Combinatorics
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Lecture 43
Map Colouring Problem - History

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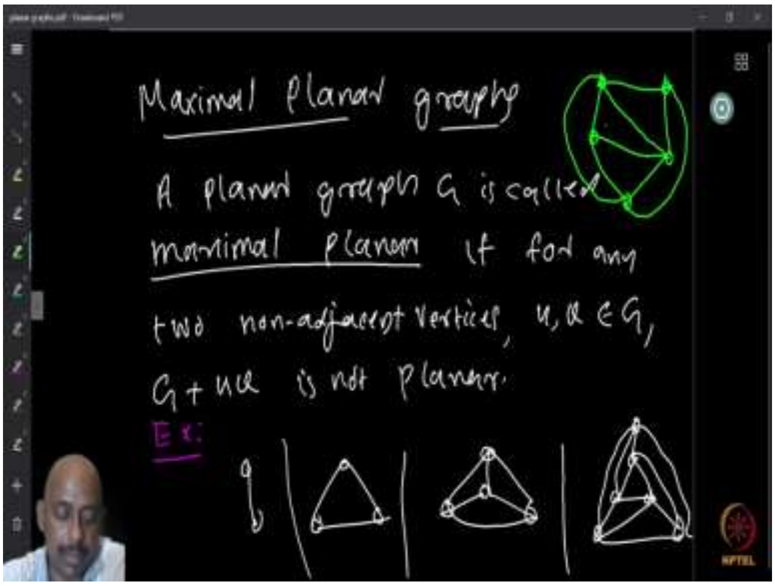

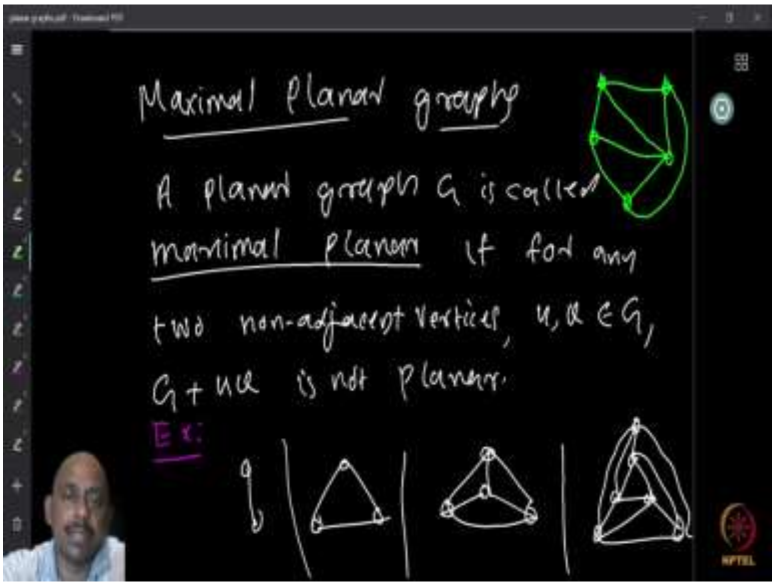

We now look at what are called maximal planar graphs. So, a planar graph is given to you let us say. Now, one question that you can ask is that, can I add any more edges to this graph, any edges between non-adjacent vertices, so that the graph remains planar? So, if I can do that, then the graph is not maximal. So, I can still add more edges. If that is not possible, then we say the graph is maximal planar.

So, let us look at one example. So, here the graphs given below, all are maximal planar. Of course, the single edge you cannot add any more edge it is maximal planar and if you take the triangle, which is a complete graph again you cannot add any more therefore that is also maximal planar. Similarly, K_4 is maximal planar because it has all the possible edges and it is planar and now you have another graph which is a planar graph.

I claim that this is maximal planar you cannot add any more edges. So, I want you to think about why this graph is maximal? You cannot add any more edges to still make it planar. On the other hand, let us take another graph. Let me say this. This graph is not maximal planar because I can add more edges to it like this edge. Can I add any more edges?

I can still add more edges and to make it still planar, for example, I can add this edge. Now, is it maximal planar I say that it is not I can still add more edges for example I can add this edge. Or I could have added for example this one. So, it is also still maximal, I mean it is also planar. Now, can I add any more edges? I claim that I cannot add any more edges so that the graph will still be planar. If I add any other edge, for example between these two or any other non-adjacent vertices, you will see that the graph is not planar anymore. So, this has become maximal planar. So, what is this property that makes it maximal planar? Can you think about this? But our definition says that if you cannot add any more new edges to make the graph, keep the graph planar then the graph is maximal planar.

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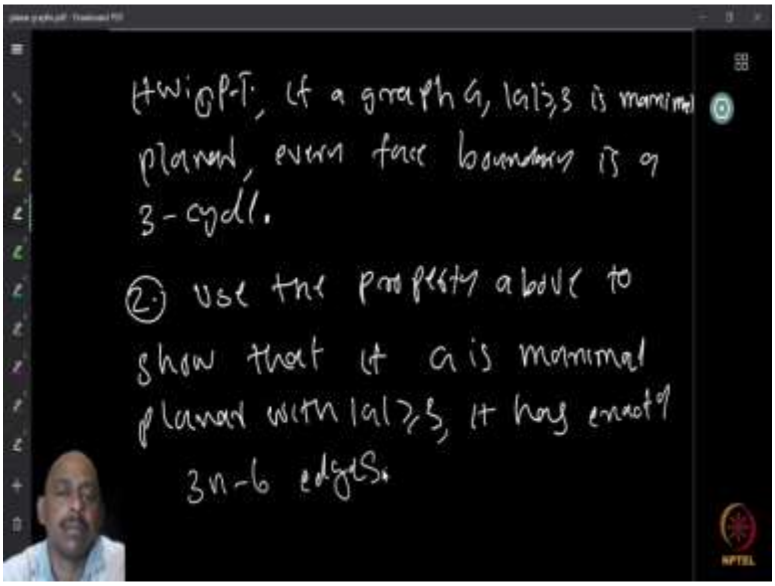

So, as a homework prove that if a graph on at least 3 vertices maximal planar then every face boundary is a three cycle. This is a nice property if the graph is maximal planar you take any embedding of the graph you will see that every face boundary will be a 3 cycle. If every face is a 3 cycle you cannot add any edges in this cycle. Because any three are adjacent there. And if you do anything else you will see that, that is not going to be keeping the planarity.

On the other hand, if a face boundary is not a 3 cycle, you can find 2 non-adjacent vertices and within the face you can draw a curve connecting these 2 therefore the graph will be still planar. So, this is the idea. Second question is to prove using the property that we just showed that, to show that G is maximal planar with at least 3 vertices then it has exactly 3n - 6 edges.

A planar graph which is maximal planar has exactly 3n - 6 edges. Now, if you want to prove the earlier result that a planar graph has at most 3n - 6 edges you can also first start with the graph G, add as many edges as you can to make the graph maximal planar and then show that now every face boundary is a triangle and show that using this you will be able to do a counting where the number of edges is actually equal to 3n - 6.

And therefore, since I added more edges without adding any vertices, I added more edges to make it maximal planar graph and you will still have only 3n - 6 you will see that any planar graph that we started with has less than or equal to 3n - 6 edges. So, this is another way to prove it.

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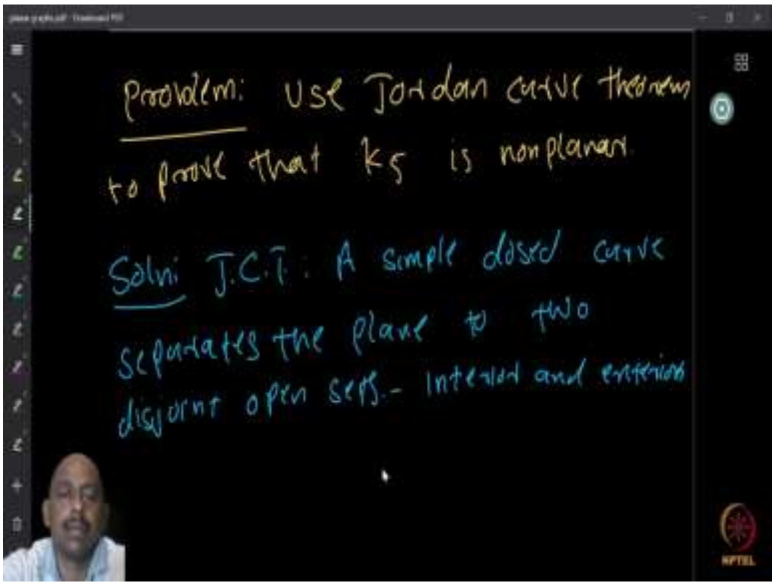

Now, here is another question. Use Jordan curve theorem to prove that K_5 is not planar and so, again it is a very nice exercise if you can do it yourself. So, think about this. Can you use Jordan curve theorem alone to prove K_5 is not planar, without using the Euler identity? So, here I present a solution. So, what does the Jordan curve theorem say? It says that a simple close curve separates the plane to two disjoint open sets. One is the interior part and the exterior. So, how do you use this to prove nonplanarity of K_5 and so here let us see.

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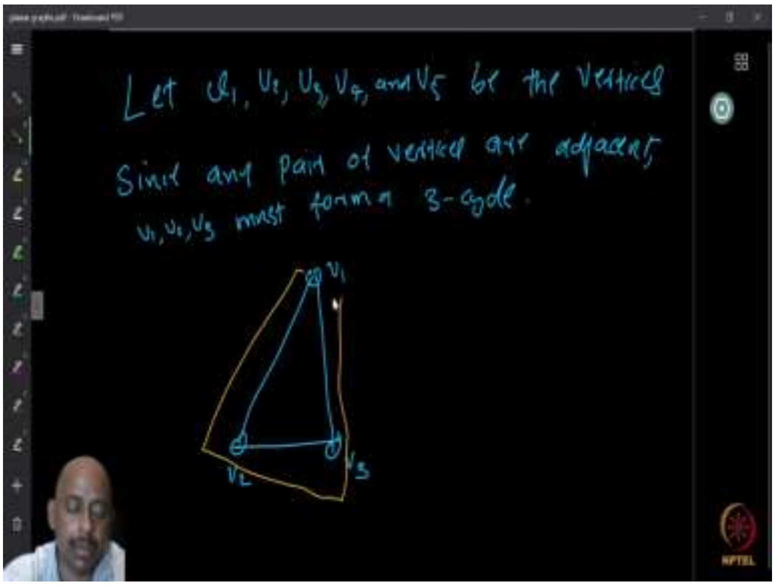

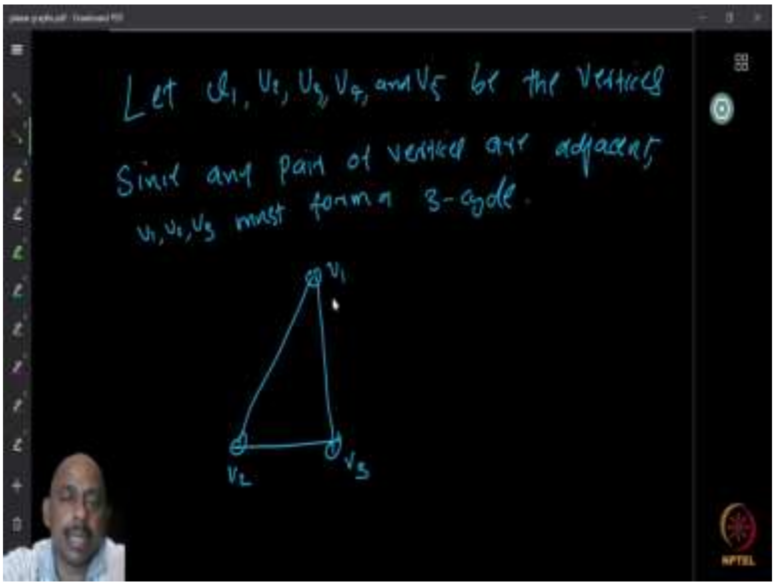

So, what we do is that we named the vertices as $v_1, v_2, \dots, v_5$. Now, we know that the graph is complete which means that between any two vertices there is an edge. So, if you take v_1, v_2, v_3 they form a triangle, there is a complete subgraph therefore, they form a triangle. So, v_1 is connected to v_2 and v_3. Now, this forms a close curve because you have this 3-cycle. Now, this 3-cycle divides the region, the plane into two parts; one is the interior part one is the exterior.

Now, since you are looking for an embedding of K_5, you have to add the other vertices. So, you want to place the vertex v_4 on the plane of course, the only possible places where you can place this are either inside or outside, both are identical. So, we will not worry about this we will see that one of the cases is sufficient by symmetry, we will assume that the vertex v_4 is kept inside. So, we can assume that the fourth vertex is either inside or outside so it can be inside.

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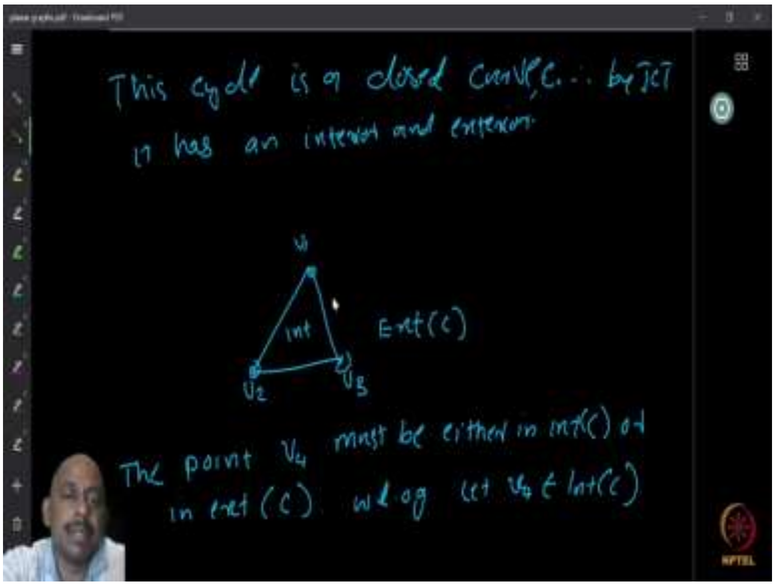

So, you have the interior and exterior. So, look at the interior part, we will assume that v_4 is sitting in the interior. So, we are going to keep v_4 in the interior.

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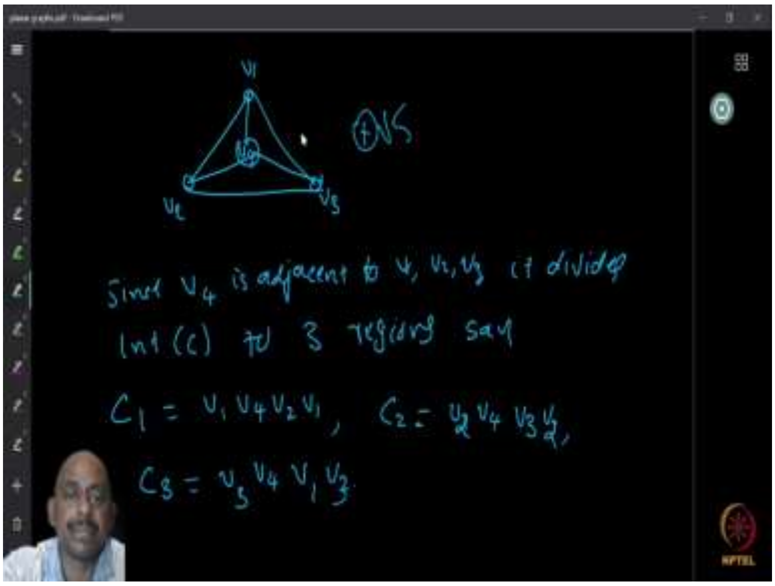

Now, if you keep v_4 in the interior, the v_4 is connected to v_1, v_2 and v_3 because the graph is complete. So, we will get v_4 connected to v_1, v_4 connected to v_2 and v_4 connected to v_3. Now, this again divides the inside internal region of the triangle that we started with, into three sub regions. You have these three different faces. You get the three different faces. Each one is a cycle, boundary is a cycle, which is a v_1, v_4, v_2, v_1 or, you have v_2, v_4, v_3, v_2 or you have v_3, v_4, v_1, v_3.

We have these three sub regions. Now the question is that where are you going to keep the vertex v_5? Suppose you take the vertex v_5 we know that v_5 is adjacent all other vertices. In particular it is adjacent to v_1, v_2 *and* v_3. Now, the only connected area where which is having all these vertices is precisely the exterior face.

So, our last vertex must be in the exterior face. The only place where I can keep v_5 is the exterior because it must be connected to v_1, v_2 and v_3 all three. But then, if you keep it in the exterior of C, v_4 is in the interior of C and therefore, the curve C which is v_1, v_2, v_3 cycle that curve divides the region into two; interior and exterior and therefore, any path from v_5 to v_4 must necessarily cross the curve that we have just created, v_1, v_2, v_3. But this is not possible.

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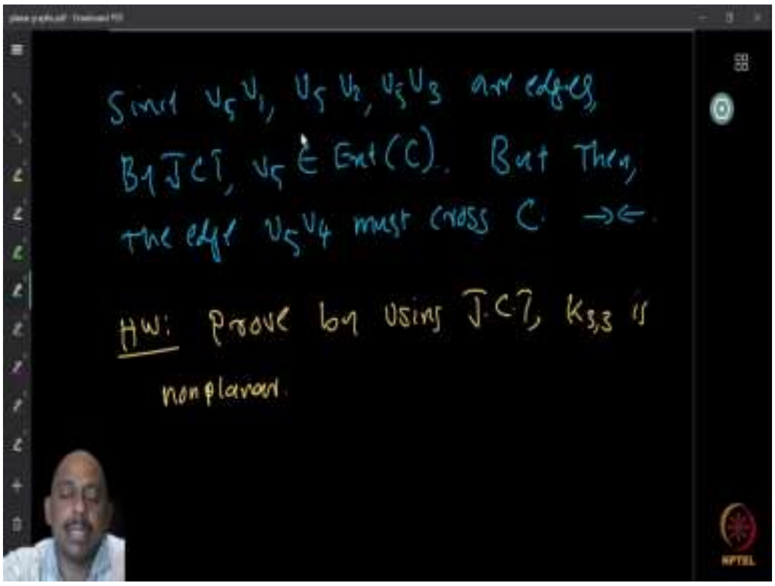

Therefore, the edge must cross the curve. So therefore, the embedding is not planar. Hence, yes, you can use your Jordan curve theorem to prove this. So, similarly, I want you to do as a homework prove $K_{3,3}$ is not planar by again using the Jordan curve theorem. So, think about this. Now, there are there are many other things one can do with these things, but for the time being, I want to look at a different topic, which is very interesting and very powerful.

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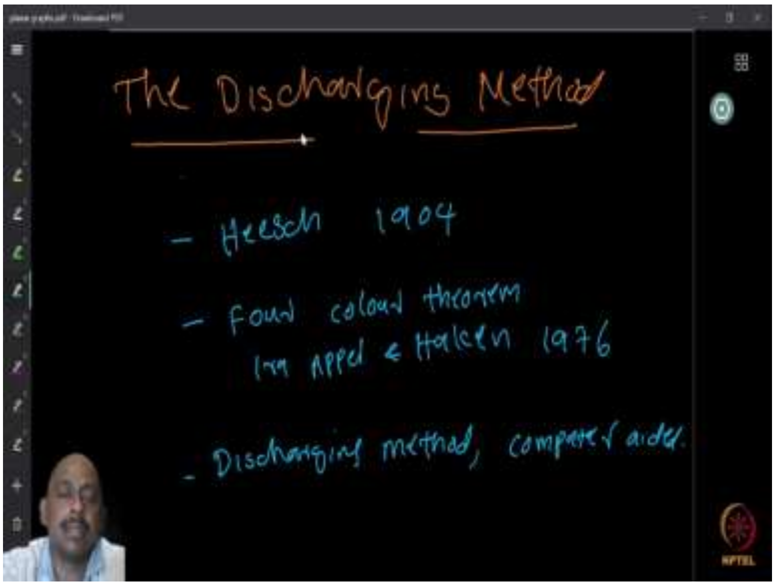

This is called the discharging method. Now, before going into the discharging method, let me tell you a little bit of history. So, this method was developed by Heesch in I think, if I remember correctly, in 1904. Now, this result has been used very less for like maybe around 50 or 60 years and after this or maybe 70 years after this, this was used in a very famous paper by Ira Appel and Haken, Wolfgang Haken and Ira Appel who proved in 1976, a very long-standing conjecture called four colour conjecture.

So, this four colour conjecture came from the following question. There was a, I think it was in probably 1850 or something 1850, 51 if I remember correctly. Francis Buthray, he was a Map Maker. So, he asked the following question. Suppose you have a map of a country let us say you will assume for simplicity that all our countries are connected regions that you do not have several parts in the world which are part of the same country which is actually true, but we will assume that is not the case.

Now, if you have connected regions and you want to colour the map of the world map like say giving different colours for different countries. Not necessarily different colours you want different colours for adjacent countries, so that you will see a, immediately like, these two are different countries or what is the border etcetera. Now, of course, one can say that, okay, use a different colour for each country. This is definitely possiblity there are two issues with this.

One is that if the number of colours increases, then the difficulty of resolution, looking at a colour and differentiating becomes more and more. When the number of countries is high, now, you will have to use different types of colours which are close to each other. So that many people may not be able to distinguish between the small difference in the colours. so, we cannot use too many different colours.

It is always better to use as low, as the number of colours possible. We use the minimum number of colours, like let us say there is two or three then, it is very easy you can just use like, red, blue, green or some very contrasting colours. So, that, it is very pleasing and immediately clear what is the borders of the countries so you want to. So, one natural question for a Map Maker is to see what is the minimum number of colours with which you can do the colouring.

So, what this Map Maker observed is that he can do with four colours for all the cases that he has seen. Now, he wanted to ask is it always possible, no matter how the country's borders are, can you always say that we can colour the graph with at most four colours. Now, this question was asked to some mathematician friend and then he kept it at a log, he mailed to some famous mentions who looked at it then replied that he is not clear he will look into it and then asked others and this problem has been in circulation and many people have started looking at it and there were several attempts to prove this.

So, after 125 years. In 1976 finally, it was proved by Appel and Haken. These two people proved this method, proved this result using a technique called the discharging method. Now, this discharging method proof became controversial, because the number of cases in this was two huge. So, they had to use computer help, they had to write some algorithm to verify whether the result is correct.

So, the computer basically verified the calculations and therefore, many mathematicians said that, no, this result is not acceptable. Now, what is the reason for them to say this? This has nothing to do with like the persons or the that it has been used with computers, but the computers has an inherent issue that a computer is an electronic device which usually in the manufacture people see that after some time there could be some bugs.

Now, there are certain kinds of bugs which may be triggered only in the execution of certain specific sequences of instructions. Now, there are several such bugs which appears in computer science. Everything will work fine only if you give a specific sequence of instructions, then computer produces or the algorithm produces a wrong result. Now, the mathematicians asked, okay, what happens if this particular sequence of instructions was the algorithm that you use to solve the cases of the planar or the proof of the four colour theorem?

So, therefore, they know, they said that, okay, we do not want to take it for granted. But of course, without any other proof, we have to take it because, this is a serious attempt and, we cannot find any problems with it other than the fact that it is using computers. So, then, after some time, other people wanted to see if they can come up with a different set of rules and very famous mathematicians, four of them in fact, Robertson, Seymour, Robin Thomas and, and one more Person Steal and Thomas Yeah.

So, these four people tried to come up with a different proof without using the help computers. So, they came up with a very different idea they simplified the arguments of Appel and Haken they came up with several things and finally they said that okay they also need to use discharging method, but they they also had to use the help of computers.

So, after, this was in 1996 if I remember correctly and then there were a couple of more attempts and each of the attempts so far to prove this always use the discharging method eventually and also always used the help of computers to do some computation. So, if you can come up with a proof without using computers that would be great. Now, the question is about discharging method.

Now, the question is, is the discharging method, so powerful that, it is required to prove four colour theorem? We do not know. So far, all the results and all the proofs have used discharging method. But this method is very powerful in the sense that it can be used to come up with several structural properties of classes of graphs including planar graphs. And not all applications of this method requires the help of computers. So, we are going to see some simple examples and study this method. And of course, we do not have time to look at the proof of the four-colour theorem, but we will learn the technique that is used to prove the four-colour theorem. So, to look at the question again.

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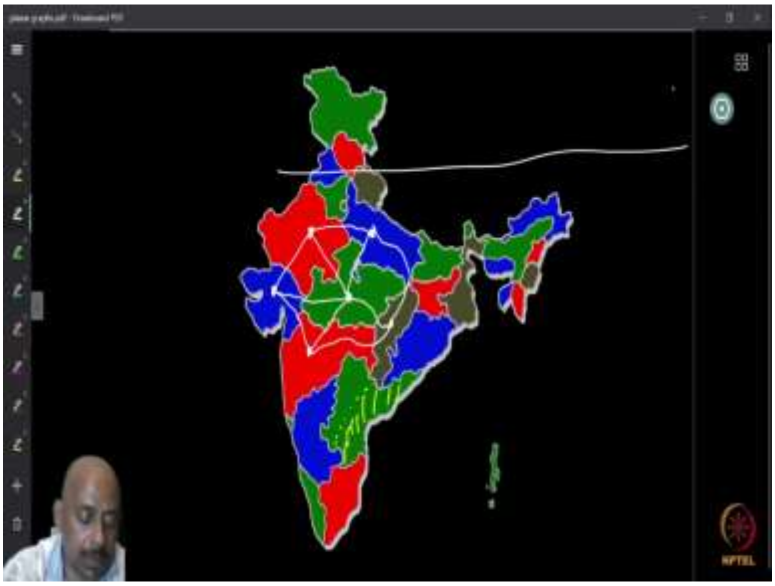

Let us look at this map of India. So, we look at the map of India and we want to colour the map using the minimum number of colours. So, the aim is to try to use as least colours as possible and then I observed that I have to use at least four colours. So, why is that let us look at the map of current map of India and then look at the cases why we require four different colours? Why we cannot do with three colours?

So, we will start with the assumption that or we start the attempt by using only three colours. So, we will start with let us say one of the states, Madhya Pradesh. So, Madhya Pradesh definitely needs one colour. So, I will start by giving it the colour green. Now, since adjacent state should get different colours now, immediately I see that Uttar Pradesh is adjacent Madhya Pradesh.

So, therefore Madhya Pradesh must get a different colour. So, other than green I have to give some other colour. So, let us say that I gave blue. Now, we observed that Rajasthan is bordering both Madhya Pradesh and Uttar Pradesh. So, since it is bordering both of these I cannot use either blue or green. So, I use red to give Rajasthan. Now, when I go to Gujrat, I see that okay Gujrat can be coloured blue because it is adjacent to of course Rajasthan and Madhya Pradesh.

So, I cannot give red or green but I can use blue So, I will give you blue Gujrat. Then I come to Maharashtra, Maharashtra is adjacent to both the Madhya Pradesh and Gujrat so, I have to use a colour other than green and blue. So, I give red. Then I come to okay so this is not the current map I think this is a map before the division of Telangana and Andhra Pradesh. So, let us say that this was not green.

So, then I come to Telangana, so Telangana can be coloured with let us say green or blue actually, because we are assuming that we have not coloured anything else. But let me say that I coloured Telangana with green. Now, no matter what I colour no I do not have to go to Telangana because yeah. So, now let us look at this.

So, now I look at this is I think Chattisgarh, let us look at Chhattisgarh. Now Chhattisgarh is adjacent to both Madhya Pradesh, Maharashtra and also to Uttar Pradesh. So, therefore, since I had been forced to give different colours to Madhya Pradesh, Uttar Pradesh and Maharashtra because all these colours were forced. I started with one colour green, then I had to use a different colour.

So, I use blue I had use a third colour so, I had to use red. Now, I do not have a choice to give if I am using only three colours, I do not have a choice to give anything other than blue to Gujrat and similarly, I did not have a choice to give anything other than red to Maharashtra if I am using only three colours and therefore, I had been forced to give three different colours to Uttar Pradesh, Madhya Pradesh and Maharashtra.

Now, Chhattisgarh is adjacent to all these three neighbouring all these three. So, therefore, I cannot give any of these three colours Chhattisgarh. So, I have to use a different colour let us say brown. So, this is the argument to see that we need four colours to give colors to distinct states of India, at least 4. Now, the claim is that 4 a sufficient. Why is that? So, now one can immediately convert this into a graph theory question as you can probably see now.

Just put vertices at each state, centre of a state. Then whenever the states have common border line you put an edge connecting them. So that defines the graph and this tells us how to convert the map colouring problem to a graph colouring problem. So, now I have to colour the vertices if I give a vertex a colour, any adjacent vertices cannot get that colour. So, therefore if I have a colouring of the map, I have a colouring of the graph and if I have a colouring of the vertices of the graph, I have a colouring of the map also. I have converted the map colouring problem to a vertex colouring problem.

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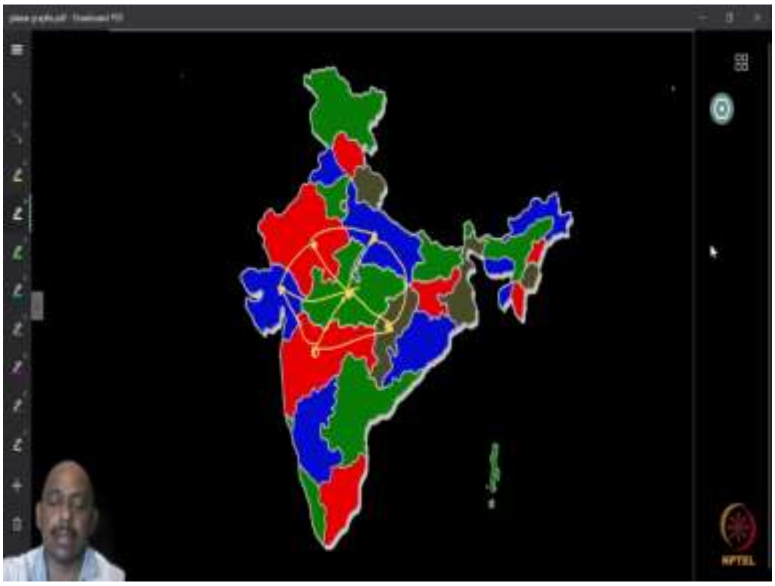

So, I have converted into a vertex colouring problem. So, now one can show that converting these kind of maps to graphs, if assuming that the regions that we are looking at are all connected single regions, then you can show that the graph that you obtain are planar graphs. In fact, there are more planar graphs that you can obtain. And there are the graphs that you obtained from the map by doing this, are planar graphs assuming that no maps are connected regions.

And but of course, there are other planar graphs which are not obtainable this way also. But we now prove that for every planar graph, we can do with four colours. So therefore, it says that maps can also be coloured with four colours. So that is the idea of the I mean that is the idea of converting this to a graph problem. Now, people are looking at solving this question on graph colouring, not just map colouring, also graph colouring.

So, this graph colouring has been active for like almost 125 years and before it was finally proved. So, there were some very interesting attempts to prove this and some of them, stood, people believed it to be correct for almost 11 years and before somebody found there is a problem in the proof. So, it is always a good idea to look carefully before accepting any even accepted within course, results as true. Because people can of course, miss some important points and then think that it is correct. But anyway, let us continue.

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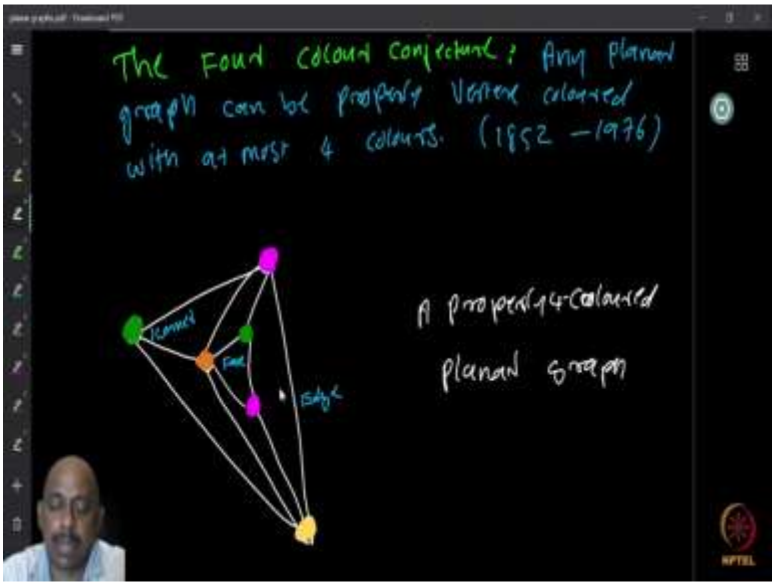

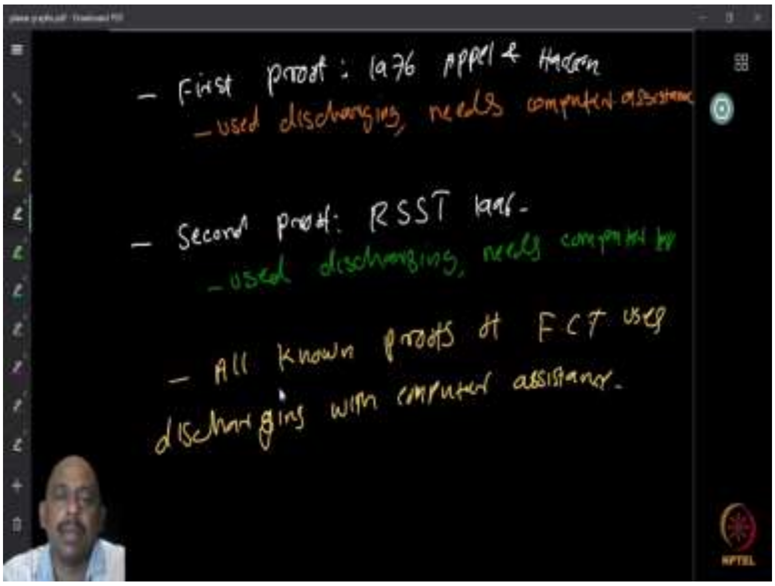

So, the four-colour conjecture was that any planar graph can be properly vertex coloured with at most 4 colours. So, this was open from 1952 to 1976. I think I mentioned, 1852 to 1976 I mentioned as 1851 I think it was 1852. Now, if you recall, a proper colouring of a graph is basically assigning colours to the vertices such that adjacent vertices get different colours.

So, here is an example of a proper four colouring of a planar graph. So, here we have a planar graph and you have coloured with four colours and can you show that four colours are necessary or you cannot do with less colours or if you can actually do with less colours, show that also. So, now, this is something that we already mentioned that the first proof of this was proved by Appel and Haken in 1976, used the discharging method and needs computer assistance. Now, this original paper was around 2000 page because of the different computations and all the calculations that were there.

Then, the second proof came in 1996 Robertson, Seymour, Steel and Thomas, and they also use discharging method and they also required computer assistance, but they reduced the number of pages to something like 600 or something. Then there were other attempts, but all the known proofs as we mentioned before, use the discharging method and computer assistance. So, if you can come up with a proof without using a computer help you will be a big name in computer science. So, let us look at a trivial example to study the discharging method.