Combinatorics
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Lecture 42
Euler Identity

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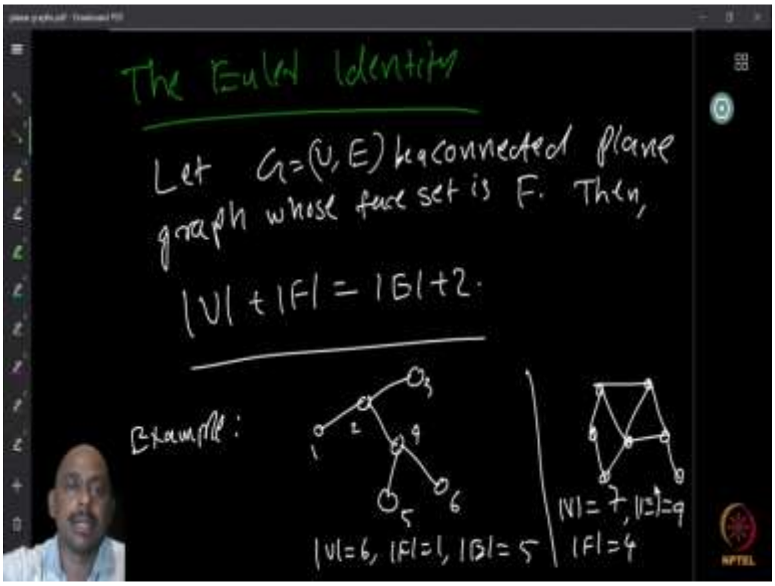

Now, this identity is called Euler identity because Euler proved this for the polyhedron he observed it and proved this and it has been generalized to graph later and it is called the Euler identity. So, let G be a connected plane graph. Plane graph means that it comes with the embedding. Because without the embedding you cannot talk about the set of faces. We talk about plane graph when you have the planar graph with a fixed embedding. Let G be a connected plane graph.

So, we will assume the graph is connected then the following identity holds. That is, the number of vertices of the graph plus number of faces of the embedding is equal to the number of edges plus 2. |V| + |F| =|E| +2. Again, examples. So, if you take any tree there is no cycle there is only one face. The number of vertices is 6, number of edges is 5 and that is it. So, you have the identity. Similarly, you take any other graph, you have number of vertices 7 here, number of faces is 4 and then number of edges to be 9. So, this is the identity. Now, we want to prove this. So, how will you prove this? So, try to see if you can come up with a proof for this by yourself if not proceed further.

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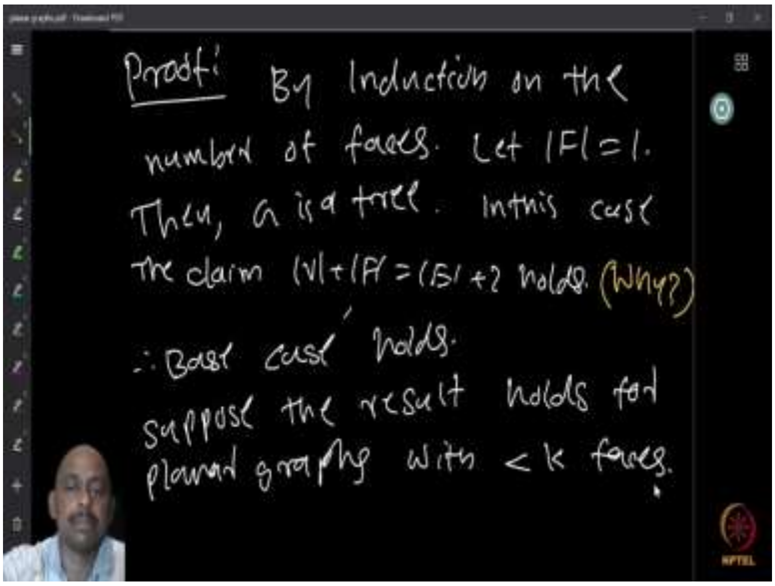

So, here is the proof. We use induction on the number of faces. There are several ways to prove this, we want to prove it by induction on the number of faces. So, if the number of faces is equal to one. So, when is the number of faces equal to one for an embedding on the plane? Only if there are no cycles because if there is a cycle that cycle already separates the region into at least 2 parts.

So therefore, when the faces are equal to 1 then you cannot have any cycle. So therefore, the graph is acyclic and since we are assuming the graph to be connected the graph must be a tree. So, when the graph is a tree, the claim is that the identity always holds. So, can you tell me why? Think about this and see why the identity holds if the graph is a tree.

The identity that number of vertices plus number of faces is equal the number of edges plus 2. Because we already know that for a tree the number of edges is equal to number of vertices minus 1. Number of edges is equal to number of vertices minus 1 so therefore, number of vertices plus the number of faces which is equal to 1 is actually equal the number of edges plus 2 because number of edges is number of vertices is minus 1. So, that is the proof, hence we have the base case. So, once we have the base case, we can assume the result holds for all planar graphs, where the number of faces is less than k for some k greater than 1.

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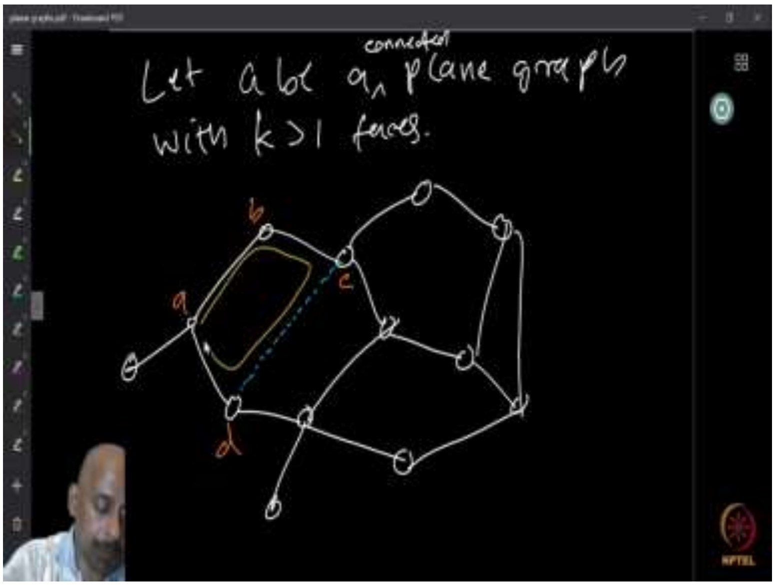

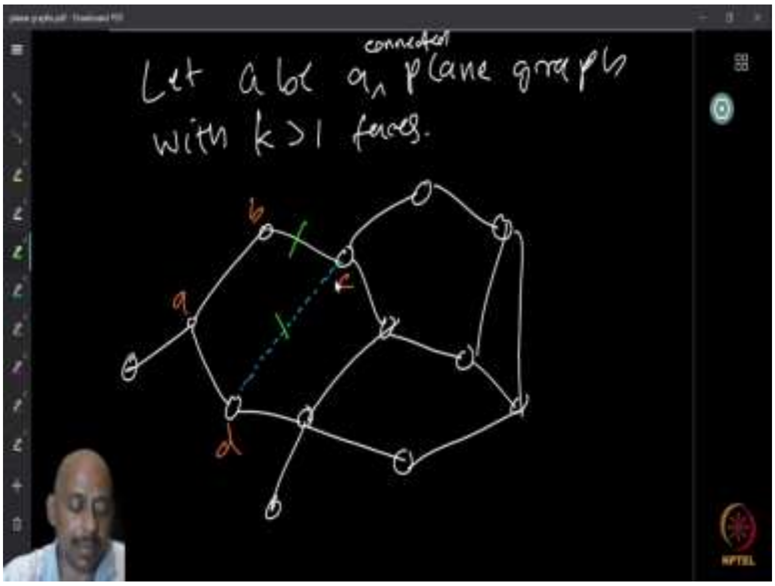

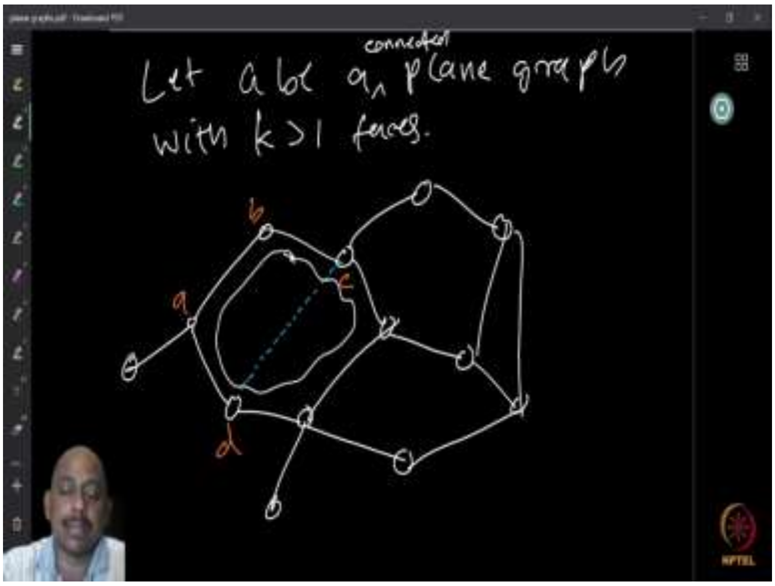

So, assume that the result holds for connected plane graphs with k greater than 1. Now, let us take any graph with more than let us say k faces. So, take any graphic more than k faces, k plus 1 faces. And then we will use the index. How do you do that? So, again, maybe it is going to be illuminating. If you think about this for a few minutes and see can you solve now this by induction, can you complete the proof by induction?

So, I will let you think for a few minutes and then pause the video and then continue after spending some time thinking about this. So, what we will do is that we will assume that the graph has k greater than 1 faces and then we will consider the cycles in this graph. Because if there is no cycle we know that there is only one face. So, take any face, any face boundary of the bounded face any face boundary is going to be a cycle.

Take the face boundary of any bounded face for example, the a, b, c, d face here in this graph. If you look at this particular face it has 4 edges right in the boundary. Now, take any one of the edges that you want. You take any one of the edges that you want and then what you do is that you look at this particular edge. So, if you take the edges in the boundary of the cycle that edge separates exactly 2 faces that is what you should see.

If you take any edge no matter which edge is that if it is in the boundary of the cycle of the face then that edge belongs to exactly two face. The face which is one of the internal faces and then the one which is the external face in this case and if you take for example, the other edge like this one then it is separates 2 regions, again 2 faces and these 2 faces are not joined by I mean like in the embedding.

If you remove the points of the embedding, then these are disconnected components in the R minus the graph G. So, if you look at these components, they are basically separated by this edge and this edge belongs exactly 2 face right in the boundary of exactly 2 face. So, we have this observation. Now, what happens if I remove this one edge from the graph. So, look at the graph G minus this edge.

If I removed the edge from the graph then I have removed let us say the edge cd. If I removed cd from the graph then the edge which was separating these 2 regions is no more present and therefore, these 2 regions gets collected into a single region. This entire area becomes one single region. So, what happens to the number of faces in the graph when I remove one edge of in the boundary of the cycle? The number of faces decreases by exactly 1.

Because 2 faces have been merged into 1 face and nothing else happened to other faces and therefore, I get a graph with one less number of face. Now, if the number of faces is strictly less I can use induction now. I use induction because by induction hypothesis anything less than k the result holds and therefore, I started with a graph with exactly k faces and now, I have removed one edge the new graph has exactly k minus 1 faces. So, therefore, I can use induction.

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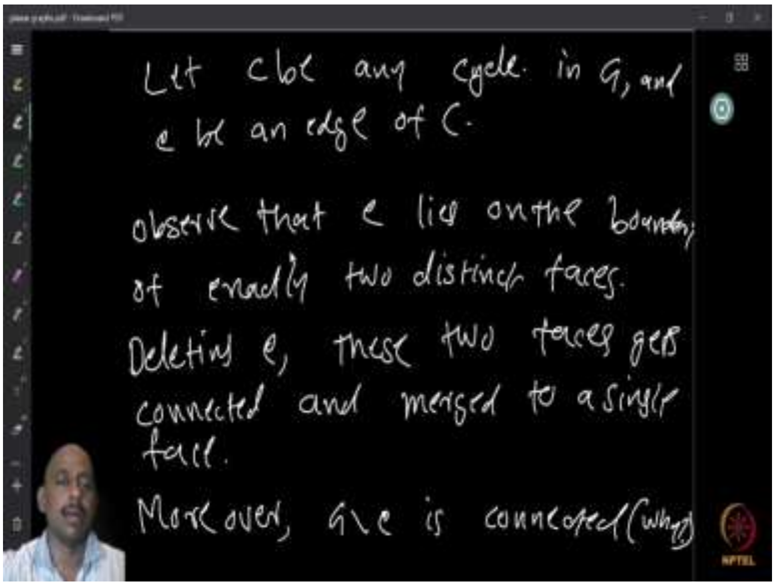

So, how do I use induction? Now, if you look at the cycle. So, here is a formally written down argument. So, let C be any cycle and e be any edge of the cycle. So, the edge e lies on the boundary of exactly 2 distinct faces and deleting the edge e, these 2 faces gets connected and merged into a single face. Now, the graph G minus e is connected. And can you tell me why?

So, the claim is that after removing the edge, the graph is still connected. This is not necessarily true always. But in this particular case, I claim that the graph the G minus e is connected. So, again pause the video and think for a few minutes before continuing. Now, the graph G minus e is connected because e was part of a cycle. So, if I remove one edge from part of one cycle then it is not going to disconnect the graph. Again, you can formally prove this using the arguments that we have studied before and once you have this,

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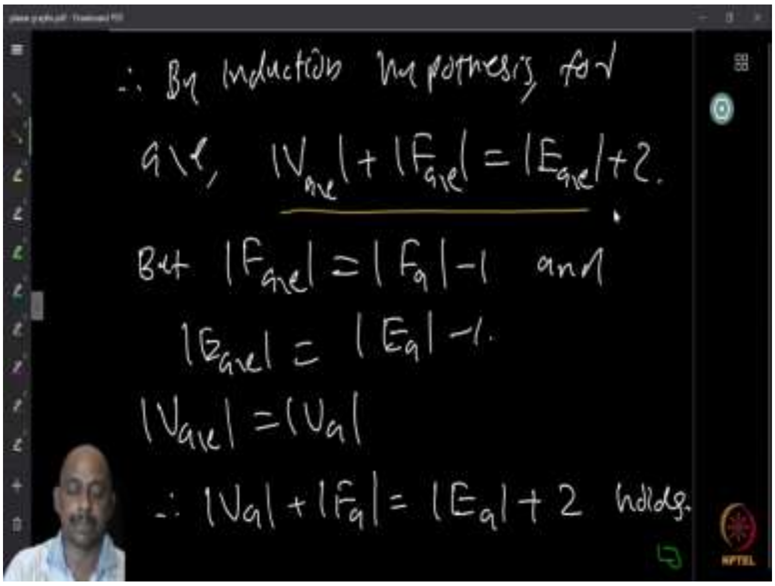

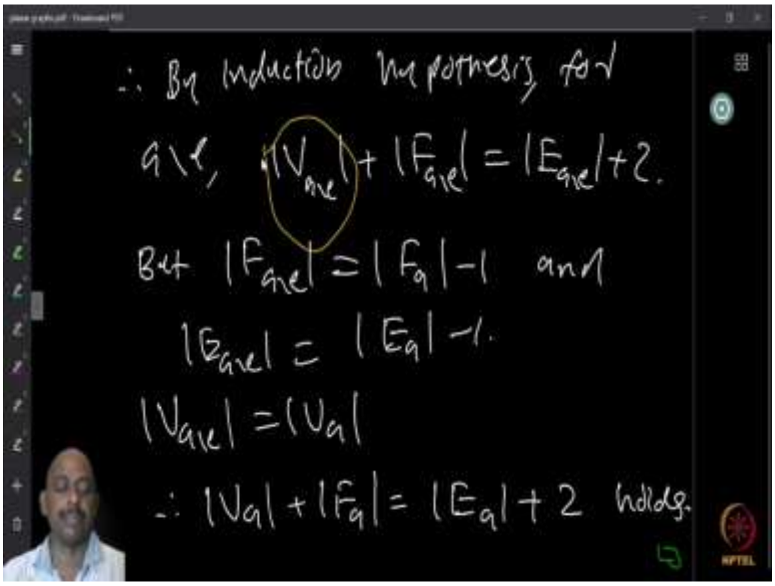

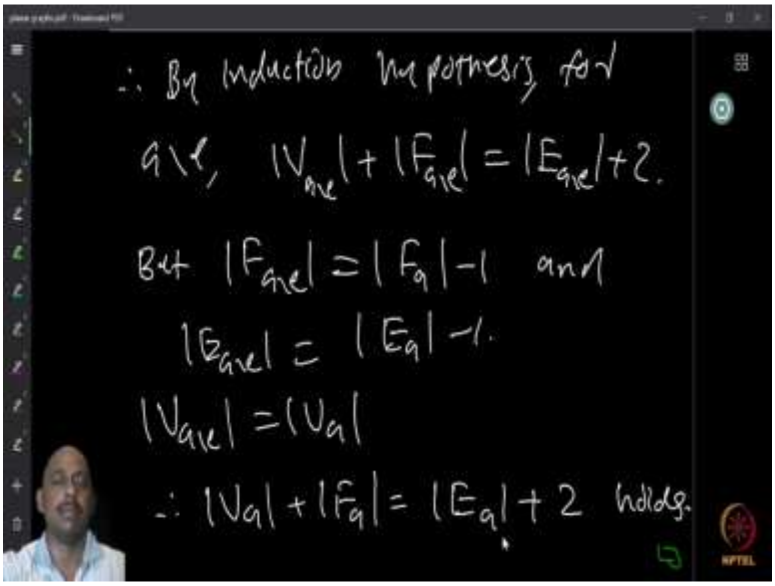

It is easy. You have a graph which is connected and the number of faces is less. So, therefore by induction hypothesis it is actually the Euler identity. So, therefore, for the graph G\e the number of vertices in G\e plus number of faces in G\e must be equal to the number of edges in G\e plus 2, ($|V_{G\setminus e}| + |F_{G\setminus e}| = |E_{G\setminus e}| + 2$), the Euler identity holds. Now, what we know is that the number of faces is exactly one less than the number of faces in G.

So, these equations are true. Number of edges also decreased exactly by 1, that is how I got G\e, deleted exactly one edge. But what happened to the number of vertices? It remains the same. Now, if you look at the earlier identity, the number of vertices in G\e is equal to the number of vertices in G. This is actually equal to the number of edges in G. On the other hand, I subtracted exactly one from the one term in the left side and exactly one from the one term in the right-hand side, so, the identity remains the same.

So, therefore cardinality of $|V_G| + |F_G| = |E_G| + 2$ and this also holds. I just added or subtracted, whichever way you want to look at one from both sides of the identity. Therefore, we have the proof of Euler identity. So, for any connected plane graph the number of vertices plus number of faces equal the number of edges plus 2. Now, you can ask what happens when the graph is not connected. So, I leave it as an exercise for you to figure out what happens if the graph has several components.

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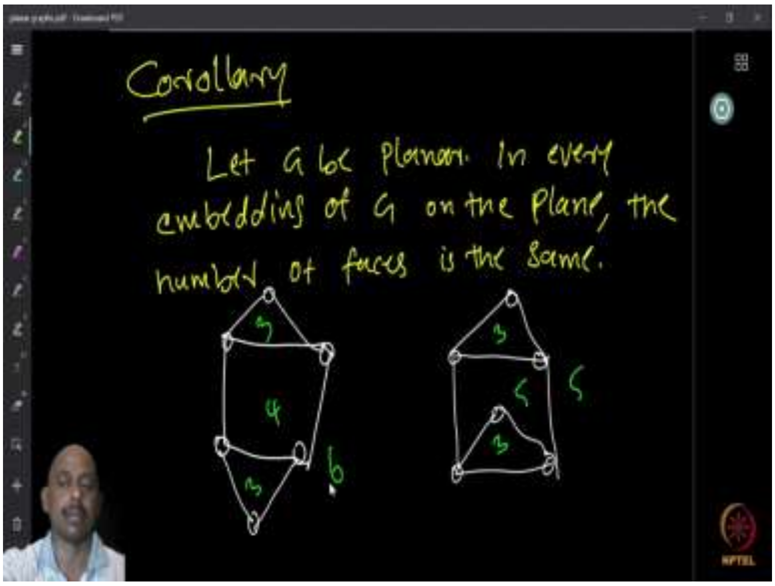

Now, here is a corollary. So, let G be planar in every embedding of the graph G on the plane the number of faces remains the same. You can have several embeddings of the graph. But, for example, I will show you the graph with the different embeddings. Let us take one example. Let us look at this graph. This has this embedding. It is a planar graph with a plane embedding.

Now, look at another embedding of the same graph. Now, this is a different embedding of the graph because if you look at the cycles of the face length, so, look at the length of the faces in this embedding, you have a 3 cycle here, there is a 3 cycle here, there is a 4 cycle here and then there is a 6-cycle boundary on the outer outer face. On the other hand, in this embedding, you have other faces, how length, 3, 3, then 5.

And, again, the boundary of this face is 5 here it was 3, 3, 4 and 6. So, the now the embedding is different. So, the embeddings are different, but then you can have several possibilities for different graphs, but in, no matter what this earlier theorem tells you that, the number of faces in the plane are embedding must always be the same, because the identity says that the number of vertices plus number of faces is equal to the number of edges plus 2. So, therefore, since all the other parameters are the same, number of vertices and number of edges remain the same. It must also follow that the number of faces remain the same. It is a direct corollary of the result.

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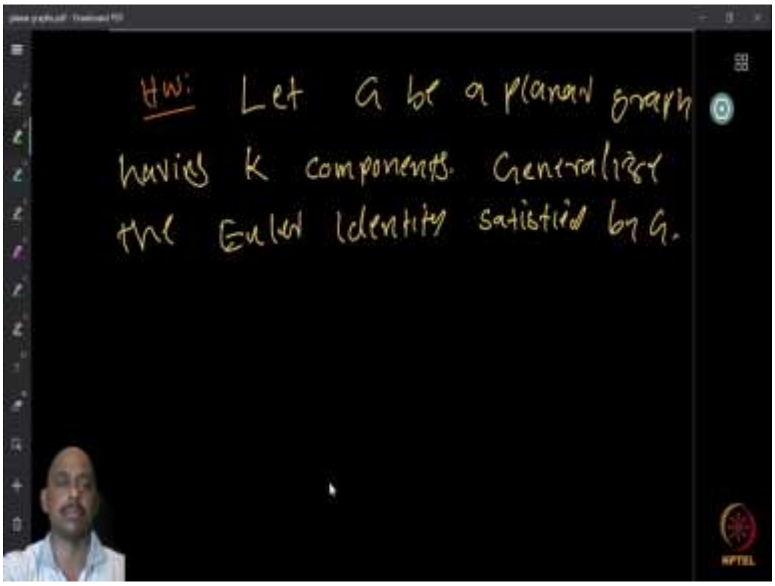

Now, here is a nice homework question. Let G be a planar graph with k components. Generalize the Euler identity satisfied by this graph. So, the Euler identity for a graph with several components will be slightly different from the Euler identity for a connected graph. So, can you find out this identity? So, this is a nice homework.

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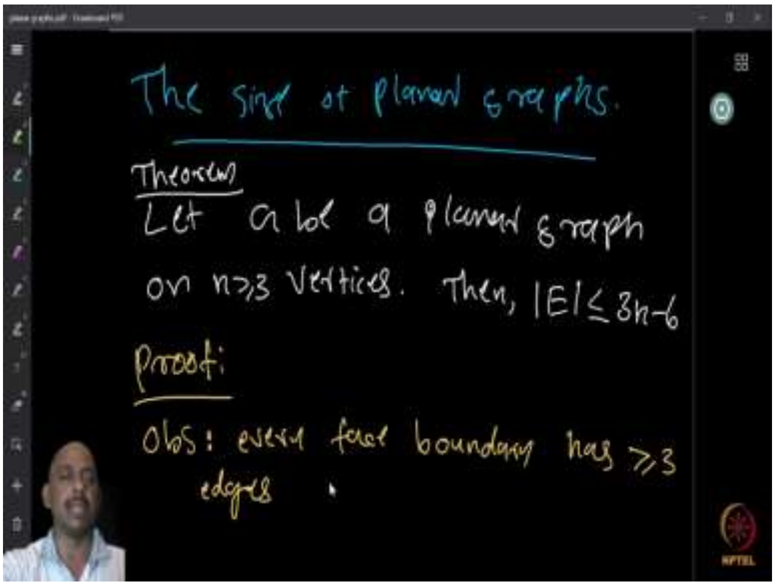

Now, using this Euler identity we can prove several interesting results. It is a very powerful tool actually. And what we are going to do with this? First is to prove that the planar graphs cannot have too many edges. So, if you if you recall graphs, an arbitrary graph like for example, if you take the complete graph on n vertices, you can have n choose 2 edges which is equal to $\frac{n(n-1)}{2}$, its close to $\frac{n^2}{2}$.

So, you have, many edges possible in a graph. On the other hand, one can prove that if a graph is planar you cannot have too many edges for this graph. So, if the number of vertices is fixed, then number of edges is bounded by a factor. Now, how do you prove this and why is this the case? So, an intuitive way to look at this why the number of edges is small is that, when you have an embedding, the faces, the boundaries of the faces must all have these edges that define the boundary of the face.

But now, if you take any face, the face boundary has at least 3 edges, for any face to be defined, you need at least 3 edges, because you need cycles to define. Now, this is that like, when you have certain number of faces, then the number of edges must be related to the number of faces.

Because you have the Euler identity, which says that number of vertices plus number of faces is equal to number of edges plus 2. So, since every face has at least 3 edges to be in the boundary, this puts a restriction on the number of edges. Now, we want to use this idea to prove formally, that the number of edges in any planar graph is at most, 3 times the number of vertices minus 6, for n ≥ 3. That is, $|E| \leq 3n - 6$).

For n = 2, you have this obvious counter example, the trivial case, where you have 2 vertices 1 edge, where the identity is slightly violated. So, if n is at least 3, then we have the number of edges in the planar graph is at most 3 times the number of vertices minus 6. So, the proof uses the observation that every face boundary has at least 3 edges.

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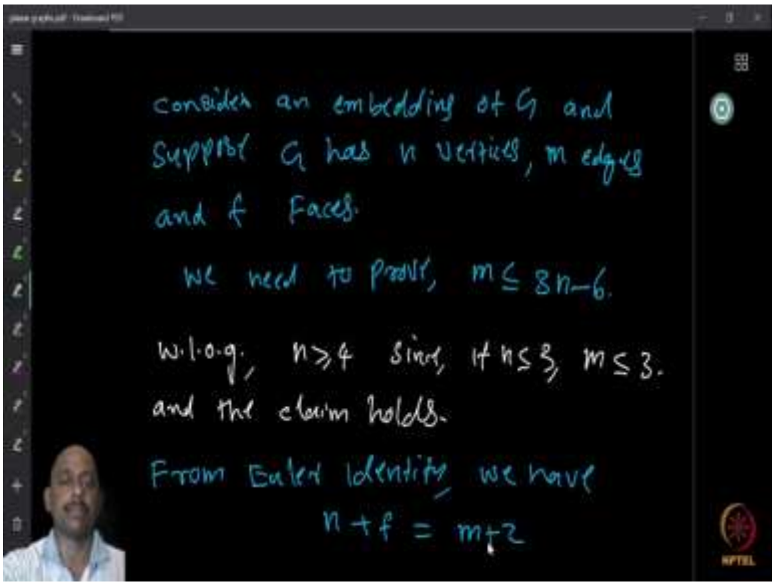

So, what do we do? We consider an embedding of the graph G and we suppose that the graph has let us say n vertices and the number of edges is denoted by m and number of faces denoted by f. So, the graph has n vertices, m edges and f faces. So, what we need to prove is that, m $\leq 3n - 6$. So, first of all, we observed that if the number of vertices is actually equal to 3 then the number of edges possible is also at most 3.

Because even the complete graph on 3 vertices can have only 3 edges. And for n = 3 the identity is clearly true. So therefore, no because 3 × 3 - 6 is actually equal to 3. So therefore, 3n - 6, m is at most 3n - 6. Now, therefore we will assume that n ≥ 4 because the case 3 is done. So, let us start with the assumption that we have at least 4 vertices and we have the identity Euler identity says that number of vertices plus number of faces is equal to number of edges plus 2. That is, n + f = m+2.

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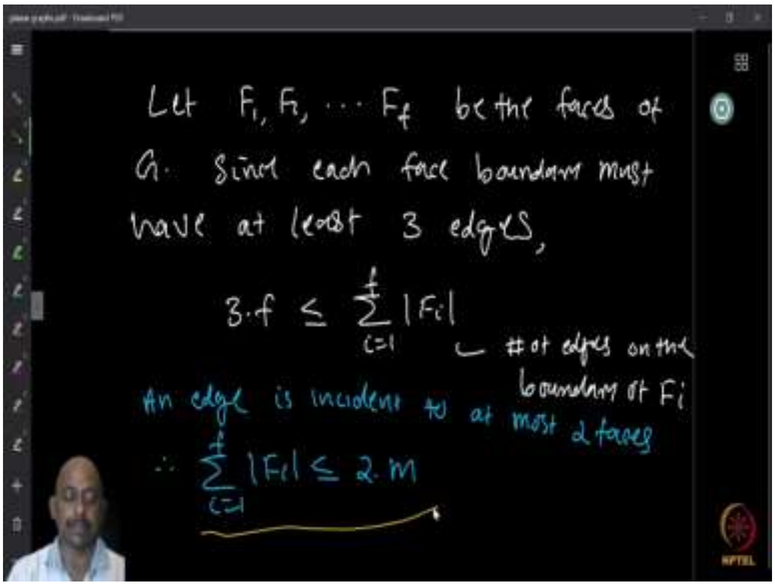

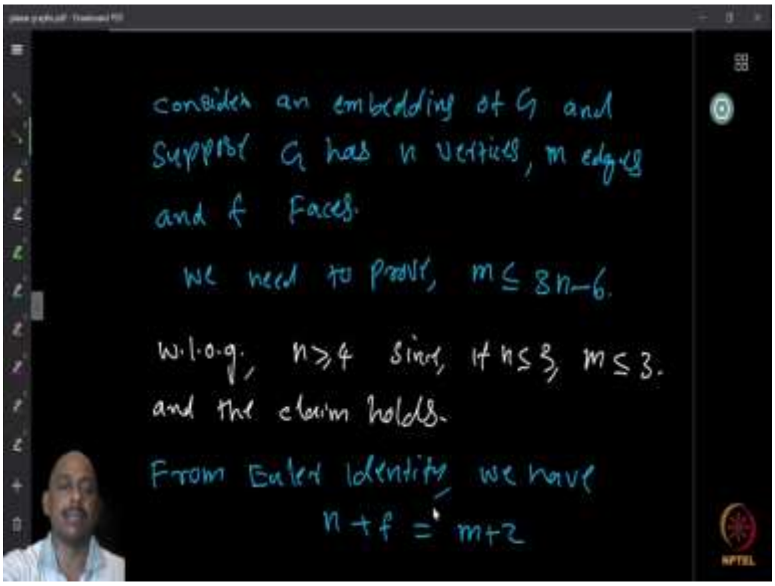

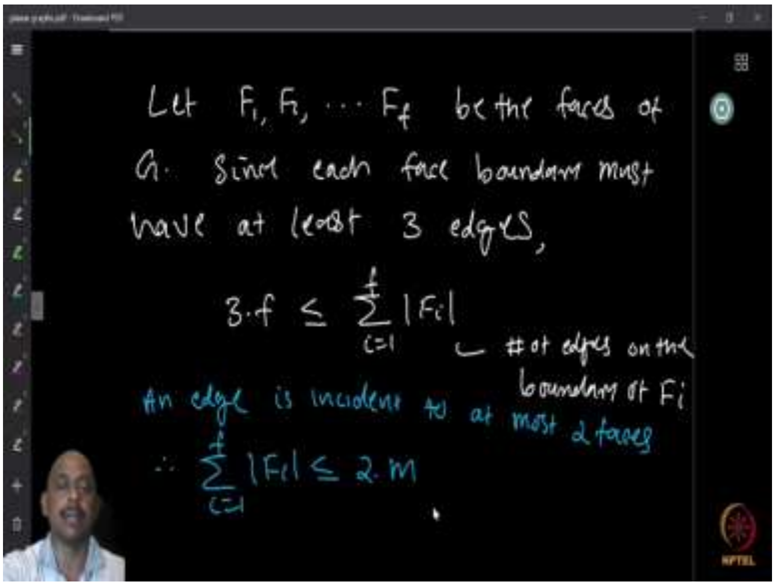

Since there are f faces, let us call these faces by name, $F_1, F_2, \ldots, F_f$ be the faces of the graph G. Now, since each face boundary must have at least 3 edges, we know that

$3f \leq \sum_{i=1}^{f} |F_i|$, where the $|F_i|$ is the length of the face F_i, length of the face is the number of edges in the boundary of that face.

But now, what we know about the relation between the faces of a graph and the edges is that if you look at the boundaries of faces, every edge is incident to at most 2 face. So, an edge cannot be contributed or counted by more than 2 faces.

In the summation, an edge is only counted at most 2 times. Therefore, if you look at, length of the boundary of a face is basically the number of edges. Therefore, you will see that since an edge cannot be counted more than twice, $\sum_{i=1}^{f} |F_i| \leq 2m$.

Now, we have the identity $3f \leq 2m$

We can now substitute this into the Euler identity, n + f = m + 2.

So f = m-n+2.

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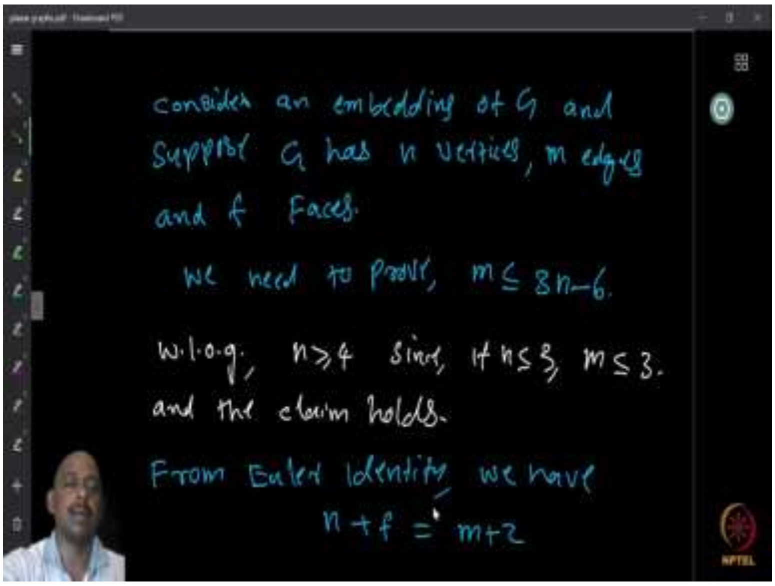

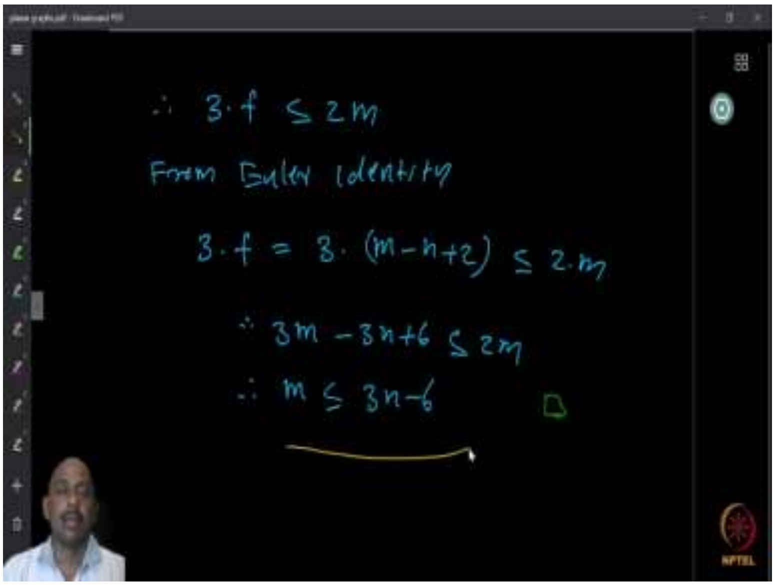

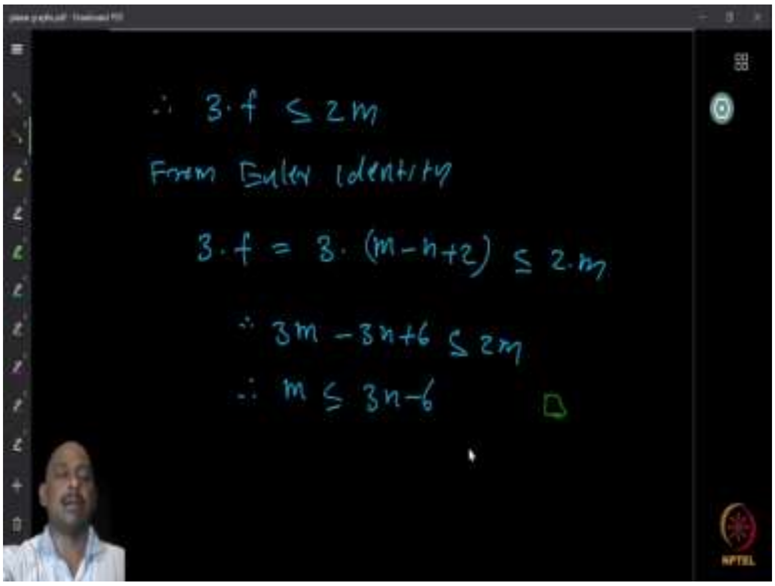

$$3f = 3(m - n + 2) \leq 2m$$

$$3m - 3n + 6 \leq 2m$$

$$m \leq 3n - 6$$

So, for any planar graph G we have shown that you can have at most 3n - 6 edges. Now, we will use the identity because we can assume that the graph is connected because if the graph is not connected what we can do? We can add edges to make it connected. We can add edges to make it connected and which will only increase the number of edges. So, even after adding edges to make it connected, we have the identity therefore, for any planar graph you will see that the identity holds too.

So, by proving it for connected graph we could generalize to other graphs. For arbitrary planar graphs now, we have the identity. We can now use this to prove several results, this idea. So, one way you can immediately think of using this theorem is that if a graph is given to you and you are asked to check whether the graph is planar only how to I mean the first thing that you can do is to check if the number of edges is more than 3n - 6. Because if the number of edges is more than 3n - 6 you know immediately that the graph is not planar, because by this result we know that any planar graphs will have atmost 3n - 6 edges. So, this can be used to prove some interesting results.

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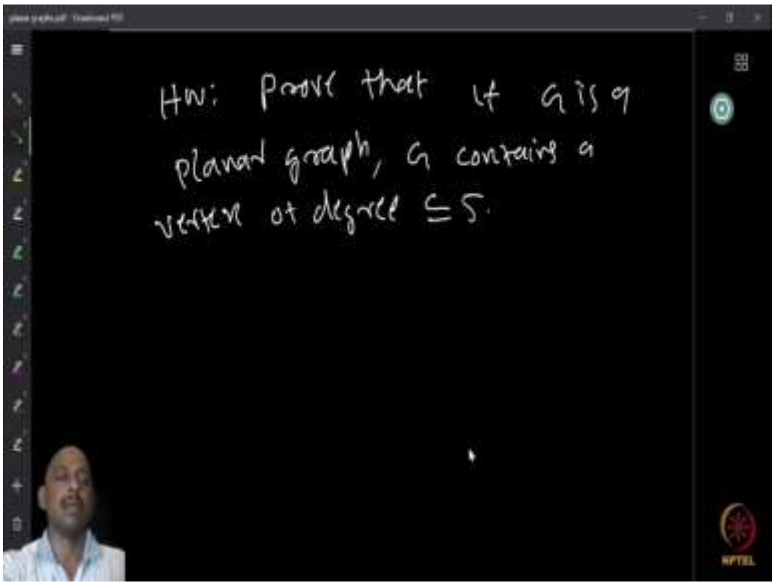

So, here is a homework, prove that if this planar graph, then G contains a vertex of degree less than or equal to 5. So, think of this and try to come up with that.

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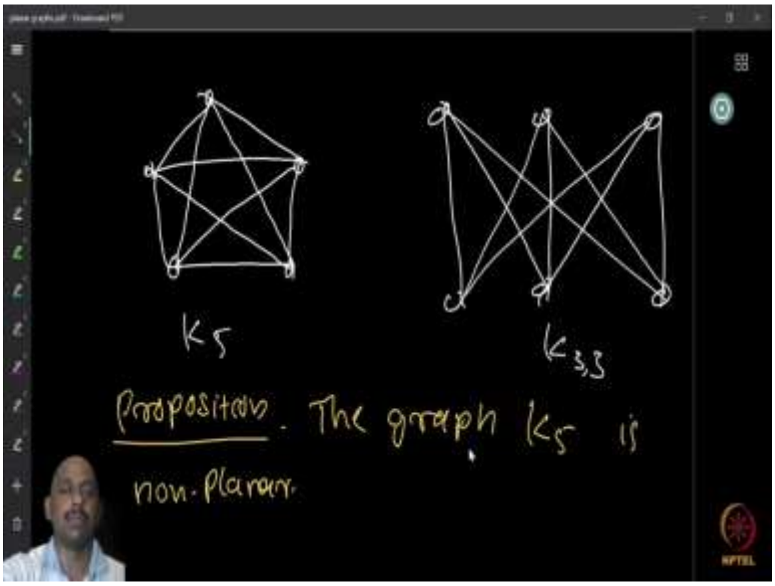

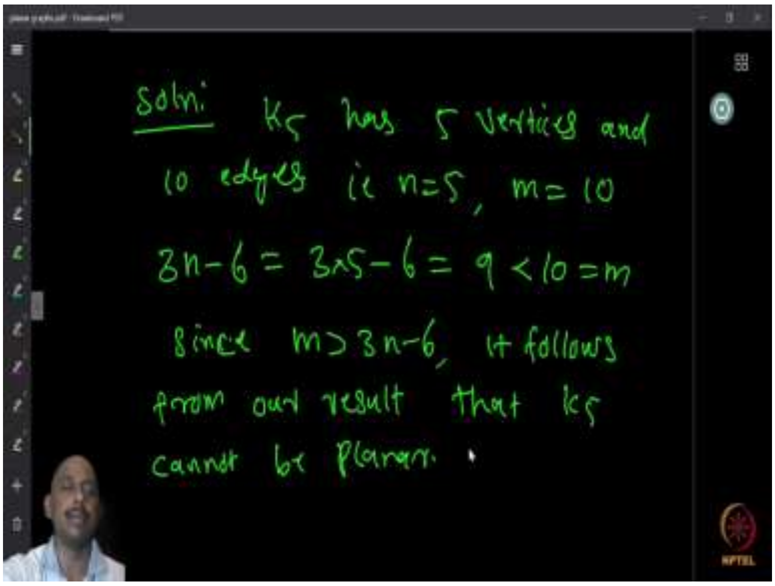

So, now we have the tool to prove that the graph K_5 is not planar. So, you have these 2 non-planar graphs that I introduced to you earlier and I claim that these 2 graphs are not planar and we did not prove it. Now, we can prove it by using the Euler identity. So, let us prove the first one, K_5 is not planar. So, K_5 case is very easy; K_5 has 5 vertices and 10 edges.

So, in our Euler identity, n = 5, m = 10 and now 3n - 6 = 3 × 5 - 6 = 15 - 6 = 9 < 10 which is the number of edges in the graph. So, since you have more edges than 3n - 6 the graph is definitely not planar by Euler identity, using Euler identity we are able to prove this a number of edges bound. So, the graph is immediately not planar, K_5 is not planar. Now, let us try to apply the same for $K_{3,3}$.

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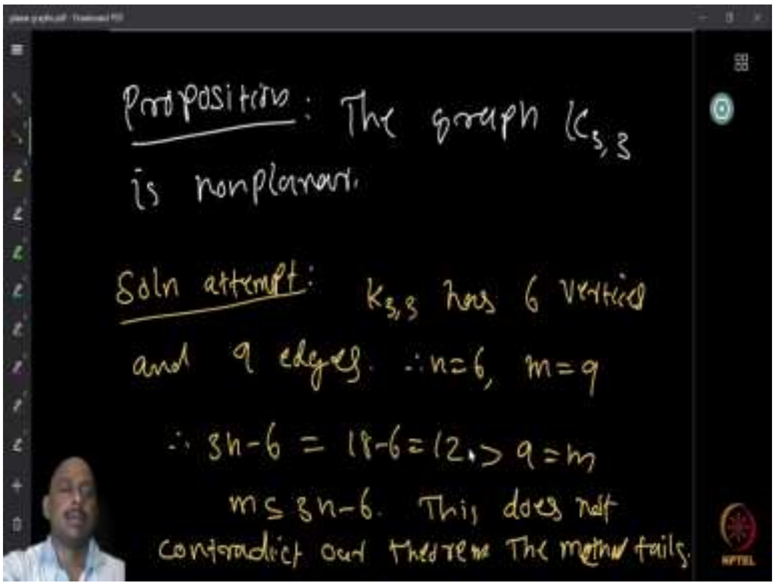

So, we have $K_{3,3}$ the graph $K_{3,3}$ is not planar. So, let us try to use the same idea $K_{3,3}$ has 6 vertices and 9 edges. And what is 3n - 6 = 18 – 6 = 12. But, you know the graph on 6 vertices allow 12 edges to be there at most. But we have only 9 edges in the graph $K_{3,3}$. So, the identity does not tell us whether the graph is planar or not. Because the, the result that planar graph can have at most 3n - 6 edges is a one directional result.

It does not tell that if the number of edges is less the graph is planar, it only tells that the number of edges is more than the graph is definitely not planar. So, using this idea, immediately is not helping. Directly we cannot use this bound to show whether the graph $K_{3,3}$ planner or not. Now how do you go about proving $K_{3,3}$ is not planar. So obviously, a number of edges is not going to help you or directly by using this formula. Can you think of something about $K_{3,3}$ and how you can improve upon this. So, think about this for some time. And then we will look at the proof.

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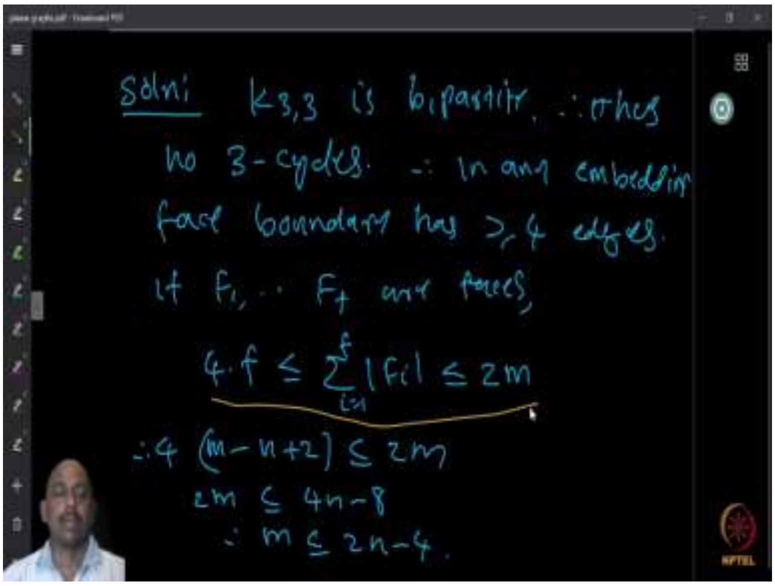

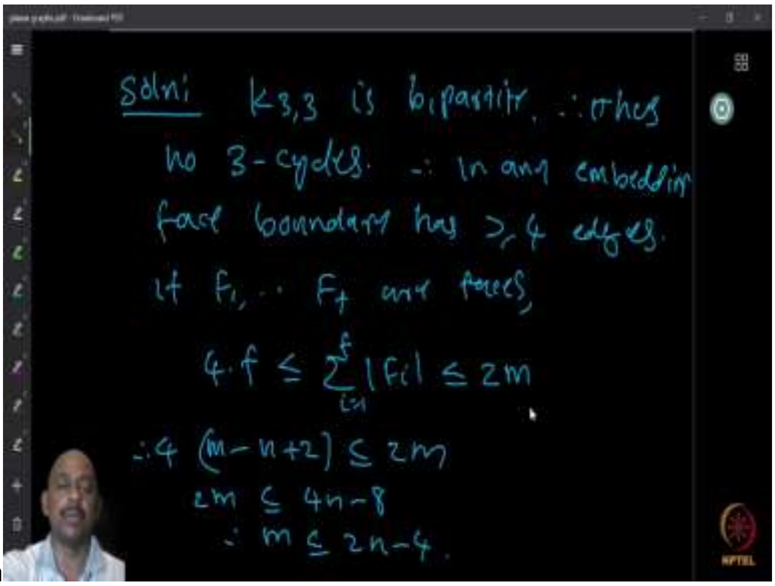

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Here is a different approach. So, you have the graph $K_{3,3}$,but the $K_{3,3}$ is a bipartite. It is a complete bipartite graph. Now, a bipartite graph has no odd length cycles. So, therefore $K_{3,3}$ has no 3 cycles. When we were trying to prove this relation between the number of vertices and edges, we use the fact that every face boundary has at least 3 edges. But now, since the graph is bipartite, you know that any face boundary must have at least 4 edges.

So, can we use this idea now, to improve upon the number of edges itself? So, let us take a bipartite planar graph, we know that, if it has f faces, then every face boundary has at least 4 edges and therefore, $4f \le \sum_{i=1}^{f} |F_i| \le 2m$. Because, if you sum over all the cardinalities of faces, each face boundary has at least 4, but each edge is counted at most twice.

So, 4f ≤ 2m. Now, directly using Euler identity, 4 (m-n+2) ≤ 2m. And this tells you that 2m ≤ 4n – 8. Therefore, m ≤ 2n - 4. Which is to say that any planar bipartite graph we did not use it is $K_{3,3}$or anything, any planar bipartite graph has at most 2n - 4 edges. So, any planar graph has at most 3n - 6 edges, but any planar bipartite graph has at most 2n - 4 edges. Now, this is a much improved bound for the number of edges. So, therefore, let us try to use it on $K_{3,3}$

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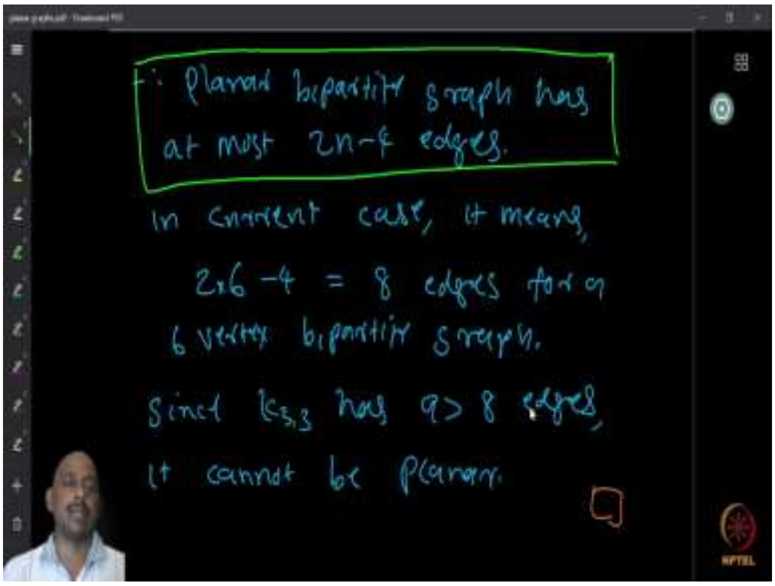

So, $K_{3,3}$ is bipartite and 2n - 4 = 2× 6 - 4 which is 8. But we have 9 edges in $K_{3,3}$.Therefore, bipartite planner graph cannot have these many edges and this tells you that $K_{3,3}$ is not planar. So, this way you can use Euler identity in several forms and with more improvements to show several results about planarity. So, here is a result that you should remember. We proved one theorem in between. As part of proving attempt, we proved another theorem which is that the planar bipartite graph has at most 2n - 4 edges. So, keep that also in mind.