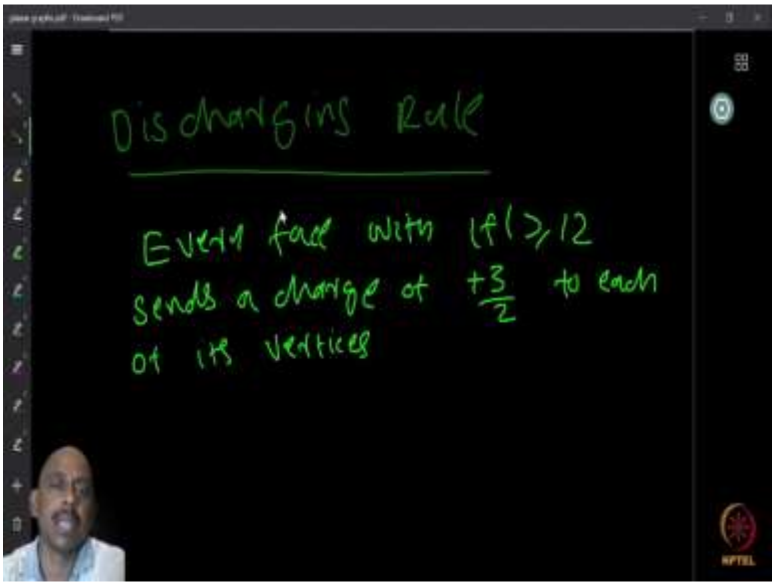

Now, we come up the discharging rule, right? So, the discharging rule says that, every face with length at least 12 sends a charge of +3/2 to each of vertices. Now, this might look like a kind of halt idea, I mean how do you come up with the discharging rule. To learn the discharging method, the important fact is to see, how you come up with the discharging rule. This is not always an easy phase, but here is the idea, what we want to do here. So, the idea of the discharging method, as we mentioned before is to show that we first start with a charging and take the total charge.

The total charge, I will show that is going to be a fixed number or either negative or positive. Here we showed that, the total charge is -12, which is negative number. Then the idea is, you move around the charges, to show that there is some structural properties like the reducible configurations are present, what we are going to show is to move around the charges and say that, okay after moving around the charges, if the particular property that we want is not there, then the total is going to be different.

Now, if I can show that the total is going to be different, then, that says that there is something wrong with the assumption. We assume that there is no reducible configuration percent, that was the problem and therefore, we can say that, it must be present so it is unavoidable. Now, what we want to show in this case is, since the total charge is negative, I want to somehow show that the total charge is going to be non-negative if there is no cut vertex of vertices of degree less than equal to 2.

The idea of charging is to make sure that, the total charge will be non-negative if we assume that, there are no cut vertex of vertices of degree less than equal to 2. So, for that purpose, we have to design the discharging rule, so how to move around the charges, depends on that.

The ones with the least negative charges should get a lot of positive charge so that it will become non-negative and the one with already positive charge does not need to get anything maybe. And others depending on the cases, you can decide how much charge it should get. So, this is how we design the rules, so in this case, we look at these questions and then come up with the following rule that every face with length at least 12, sends a charge of +3/2 to each of its vertices.

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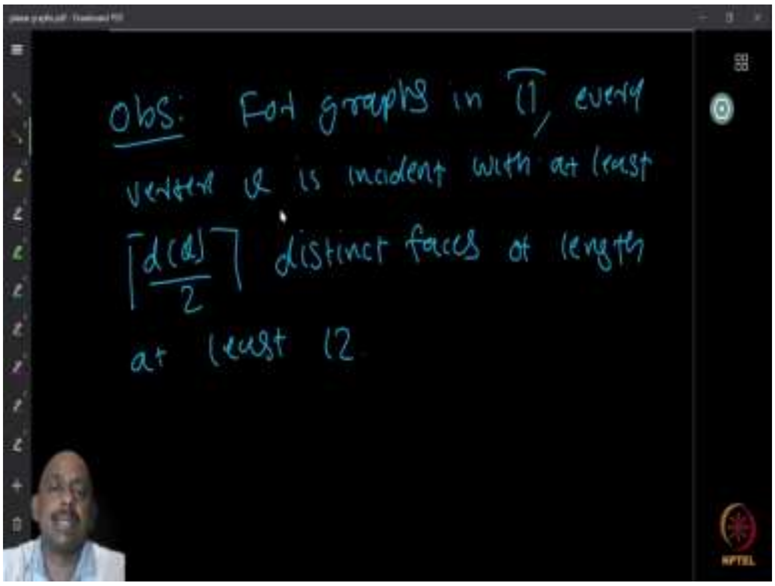

Now, let us see what happens, first observation is that if there is a vertex v, which is not a cut vertex then, we observe that it is incident to at least degree many distinct faces, all the faces are distinct faces. If the vertex is not a cut vertex, then we say that each of the faces are distinct faces. Now, our claim is that, for the class of graphs that we are looking at, every vertex is incident with at least $\lceil \frac{d(v)}{2} \rceil$ distinct faces of length, atleast 12.

So, I call a face large, if the length is at least 12. We are saying that, for every vertex, it has at least $\lceil \frac{d(v)}{2} \rceil$ many distinct large faces incident with it. Now, why this is true? This is true because of this property, that if you look at any vertex which is not a cut vertex, it has exactly d(v) many face.

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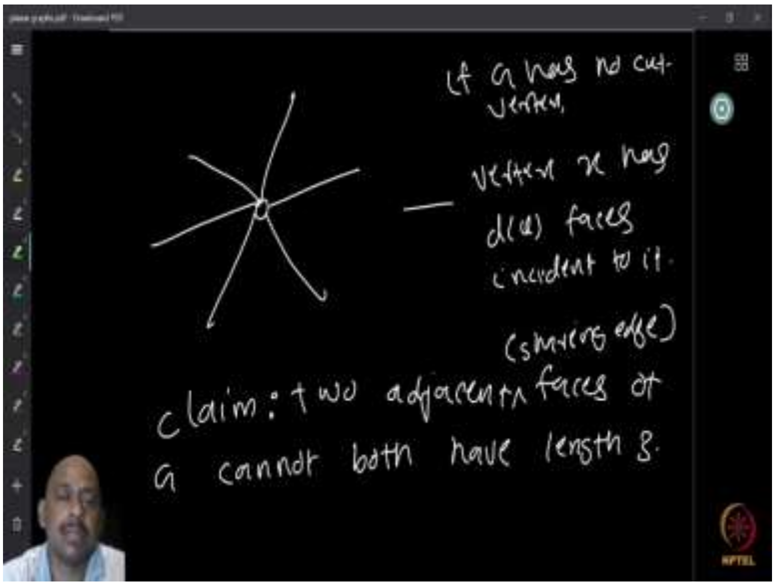

But now two adjacent faces of the graph cannot be triangles. So, I claim that, 2 adjacent faces cannot be triangles, this is not possible. Why is that not possible? Can you think of why? 2 adjacent vertices can be 3-cycles. Think for a few minutes before we proceed.

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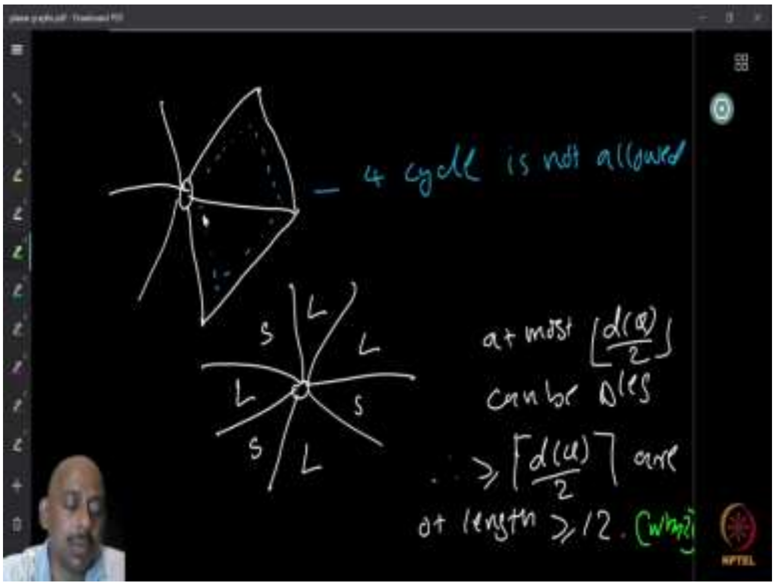

So, if you have thought about this, you must have observed that, if I have 2 adjacent 3-cycles or triangles, then they together form 4-cycles. These 3-cycles plus these 3-cycles, this forms 4-cycles in the graph. But we look at graphs which does not contain any cycles of length 4, 5, 6 or upto 11.

So, since there are no 4 cycles in the graph, you cannot have 2 triangles sharing an edge. But now, since I have degree many distinct faces, only alternate faces can be at most small faces.

So, if I have a small face, then the next immediate faces can be only large and then, the alternate ones can be again small but then again this must be large, and this also must be large. Therefore, atmost $\lfloor\frac{d(v)}{2}\rfloor$ many can be small triangles, therefore, the remaining, which is the $\lceil\frac{d(v)}{2}\rceil$ must be of length greater than equal of 12. And why is that? Because we do not allow cycles of 4 to 11 in the graph anyway. So, if there is no 3 cycle then the next is going to be last side.

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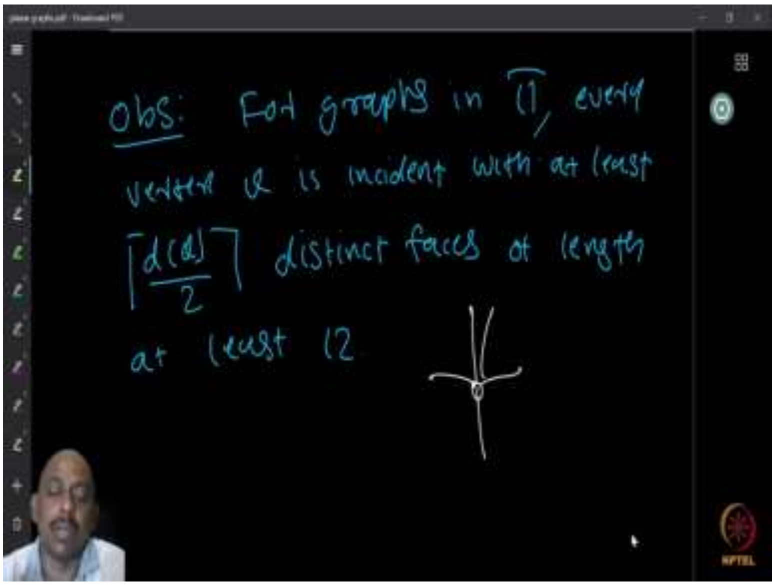

So, that is our observation that, if you are looking at graphs in the class Π, then every vertex is incident with at least $\lceil\frac{d(v)}{2}\rceil$ distinct faces of length at least 12. This tells you that, these many faces are going to give you, positive charges +3/2.

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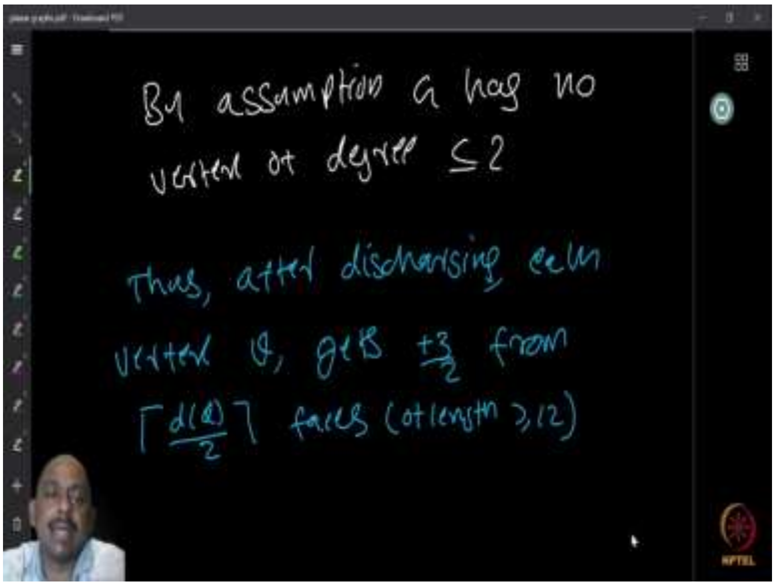

Now again, by our assumption, the graph has no vertex of degree less than or equal to 2. The second assumption was that, we are looking at graph and showing that there is no cut vertex of degree less than equal to 2. So by assumption we know our graph does not have vertices of degree less than equal to 2. If it is there, then we are already done.

Now, therefore, what happens after the discharging? Well, after the discharging each vertex gets a charge +3/2 from each of its $[\frac{d(v)}{2}]$ many faces of length greater than equal to 12, that many faces are incident. So, we see that after discharging, every vertex gets these many charges.

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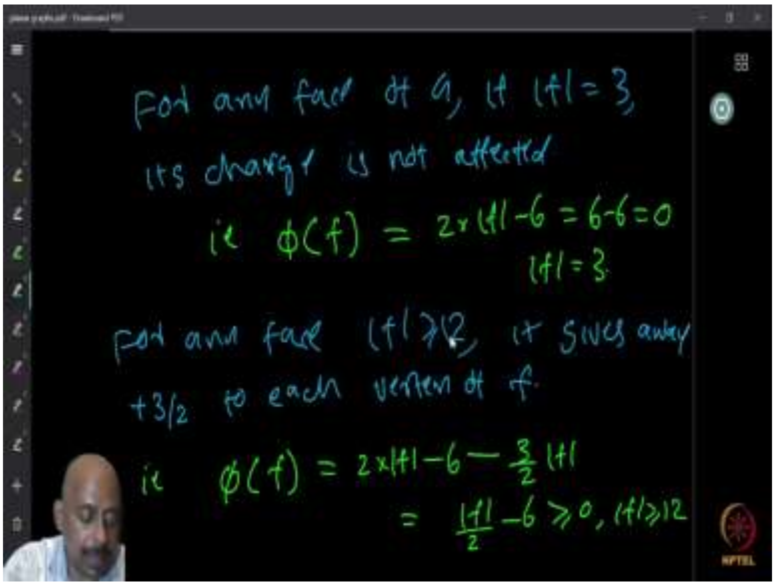

Let us see, what does this implies. They say that for every face if you look at the length of the face can be either 3 or 12 or 13 etc…. If the length of the face is 3, then its charge is not affected because, small faces does not give away charges. It does not get any charge, so therefore, its initial charge remains. So, what is the initial charge of a 3 cycle? It is 2 × 3 - 6 which is 0. So, all the triangles still have charge 0 which is non-negative.

Now, if you look at the face of the length, greater than or equal to 12, this face gives away, +3/2 to each of its vertices. So, what happens to each of these vertices, it gets +3/2 but what happens to the face? Its charge decreases by 3/2 into the number of vertices.

So, these 2| f | - 6 is the initial charge, -3/2 × | f |, which is the charge gave away. So, $\phi(f) = 2|f| - 6 - \frac{3}{2}|f| = \frac{|f|}{2} - 6 \geq 0$. $So, |f| \geq 12$. Large faces has enough surplus charges to give away and still be having non-negative charge. We have designed our number 3/2 so that this works out.

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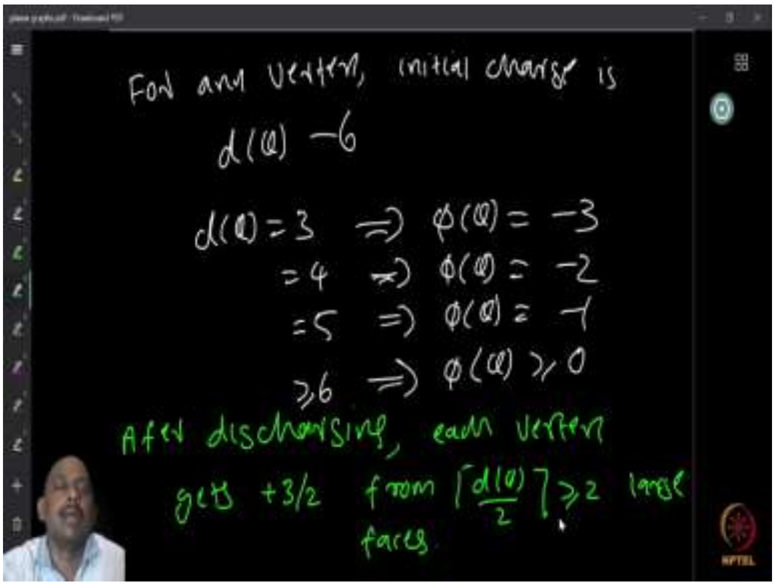

We saw that every face after the discharging has non-negative charge. Now, what will happen to the vertices? If you look at any vertex, its initial charge was degree minus 6. What happened to it? From all the large faces incident to it, it got charge +3/2.

So, what happened to vertices in the graph, that has degree either 3, 4, 5, 6 larger numbers. If the vertex degree was 3, its initial charge was -3. But then, what happened to it? It has at least degree by 2 ceiling, which is $\left\lceil\frac{3}{2}\right\rceil = 2$. It has at least 2 large faces incident to it and these 2 large faces must have given it, +3/2 charge each.

So, 2 × 3 / 2 = +3. So, the vertex degree 3 gets a charge atleast 3 from its faces. And 3 + (-3) = 0. So, its charge becomes 0. So we designed our number +3/2 so that this also have.

Then, again what happened to the vertex of degree 4? Its charge was -2 but it gets at least 3 again so that for it becomes +1. Similarly, degree 5, it gets actually 3 × (3/2), so it is going to become positive again. For degree greater than equal to 6, is going to be always greater than equal to 0 because it is not going to lose any charge. So, it is only going to get positive charges. So, therefore, every vertex of the graph, after discharging, has positive charge or non-negative charge.

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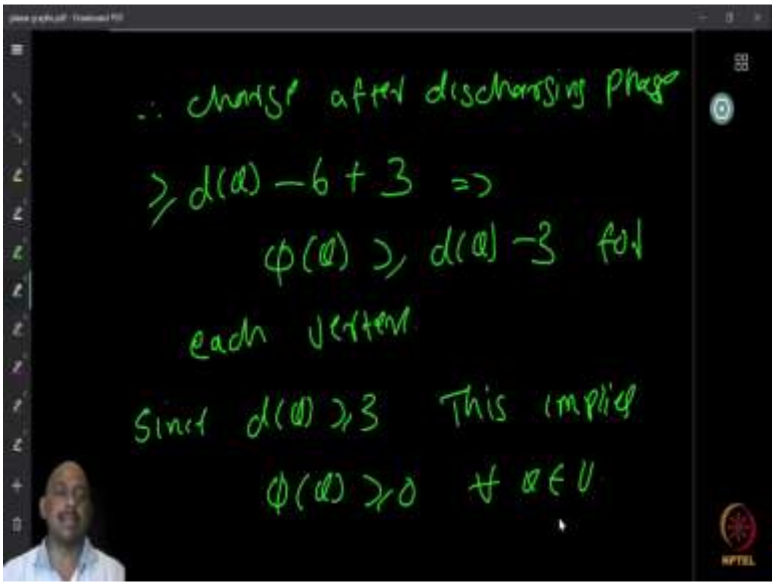

Every face, after the discharging has, non-negative charge. This is that, the total charge is going to the sum of all the vertex charges plus the sum of all the face charges which is going to be non-negative. But this is impossible, right? After the discharging, we see that, all the vertices and the faces has non-negative charge which is impossible. Now, why this impossibility came; because we assume that there were no vertices of degree less than equal to 2, if there is a vertex of degree 2, by doing this discharging its charge which was initially -4 need not become positive.

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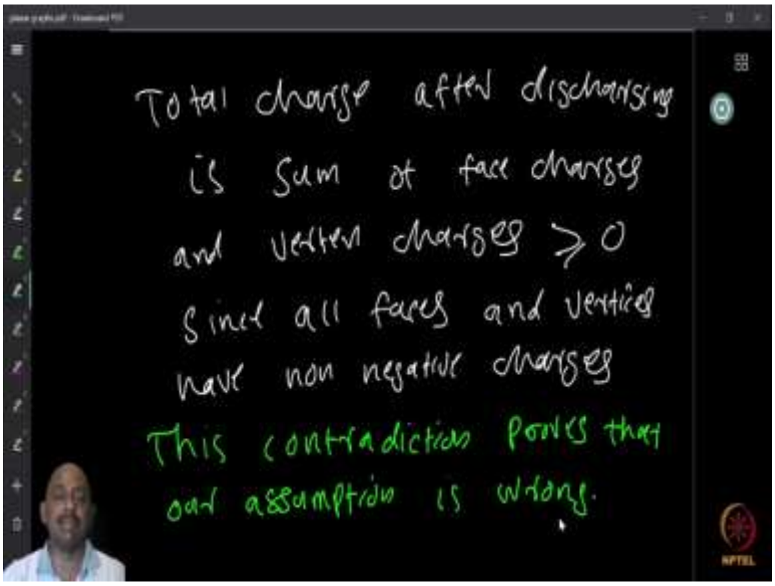

Similarly, if there was a vertex which is a cut vertex, this could also not become negative because it might have several faces that we over counted because 2 of the faces that we are thinking is different might be the same. So, it may not have enough faces to give its positive charge. So, the total charge after discharging, will be negative and that happened to be non-negative only because we are assumed wrongly that there is no cut vertices or vertices of degree less than equal to 2.

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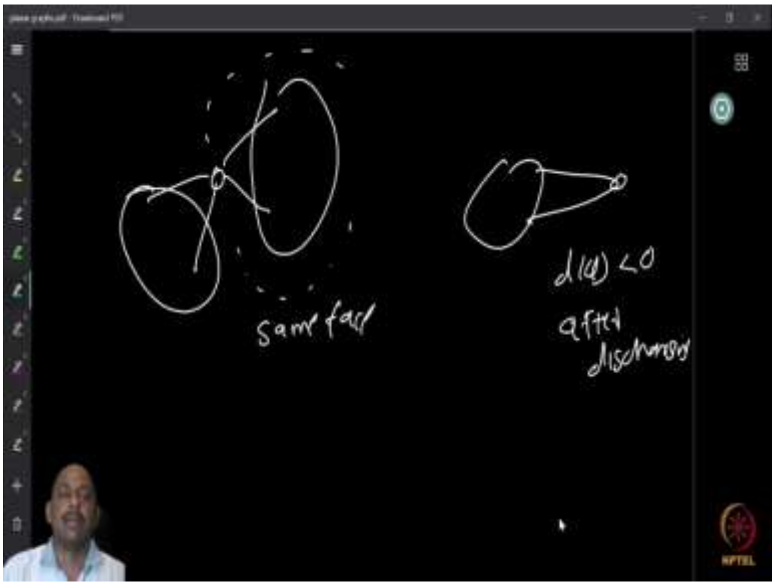

So, that is the idea and this proves that, there must be a cut vertex which gives this kind of situations where we do not have a degree by 2 many large faces or degree 2 vertices which also will not become non-negative. So, at least one of this must be present in the class because the total can never be non-negative because it was -12 so this establishes the result.

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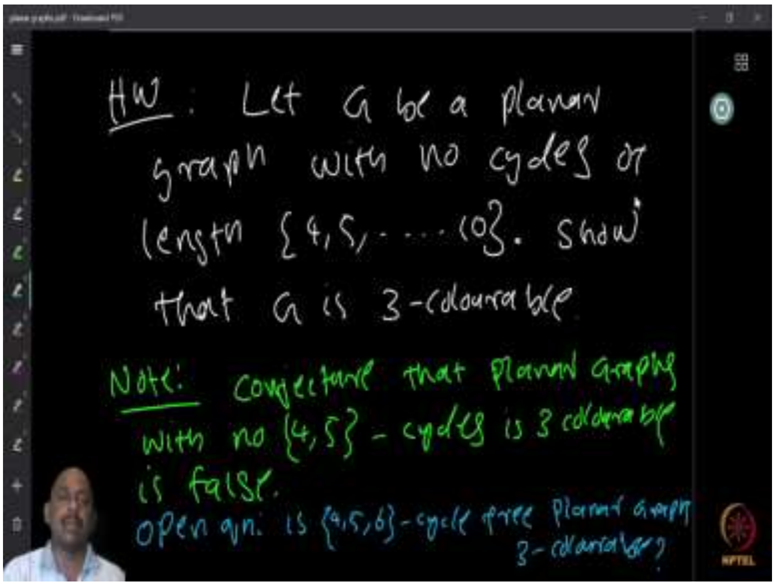

Now, here is a nice homework question. Let G be a planar graph where there are no cycles of length 4 to 10. So, we started looking at the question 4 to 11, now we say that, we allow 11 cycles but there are no cycles of length 4 to 10. Now, can you show that the graph is 3 colourable? I claim that this can be done by slightly modifying the proof that we did. So, take it as a challenge and try to do it. It may be little hard work but it will be nice.

Now again, I think I mentioned this already that the conjecture, that the planar graph with no cycles of length 4 or 5 is 3 colourable, is a false conjecture because the conjecture was disproved a couple of years back but still open question is that, if there are no 4, 5, or 6 cycles in the planar graph, then we say the graph is 3 colourable.

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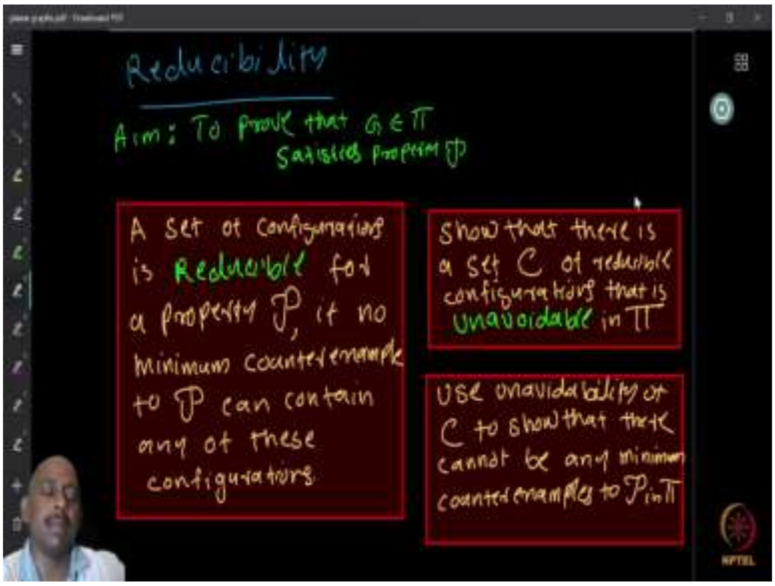

So, an over view of what we did was that, we know, the discharging method works as follows. We want to show some property is true for a class of graphs, so we have class of graph Π, we want to show some property $\mathcal{P}$, we try to come up with a set of sub-structures like the examples which we saw, cut vertices degree of, small degree vertices or some other things like that, some configurations are reducible for this property. So, try to come up with sub-structures for this property, it is reducible at, no minimum counter example in this particular property can contain this.

Now, try to show that, is it possible to show that, one of the structures must be present in the class of graphs that we are looking at or for what class of graphs we can show something like this. And if you can show that it is unavoidable for that particular class, we know that the property must be true. And to show this, we do the discharging and the discharging is by giving some initial charges, this initial charging will depend on what we will do with the discharging phase. Now, the discharging phase depends on what we want to actually show.

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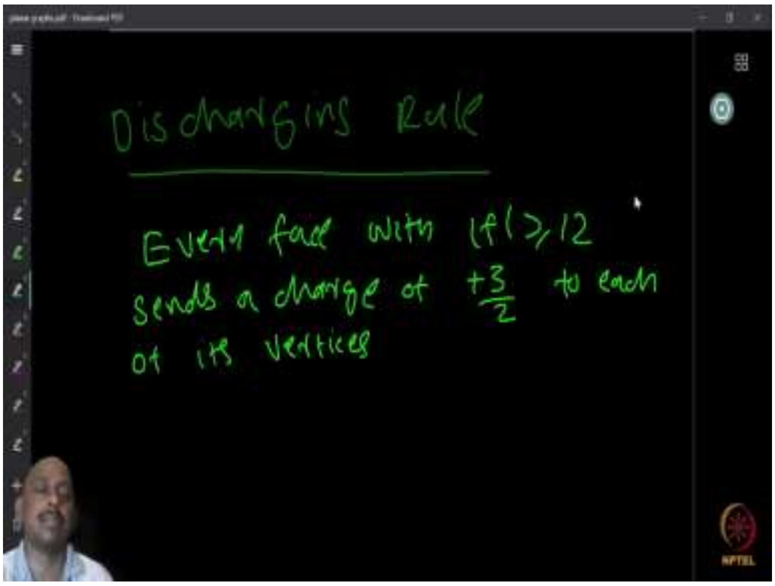

Depending on the type of property that we want to show, we have to design the rules of moving around the charges. And once this trial is there, you can try out whether it works or you can slightly modify it, then you can again try to change something like this or you can change the reducible configuration sets itself, maybe slightly and try again with different discharging rules. So, at some point it works out and then you have the result. So, that is the basic idea of discharging method. So, try this homework and with that we can stop this short introduction to discharging method. So, I can give you more interesting questions on this, and maybe look at some related topics later.