Combinatorics
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Lecture 44
The Discharging Method – Part 1

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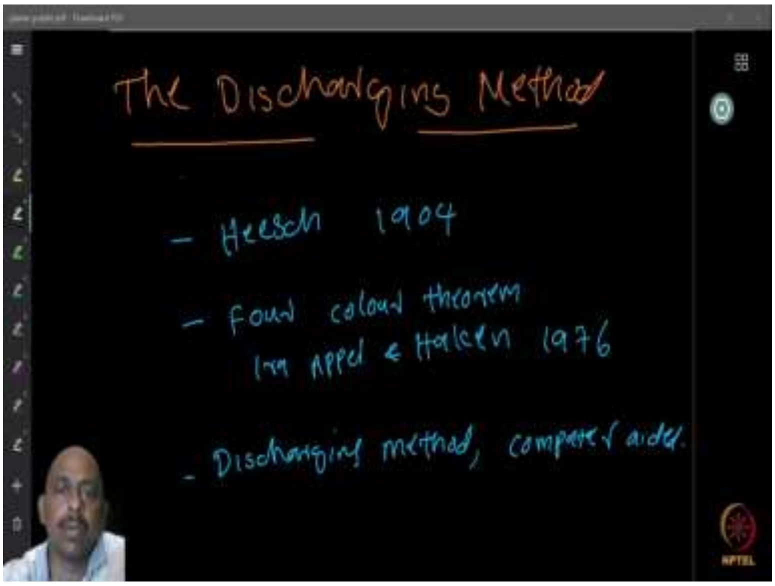

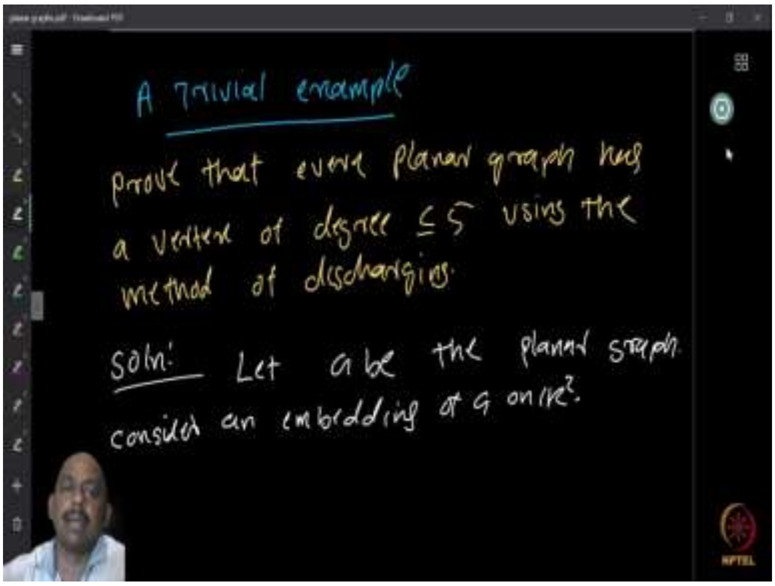

So, let us now look at the Discharging Method. We start with a very small example, a trivial example in fact, to prove this you do not really need any discharging method or anything. We can directly prove it from some of the simple results that we have studied. But we want to use it to just start an idea of discharging.

We want to prove the following: every planar graph has a vertex of degree less than or equal to 5, and we want to prove it using the discharging method. How do you do this? We start with the graph which is a planar graph and consider some embedding of this graph on the plane because it is planar, I have a plane embedding.

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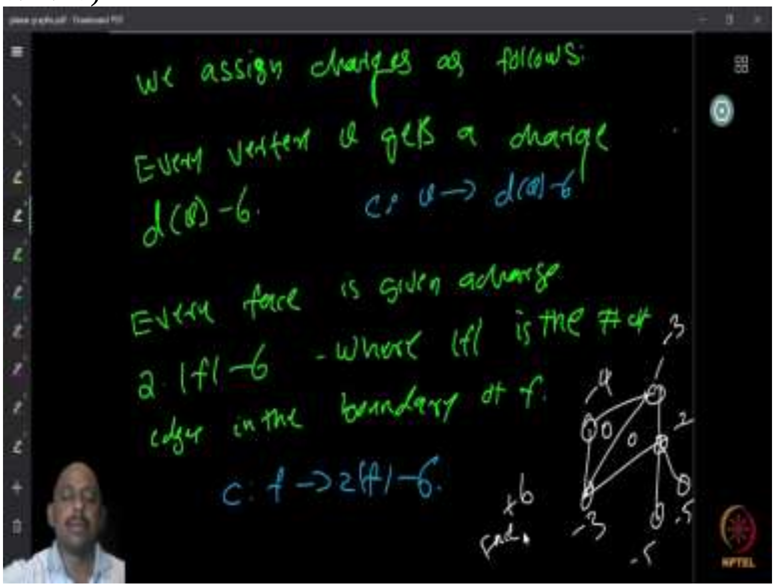

Let us look at this plane embedding of the graph, now what we do is that once we have this embedding, we assign charges to the vertices and faces as follows. So, given a graph, what we will do is that, we look at every vertex and then say that this vertex gets a charge equal to its degree minus 6. That is, every vertex v gets a charge d(v) - 6. So this vertex is degree 3, therefore it gets minus 3 charge. This has vertex degree 2 therefore it will get 2 - 6 which is -4. Here I will get 4 - 6 which is -2, 1 - 6 = -5, and -3.

So, all the vertices get a charge its degree minus 6, I mean if you have degree 10, you will get a charge 4, if degree 20 then that vertex will get charged 14. Then every face of the graph is also given a charge. It is twice the length of the face minus 6. So, length of the face is the number of edges in the boundary of the face. That is, every face is given a charge 2| f | - 6, where |f | is the number of edges in the boundary of f.

So, the number of edges in the boundary is the length so therefore if you look at this face it has 3 edges in the boundary, therefore its length is 3, so its charge will be 6 - 6 which is 0, again 2 × 3 - 6 is 0. But if you look at the outer face it has 6 so it has 2 × 6 – 6 = 12 - 6 = 6 for the outer face. So, this way you assign charges to the graph.

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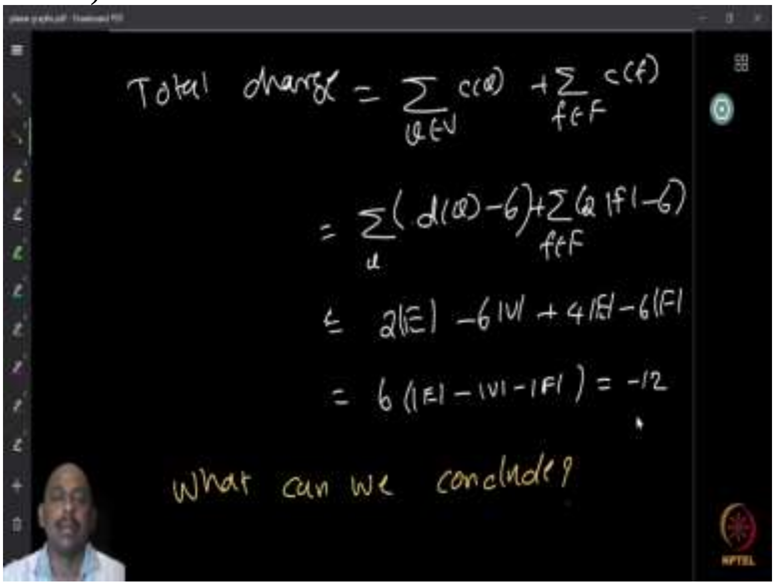

Now, I want to look at the total charge in the graph. What is the total charge? It is the total of all the vertex charge plus all the face charge.

Total charge = $\sum_{v \in V} c(v) + \sum_{f \in F} c(f)$

$= \sum_{v}(d(v) - 6) + \sum_{f \in F}(2|f| - 6)$

$\leq 2|E| - 6|V| + 4|F| - 6|F|$, since, $\sum_{v}(d(v) = 2|E|$

$= 6(|E| - |V| - |F|)$

$= -12$

So, the total charge is at most -12. What can we conclude from this? So, no matter what planar graph that we started with, we see that the total charge is going to be less than or equal to -12. But if the total charge is negative then some elements in the summation must contribute negative value. What are the negative values that is contributed by this element? If you look at any face, every face boundary is at least 3 edges, so the length of any face is at least 3.

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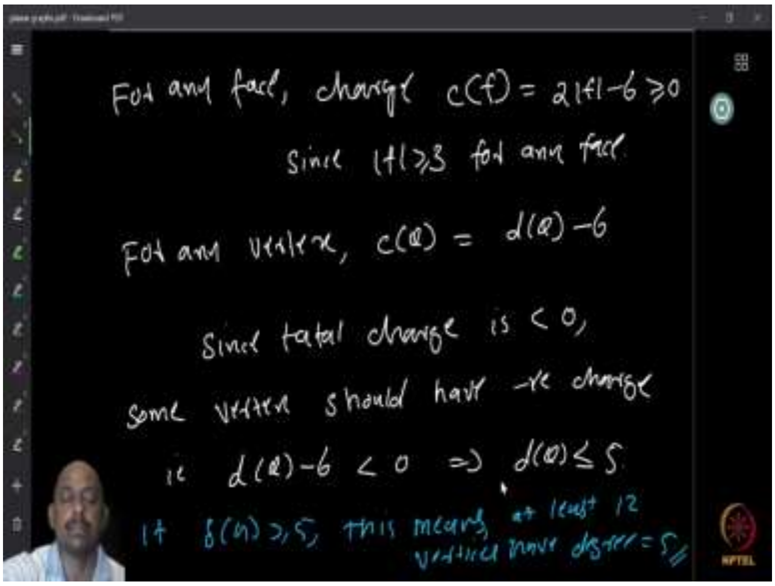

So therefore, if you look at 2f- 6, 2f – 6 ≥ 0 because the length of any face that is atleast 3 and 2f ≥ 6. On the other hand, for a vertex, its charge is the degree minus 6. Now, the total charge is negative, if you sum non-negative numbers, you are not going to get total to be negative. So, therefore some charge should come from a vertex which is negative because faces are all giving non-negative values, vertices must give negative values.

So, it says that some vertex should give negative value and a vertex which gives negative value cannot have degree greater than 5, because if the degree is 6 or more its charge is also non-negative. So, therefore there must be some vertex whose degree is less than or equal to 5. Now, this tells apart from just showing that there is a vertex of degree less than or equal to 5, it tells that if you assume that the minimum degree of a planar graph is 5, then you should have at least 12 vertices of degree equal to 5.

This is an additional information which you do not get from the other proof. Because if you look at a planar graph where the minimum degree is 5, then since the total is -12 each vertex can only give -1 to the sum. So, therefore any planar graph with minimum degree 5 should have 12 or more vertices of degree equal to 5. So, this also tells that if you make graphs with minimum degree 5 on let us say on 8 vertices or 6 vertices or 10 vertices, these are all not going to be planar. They are going to be non-planar.

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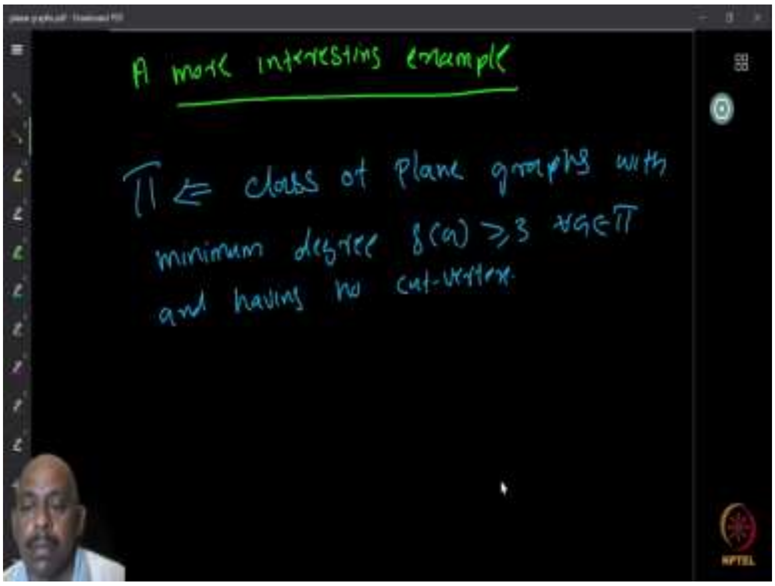

Now, let us look at a more interesting example to see what exactly is the discharging method about. So, let Π be a class of plane graphs, its a planar graph with a plane embedding. Π be a class of plane graphs with minimum degree at least 3, so every vertex has degree greater than or equal to 3 and we assume that these graphs does not have cut vertices, just to make our argument simple. So, we will assume that we are looking at plane graphs where the minimum degree is 3 and does not have any cut vertex, this is the class Π of graphs.

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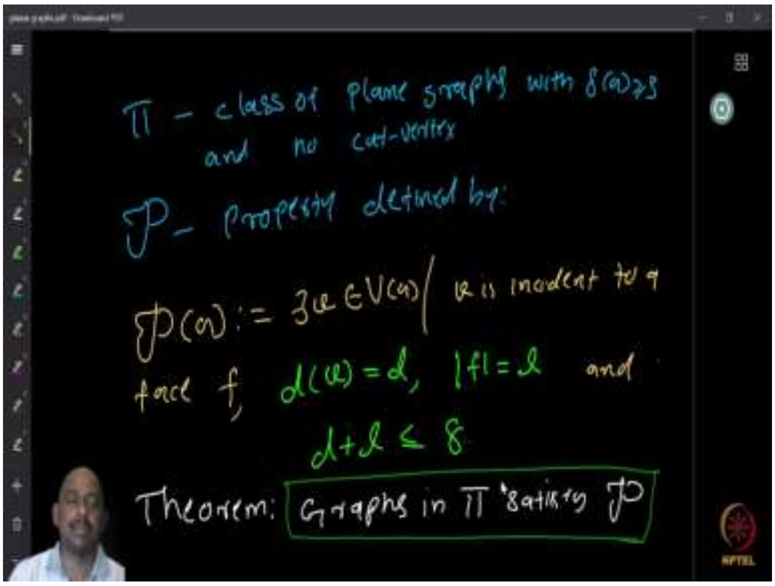

Now, we have a property $\mathcal{P}$ so the property that I define is $\mathcal{P}(G)$ is defined as follows, that there is some vertex and this vertex is in a plane graph so this vertex is inserted into a face f because once you have an embedding every vertex incident into several face. So, a vertex is incident to a face f, such that the degree of the vertex is d and the length of the face, the number of edges in the boundary of the face is l and d + l is less than or equal to 8.

That is $\mathcal{P}(G) \coloneqq \exists v \in V(G) |\ v\ \ is\ insident\ to\ a\ face\ f,\ d(v) = d, |f| = l\ and\ d + l \leq 8$

So, property P says that you can find some vertex which is incident to a face for a plane graph, a vertex incident to a face such that the length of the face plus the degree of the vertex is at most 8. You can always find if the minimum degree is at least 3 and there is no cut-vertex, this is the claim. So, the graphs in Π satisfy property $\mathcal{P}$. So if you are looking at plane graphs with minimum degree 3 and no cut vertex then it will have always a vertex and an adjacent face whose sum is at most 8. So, we want to prove this, so we are going to use discharging method to prove this.

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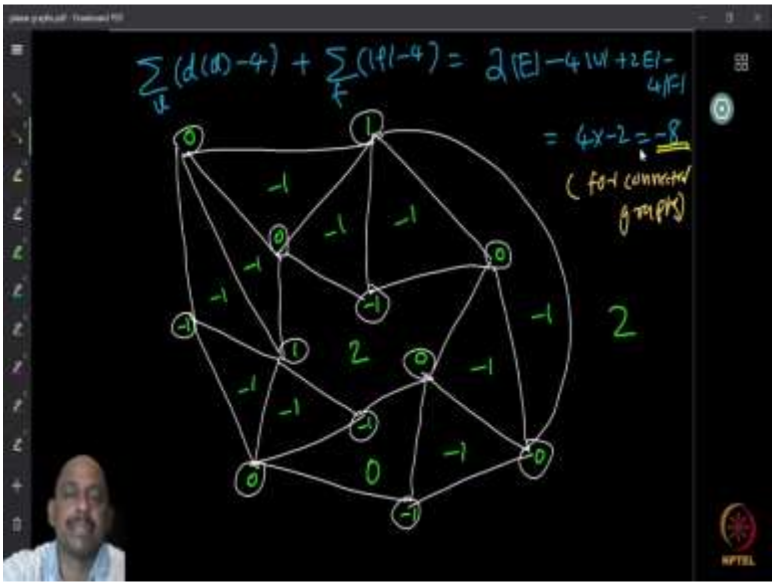

So, let us start with a simple graph example as a working example. So we start with this graph. So, it has all these vertices and edges which are defined and now I am going to give some charges as before. We are going to give charges, so for discharging we need charges. So, what is our charge? The charge is the following, so the charge rule is the following that every vertex gets a charge, its degree minus 4 and every face gets charge, the length of the face minus 4. Now, once this charging is applied what happens to the graph? Well, you can see what are the charges after charging face.

So, after charging the face, you will see that these vertices have, for example this one has initially degree 4 so the charge is 4 - 4 which is 0, it had 3 so therefore, 3 - 4 which is -1. Then similarly it had degrees 5 therefore it has charge 1, 0, 1, 0, -1 etc. And for the faces, for example this face had 1, 2, 3, 4, 5 and 6 edges in the boundary. So, therefore 6 - 4 which is 2 similarly other faces will get its charge 0 or -1 over 2 etcetera. So, all the vertices and faces get its charges.

Now, the idea of the discharging method is to somehow show some structural property of the graph. So, in this particular example we want to show some vertex which is incident to some face, so that the degree of the vertex plus the length of the face is at most 8. Now, since we are looking at vertex face incidence, let us define a vertex face incidence.

By defining the vertex face incidence of a graph as a corner. So, what do I mean is that, when a vertex like this is incident to a face like this, this is a corner for me, so vertex face incidence, because it looks like a corner? So, I am just defining a term which is not in standard graph theory just for the purpose of this problem. So, I define a vertex face incidence as a corner.

Now, before doing anything with the corner, I observe that what is the total charge after the charging face. So, if we look at the total charge it is

$$\sum_{v}(d(v)-4)+\sum_{f}(|f|-4)=2|E|-4|V|+2|E|-4|F|$$

$$=4(|E|-|V|-|F|)=4\times -2=-8$$

So, if you started with this, you will always get -8, for connected graphs with the property that we are looking at.

We will assume that the graph is connected because for each component we can see whether the property holds and we can do some small modification to the proof to do it for all graphs. So, we will assume the graph to be connected for the time being. So, for connected graph we can use the Euler identity, show that the total charge is -8.

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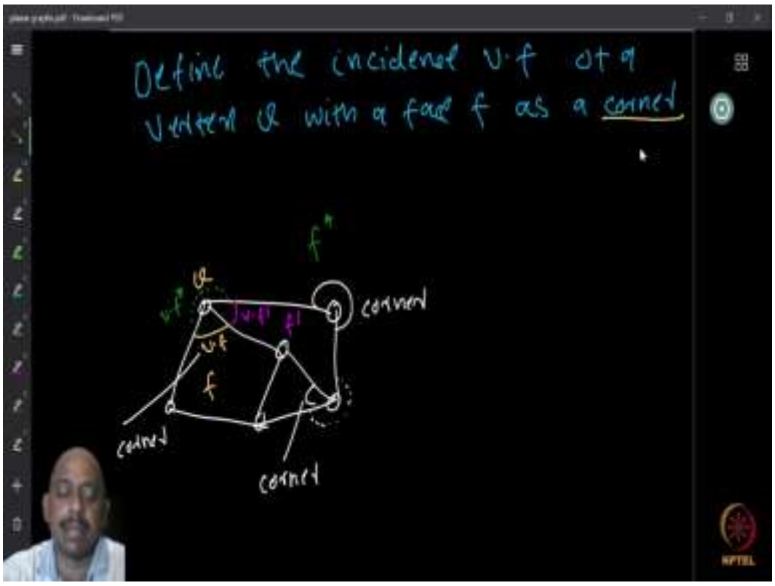

Now, we define the incidence of a vertex and a face as a corner and then here are some examples. So, you have this vertex here and you will see that its degree is 3, it has exactly 3 corners, for example this corner, this corner and this corner. Similarly, this has 2 corners, each vertex has several corners, we can define. Now, once you have this let us look at what we can do further with this idea.

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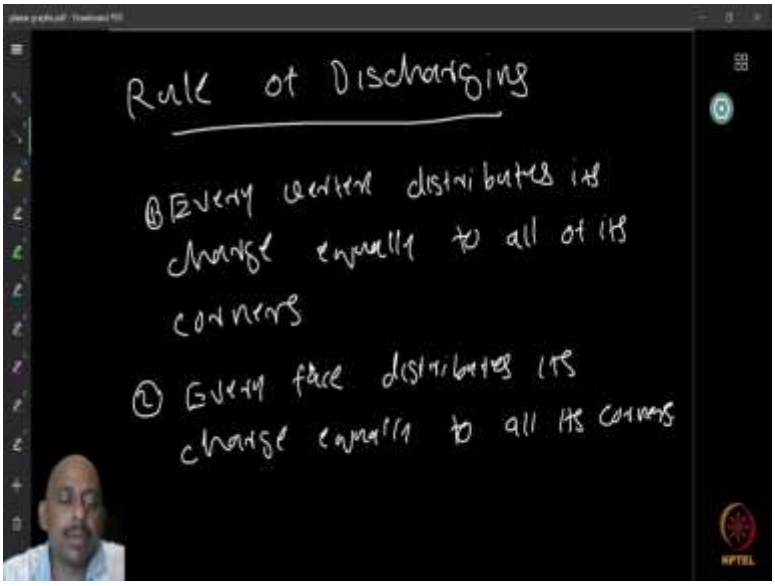

So, we are going to define a discharging rule, so this is the reason why it is called discharging method because after the charging face, we do a discharging face. So, in the earlier example very simple example, we did not do it but in general we do a discharge, so what is the discharging rule? Discharging is basically moving around the charges without losing anything like we just move around the charges. So, transfer the charges from one place to other etcetera. So, in this example our discharging rules are the following.

So, the first rule is that every vertex distributes its charges equally to all of its corners, so it gives away all its charges to all its corners, whatever I have I am going to distribute it to all of that. Every face distributes its charges equally to all its corners, so again a corner is a vertex face incidence, so when I have a corner, it has a vertex as well as a face, so the face has several corners, vertex also has several corners. Each face gives its charges equally distributed to each of its corners. This is the discharging rule.

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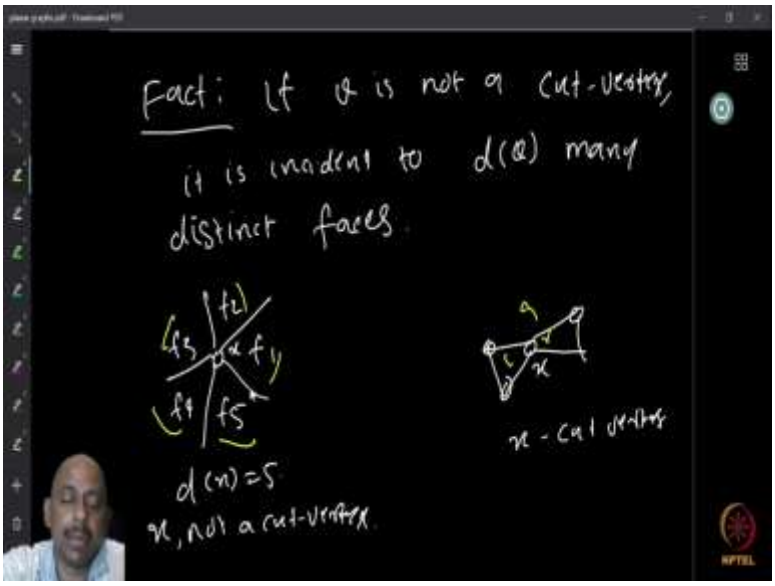

What is the meaning of this statement? This says the following. Let us observe something before we go into that. Suppose a vertex is not a cut vertex. So, if it is not a cut vertex, then I want to observe that this vertex is incident to exactly its degree many distinct faces. So, what I am saying is that, if this vertex x is not a cut vertex, then this faces f_1, f_2, f_3, f_4 and f_5 , all these faces are all distinct.

If one of these two faces is the same as the other face, then the vertex will be a cut vertex. That is the observation, you can think why? On the other hand, if x is a cut vertex, you will see that you may not have the property that for example, if the degree is here 4 but there is only 1, 2, and 3 faces because this side and this side are the same, same face only or part of the same face. Therefore if a vertex is not cut-vertex, then we will see that it has exactly its degree many distinct faces incident with. So, it has exactly that many corners also. Again, it is very easy to see that if you have a cycle or a face, then it has its vertices and the number of the length of the face and the number of corners is the same.

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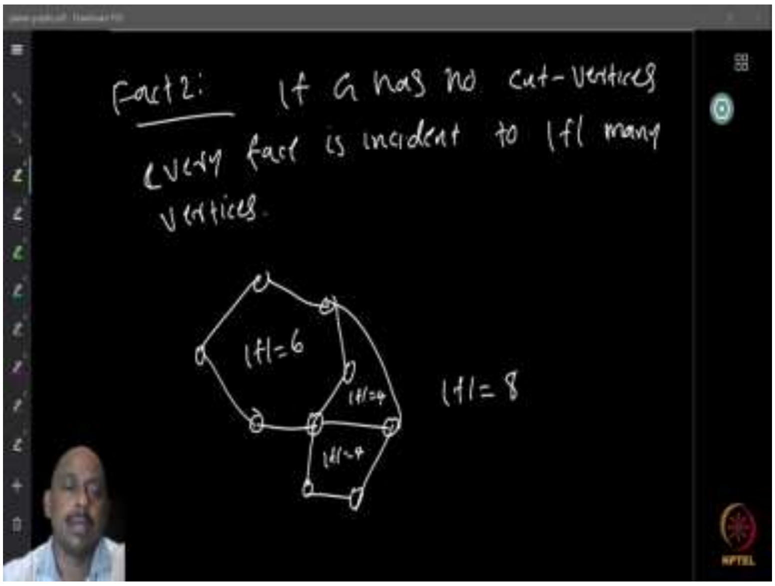

So, that is the fact 2, if G has no cut vertices every face is incident to length of the face many vertices because it is basically a cycle, boundary is a cycle. In a cycle, number of edges and number of vertices are the same. Now, from these two observations we can see what the discharging rule does to the charges?

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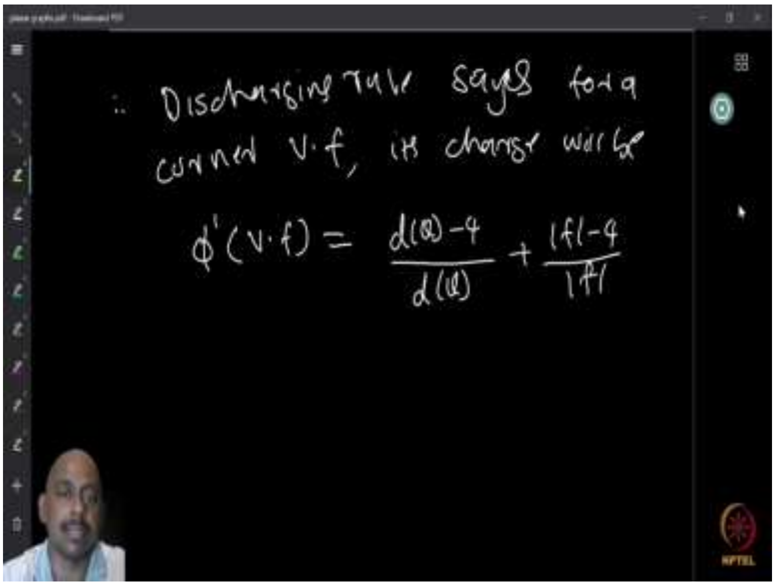

So, the discharging rule says for a corner ‘ v . f ’, its new charge, the charge of the corner will be like, if ϕ' is the charge for the corners, then for any corner v . f, $\phi'(v.f)$ will be from the vertex v the corner will get the degree of the vertex minus 4 initial charge divided by degree of v because vertex v distributes its charge equally to all its faces.

So, it has exactly d(v) many faces so therefore, it distributes its charge to the d(v) - 4 charge to each of the degree of v many corner. So, therefore it will be divided by d(v), so that is the charge given by the vertex to its corner and the face, its charge is | f | - 4 and because the face has exactly length of f many vertices, that many corners are also there . Therefore that face also gives | f | - 4 divided | f | charge to the face. So, this is the only charge the corner is going to get and therefore the charge of the corner is going to be this value. That is

$$\phi'(v.f) = \frac{d(v)-4}{d(v)} + \frac{|f|-4}{|f|}$$

So, for every corner you see what is the charge that it has. Of course, what happened to the vertices charge, it has all disappeared. It has given away its charge whatever positive or negative equally to its coordinates. Similarly, all the faces have given away the charges. So, the faces and vertices now have 0 charge.

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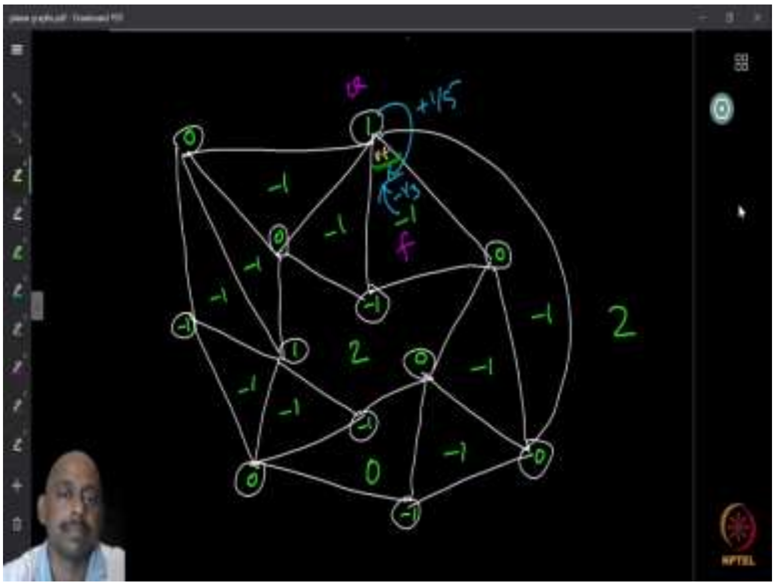

So, here is the rule in action, the vertex with the degree 5 has charge 1, it gives charge 1/5 to each of the 5 corners. So, in particular to this guy, it gives a charge + 1/5, similarly, the corresponding face f, it is also giving a charge, its charge is -1, it has exactly 3, neighbours, 3 corners, so -1/3, -1/3. So, that charge goes to this corner, so this corner gets +1/5 and -1/3 so the sum of these two will be, discharged.

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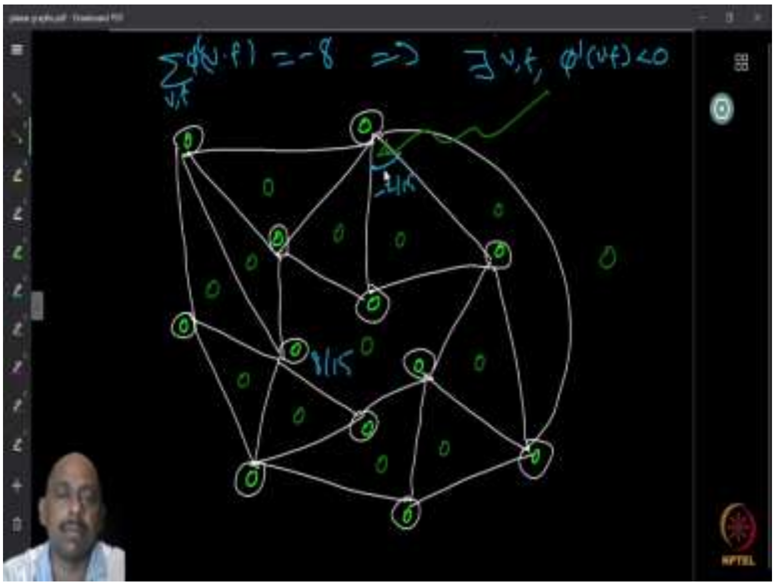

Similarly, for each corner you will get. So once the discharging completes you will get something like this, the total charges in each of the vertices are going to be all 0's and charges in each vertices will be 0's, charges in each of the faces will be 0's and then the corners will have some charge.

But now our observation is that, the charges have all moved to the corners so the total charge in the corners will be the same as in the total charge we had earlier because we just moved around the charges. We just moved or shifted the charges from the vertices to the corners but we did not change it or we did not lose it. So, the total must be the same so the sum over all the vertices and faces the charges of the corners is -8, because -8 was the earlier sum of charges.

Since our total remains the same, the total is again -8 but if the total is negative then as we observed earlier at least some of the charges must be negative because if everything is non-negative the sum is going to be non negative. So, it says that there must be some corner with negative charge, so we can find such a corner, we said that since there is a corner with negative charge, you can pick one up. So, we see that there is some corner with the charge less than 0 which we find so pick one of them.

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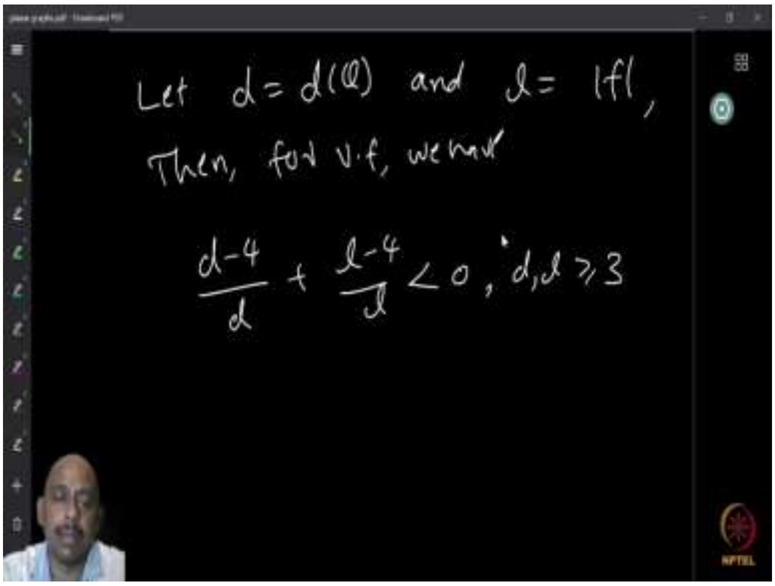

Now, if you take any such corner whose charge is negative, let us see what happens to it. So, what is the charge? The formula is $\frac{d-4}{d} + \frac{l-4}{l} < 0$, where $d = d(v)$ and $l = |f|$.

Now, this value is the charge of the face and this particular face as $\frac{d-4}{d} + \frac{l-4}{l}$ as a charge which is negative, less than 0. But we know that the degree of any vertex is at least 3 because by assumption the class has minimum degree 3 and the length of any face is of course at least 3 because faces are all cycles and the boundary will have at least 3 adjacent vertices. So, therefore d and l are at least 3. So, we have this inequality. Now we can just resolve this inequality by some routine calculation.

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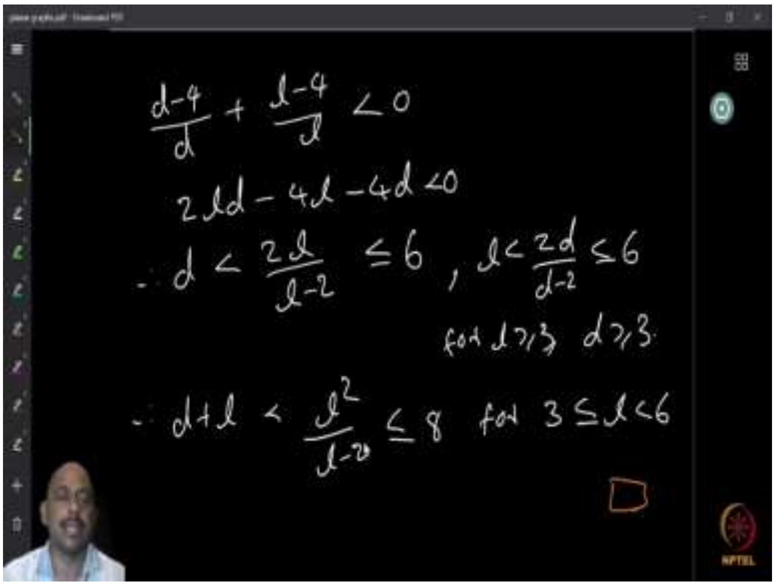

So, which is what we do here. We have

$$\frac{d-4}{d} + \frac{l-4}{l} < 0$$

$$2ld - 4l - 4d < 0$$

$$d < \frac{2l}{l-2} \leq 6\,, \qquad l < \frac{2d}{d-2} \leq 6\,, \qquad for\ l\,, d \geq 3$$

$$d + l < \frac{l^2}{l-2} \leq 8,\ for\ 3 \leq l \leq 6$$

So, we have proved that for all the corners whose charge is negative, this identity is and since there is at least one corner whose charge is negative, we get that you can find such a pair of vertex and face incidence.

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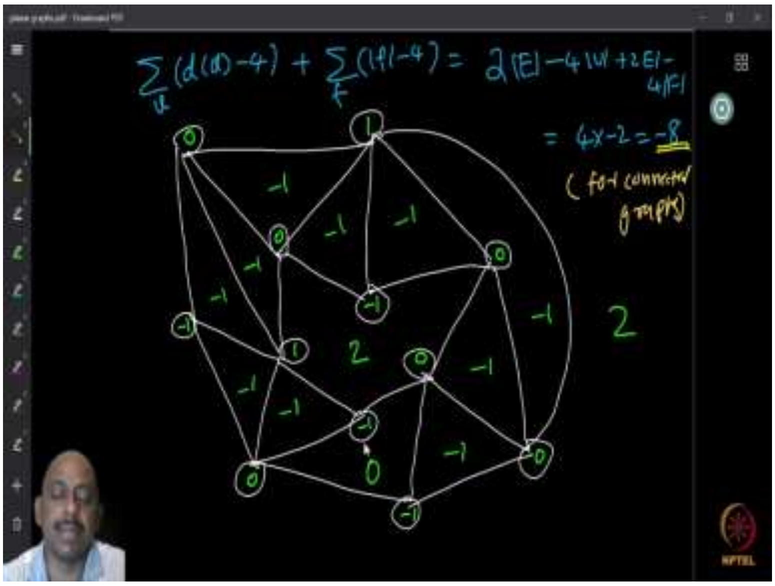

So, what we did in this was the following, we started by giving charges to the graph vertices and faces, then we use Euler identity to show that the total charge is a negative number and basically, we can show that it is less than 0 that is sufficient and then what we do is that, we use some property of the class that we are looking at, for example degrees that is 3, planar graph etcetera.

Some property to move around the charges and use it in a nice way, depending on what we want to show we have to do it but we basically define some rules of discharging that moving around the charges and after moving around the charges, we did the sum again. But what we know is that, if you move around the charges the total charge cannot be different.

So, after moving around what we want to say is that if certain structural properties are not there for example, like the one that we saw, we establish that the sum will be always different. So, we say that because the sum cannot be different the property must be there, so that is the overall idea of the proof that we were looking. So, this is how one can use discharging method to prove results on graphs and especially planar graphs it is very powerful. So, what we are going to do is we will look at one more example which is a more involved example to see a better picture of the discharging.