Combinatronics
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Lecture 41
Planar Graphs

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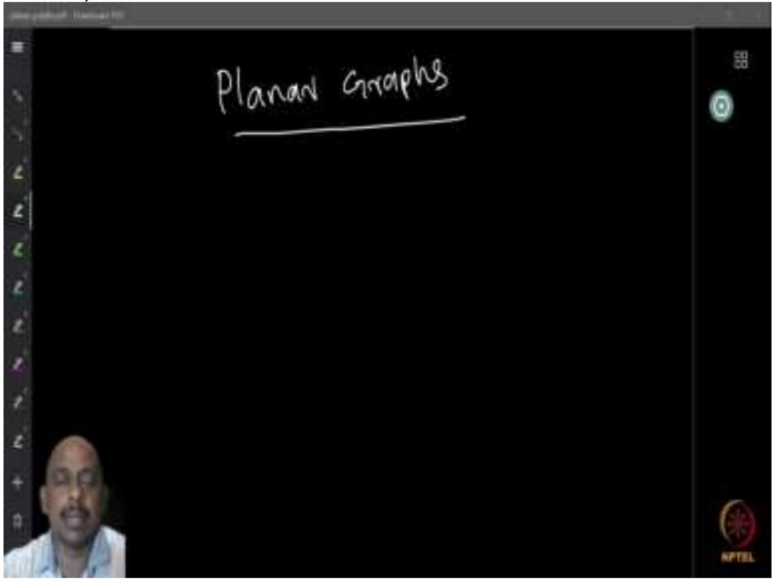

Welcome back, in this lecture we look at what are called planar graphs. Planar graphs play a very important role in graph theory. Especially, in dealing with complex problems where many other techniques fail on general graphs. We try to look at special class of graphs, and one of the most important classes of graphs that comes into mind is that of planar. So, what are planar graphs? If you look at graphs, when you think about it, one of the things that you always think about is the representation of the graph as vertices and edges that are mapped onto the plane.

Now, when we draw a graph on the plane, it often happens that like you need to draw the edges such that the edges cross each other. But, if it is possible to draw, such a way that the edges does not cross, then it is much easier to visualize. And therefore, one would like to see if it is possible to draw a given graph on the plane without the edges being crossed.

So, what we want to do is to see under what conditions we can have this property. And what can you say about such graphs, do they have some specific structure? And questions related to that. This is what comes under the study of planar graphs. So, let me define what a planer graph is more formally.

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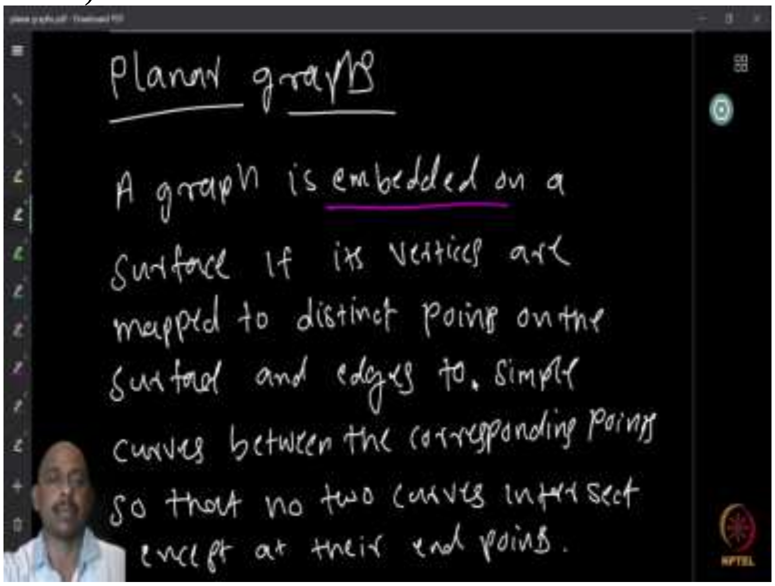

A graph is set to be embedded on a surface. So, the surface can be the plane or it can be other surfaces like torus are higher genera surfaces, a graph is said to be embedded on a surface if you can map the vertices of the graph to distinct points on the surface and the edges of this graph to simple curves, between the corresponding points so that no two curves intersect except at their endpoint.

This is what we mean by saying that it is not crossing. So, when I say simple curve, these are actually more technical terms which I do not want to go into the details. But for the time being, just take it for granted, that when I say a curve is a simple curve that is a curve which does not intersect with itself.

So, takes simple curves between the corresponding points and we do not want no two curves to intersect except at the vertices where they have their endpoints. If such a drawing is possible, then we say this is an embedding of the graph on that surface. Now, if this particular surface happened to be the plane $\mathbb{R}^2$, then we say the graph is planar.

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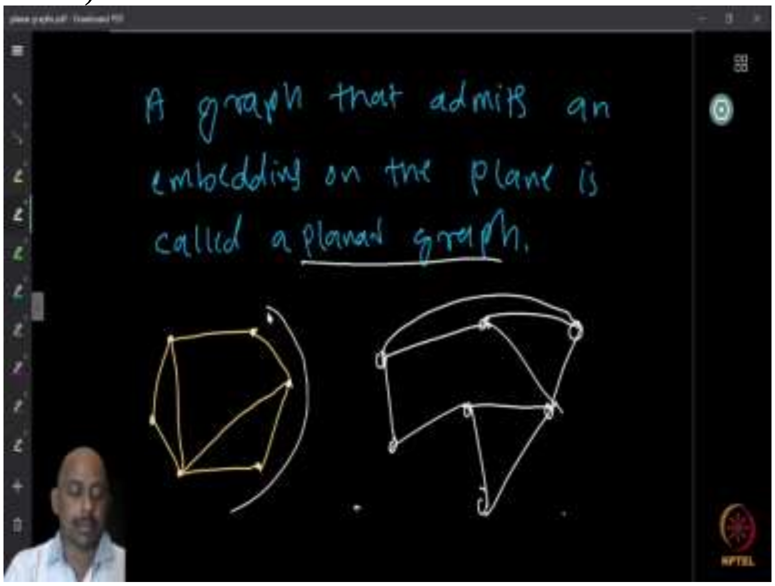

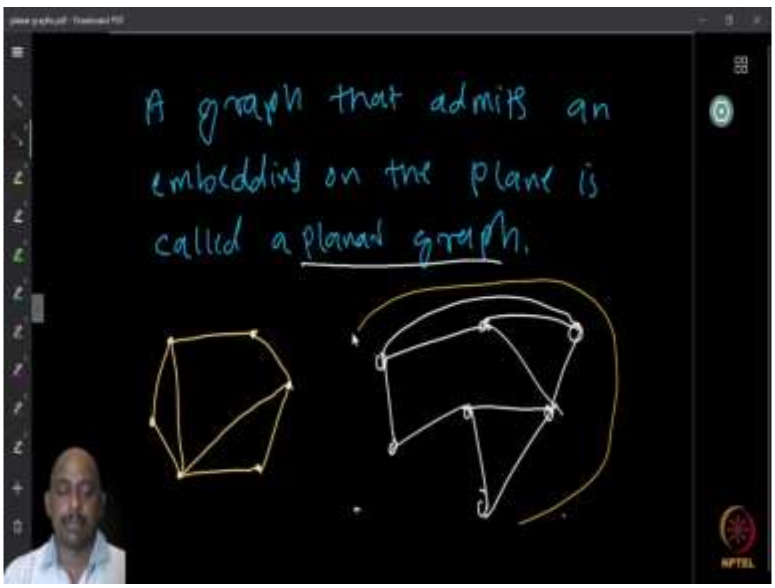

So, a graph that admits an embedding on the plane is called a planar graph. So, here are some examples. So, the first graph that you see here is a graph on six vertices, and you have 8 edges. So, you have this, this graph. And then and, you can very well see that this graph is represented on the plane where, you have the vertices as distinct points. And whenever you have an edge between a pair of vertices, then you draw a line segment connecting them.

And this has the property that the line segments does not intersect anywhere except at the vertices. Similarly, the second graph you can see again a collection of points and curves such that they do not intersect each other. So, therefore, we can see that this is also a planar graph.

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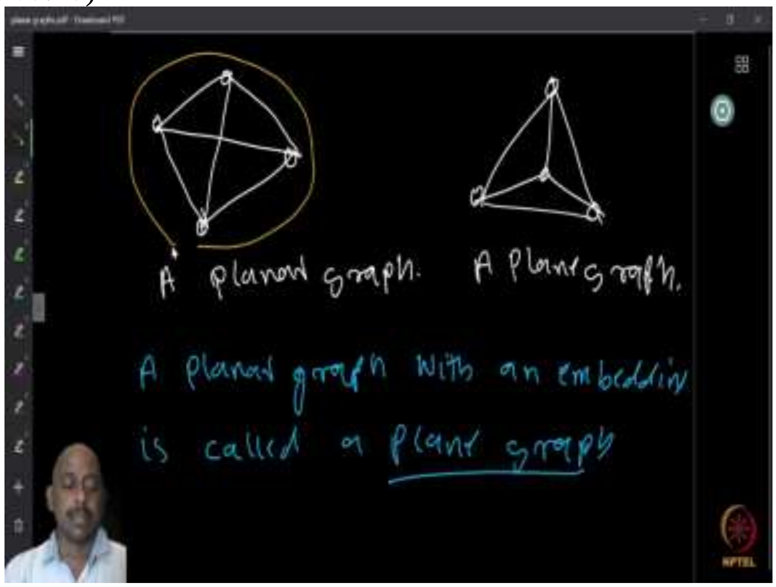

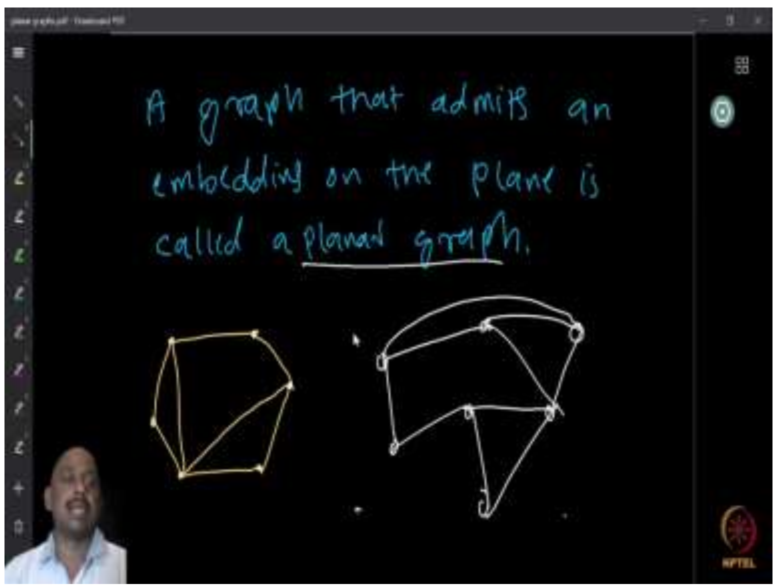

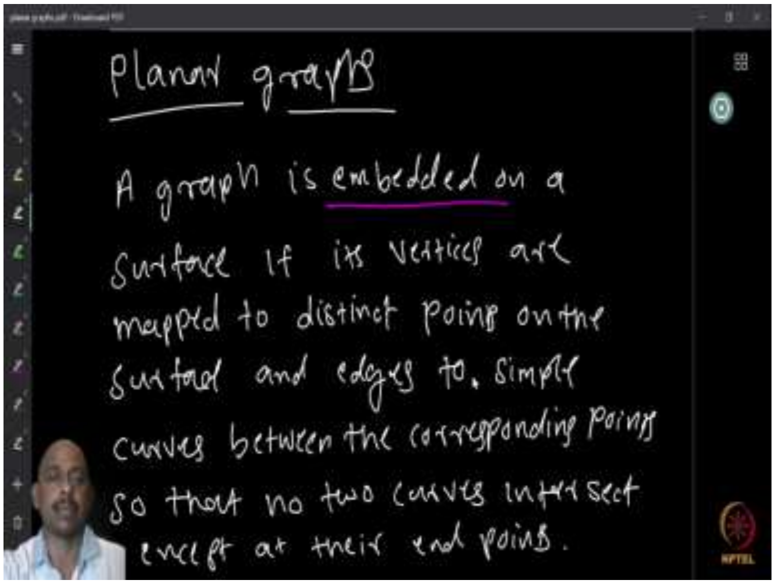

A graph can be a planar graph, even though its representation that we, draw on the plane is not satisfying this embedding property. Example, what we are saying is that a graph is planar if it admits an embedding. So, the definition of the planar graph does not say that, your embedding needs to be planar embedding for the graph to be planar, only that it has at least one embedding where it is planar. So, to see the difference, let us look at this graph, which is a complete graph on four vertices. You have this first embedding, where the edges actually intersect each other. Therefore, this is not a planar embedding. But the graph is planar.

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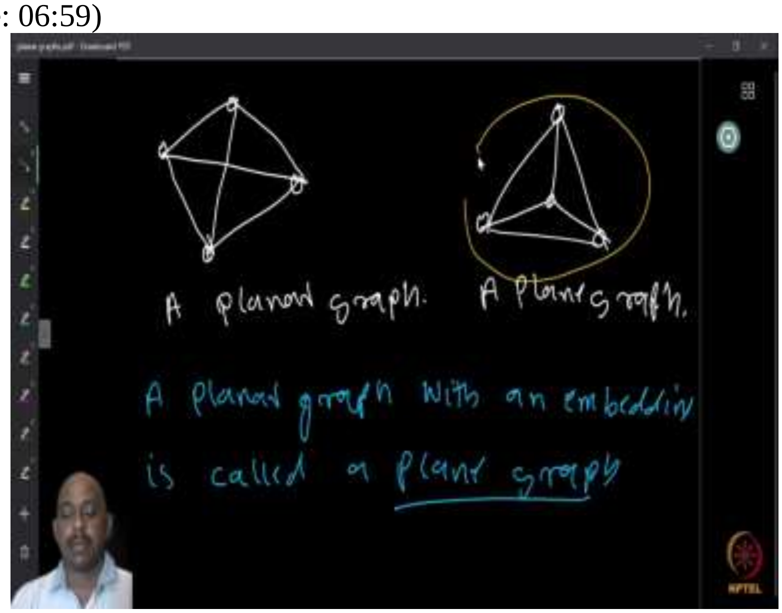

You have this planar graph whose given embedding is not a planar embedding, it is not an embedding on the plane. But the graph is planar because it admits a plane embedding. So, if you look at the second picture, it is the same graph with the different embedding, which is also an embedding on the plane. Now, consider this embedding of the graph, the second embedding, which is an embedding on the plane.

Now, the graph together with an embedding is called a plane graph. So, a particular embedding of a graph is, if it is on the plane, then we say the graph is plane graph. The planar graph together with an embedding is called a plane graph. So, the second embedding is a plane graph, because it comes with the embedding. But the first embedding, first drawing, you cannot say it is an embedding on the plane. So, the first drawing is not an embedding. Therefore, the graph is planar, but it is not a planar embedding. So, therefore, we do not say it is a plane graph, it is a planar graph.

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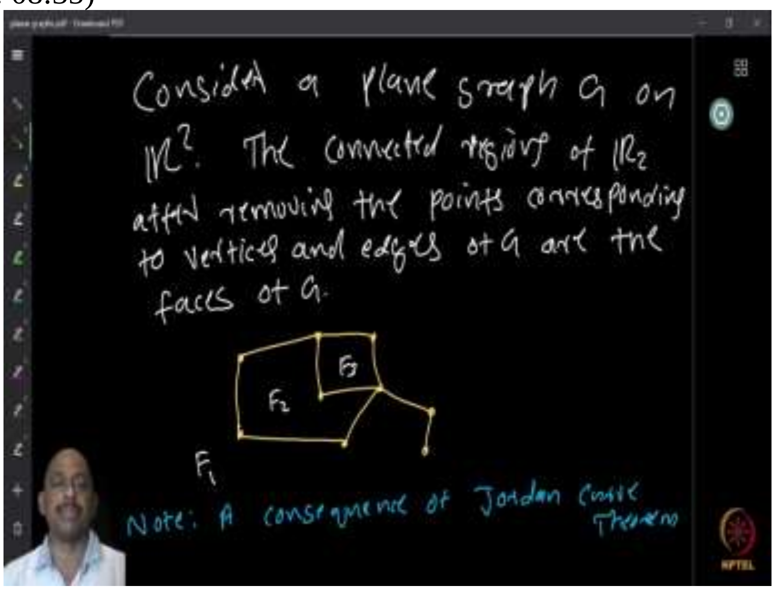

Now, let us look at an example on $\mathbb{R}^2$. Consider a plane graph on $\mathbb{R}^2$. Now, if you look at such a representation, you will see that, intuitively speaking, you will see that if you look at any cycle, if you take any cycle in the graph, the graph has cycles. Then the cycle basically forms a kind of closed curve. Now, if there is a closed curve on the plane $\mathbb{R}^2$, then this closed curve basically separates the region or the plane into two parts, the region into two parts.

One is the inside part of the curve and one is the outside part of the curve. Now, this again is intuitively clear, but to prove is, is not exactly simple. So, we will not go into it. This can be studied in another different course. But this is a very famous result called Jordan Curve Theorem. So, what is the Jordan Curve Theorem, say?

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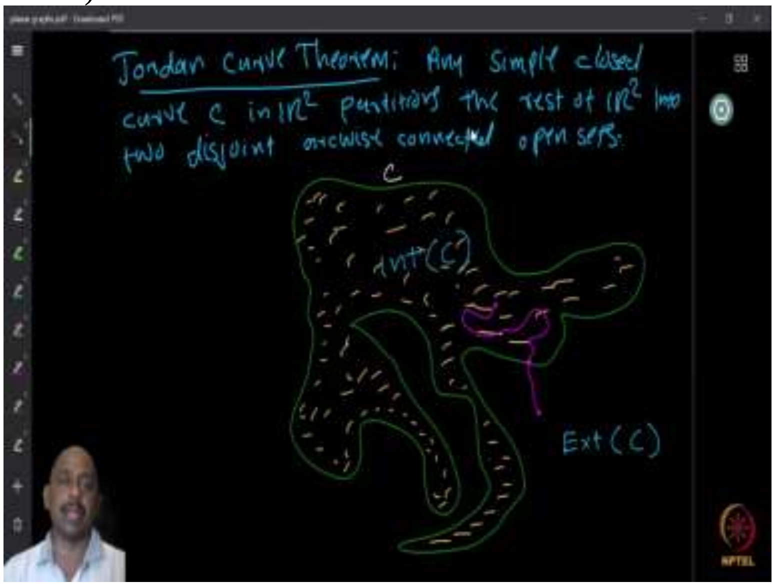

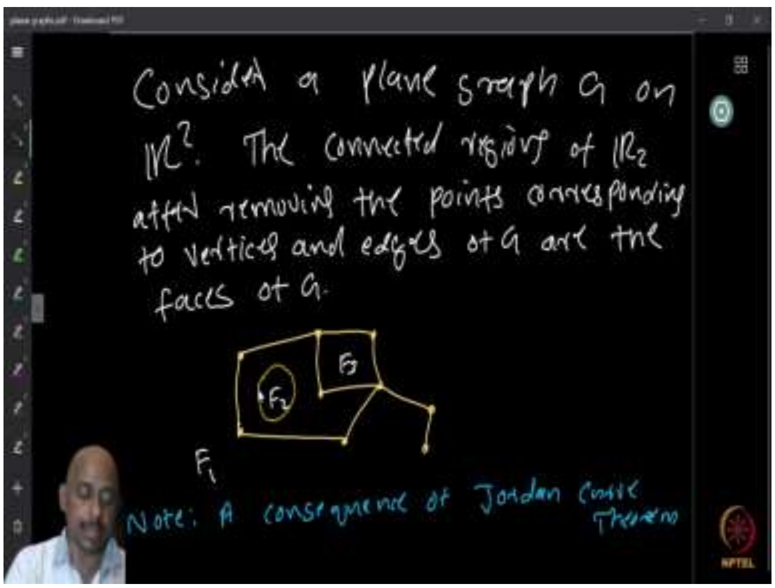

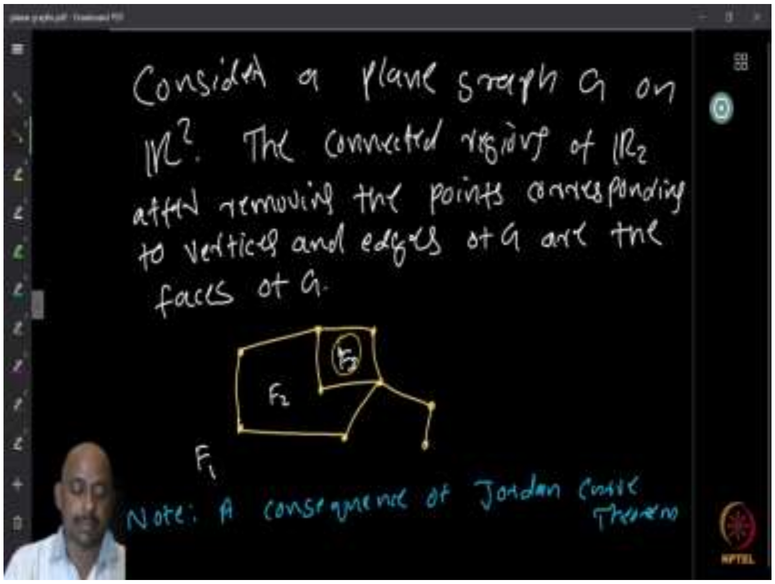

So, if you take any simple closed curve C in $\mathbb{R}^2$, then this curve partitions the rest of $\mathbb{R}^2$, except the points of the curve. The points of the curve forms a subset. If you remove this point, so, it partitions the rest of $\mathbb{R}^2$, into two disjoint arcwise connected open sets. So, again these terms are technical, but what we want to say is that if you take the entire $\mathbb{R}^2$, just put one simple closed curve, with a curve which does not intersect itself.

Then it divides the region into two parts. One is the inside and one is the outside. The inside part is connected if the graph is simple, it is connected in the sense that, you can find a curve connecting any two points of this area, it is an open set. And similarly, the outside part is also having the same property. So, this is what? Jordan Curve Theorem that, it partitions the region into two parts.

Now, if you come back to the embedding, you will see that, you take the embedding of any given graph, then what you do is that you remove all the points of this embedding, that is the point that belongs to the curves representing the edges as well as the vertices. So, if you remove all these points, then the region might have several connected, it gets in several connected components.

So, these connected parts of the region, the remaining parts are called the faces of the graph G. So, intuitively it is clear that if you look at a picture of the graph, so, you look at these connected regions, which you can find inside. They are all faces for example, this is face, this is another face.

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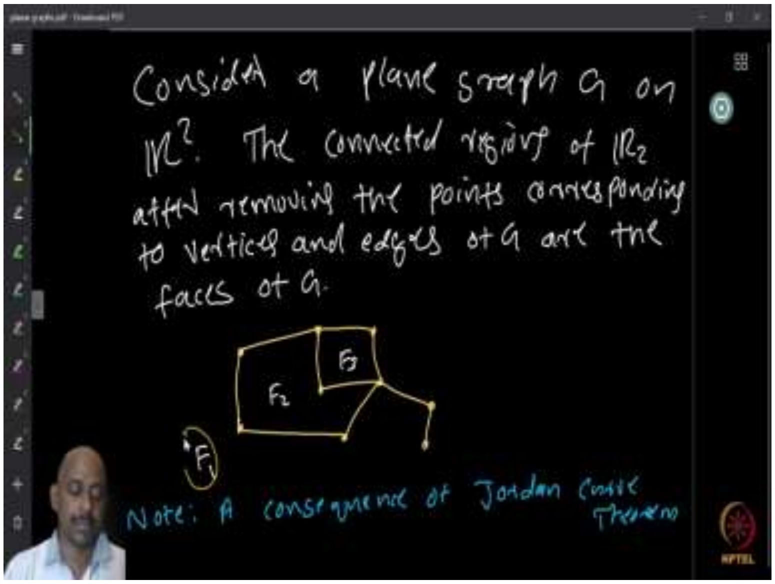

And then the outside part is also a face. Because it is also a connected region, which I call F_1. So, when you take any embedding of a planar graph on the plane, you can talk about the faces of the graph. So, the faces of the graph are basically the external face, the outside face every point other than the other faces and the curves. And then the internal faces or bounded face. There is unbounded face and there is also the other faces are all bounded. So, again this fact that, you can define these faces is a consequence of the Jordan Theorem.

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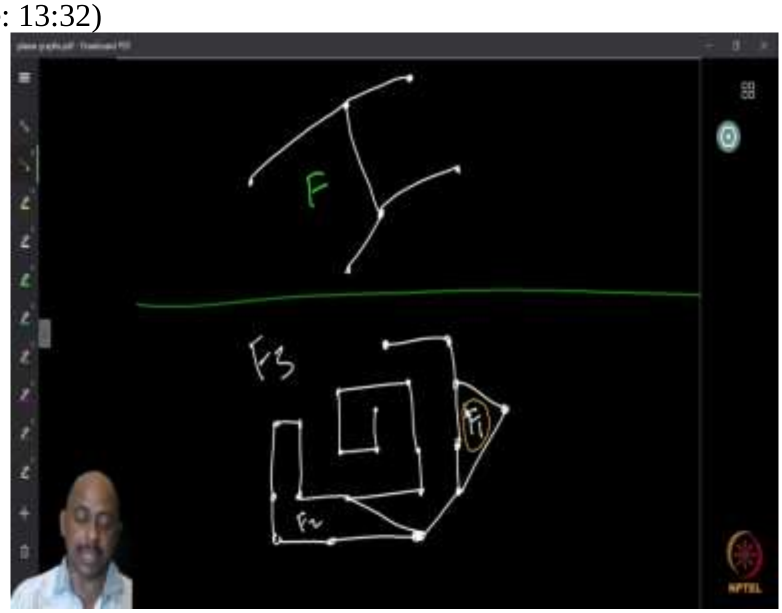

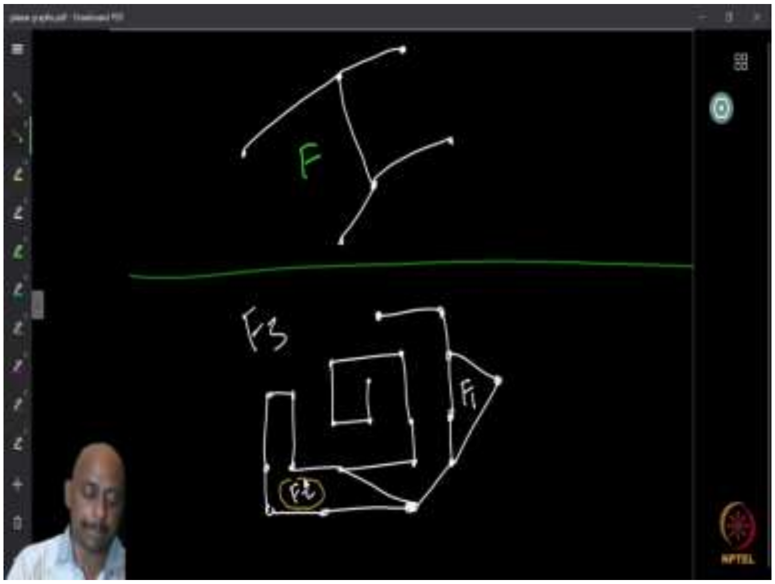

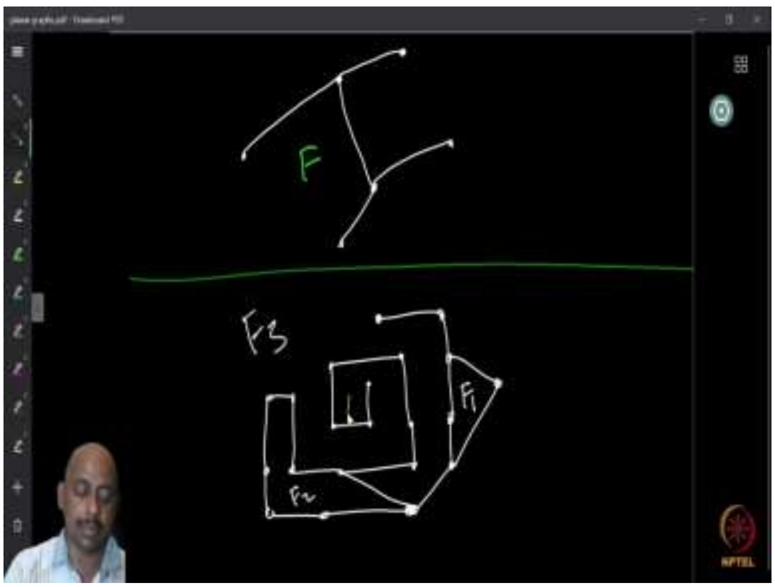

Now, here are a couple of more examples. We look at the first graph, there is no cycle. Therefore, removing these points of the embedding does not create a disjoint set. I mean, there is only one set, they do not create several sets. So, there is only one connected component. Therefore, that is the only face, which is the outer face or the unbounded external face. So, that is the first example.

In the second example you will see that you have another graph, which is a very nice-looking embedding. And you can see there are exactly three faces. So, what are these three faces? The face one is this four edges form a cycle. Then you have another face, bounded face which is F_2, mark here. Which is 8 edges. Then, the entire remaining part, is one connected area. So, therefore, that forms a face F_3.

So, even though you have these weird shapes, all these areas are connected therefore, that entire part, so it is outside, outer face. It might look like internal part, if you are not careful in certain drawings you might think that, parts like this might look like internal face. But it could be external faces. So, you have to be careful, but one can look through the embedding carefully and then you can see what are precisely the face of the graph.

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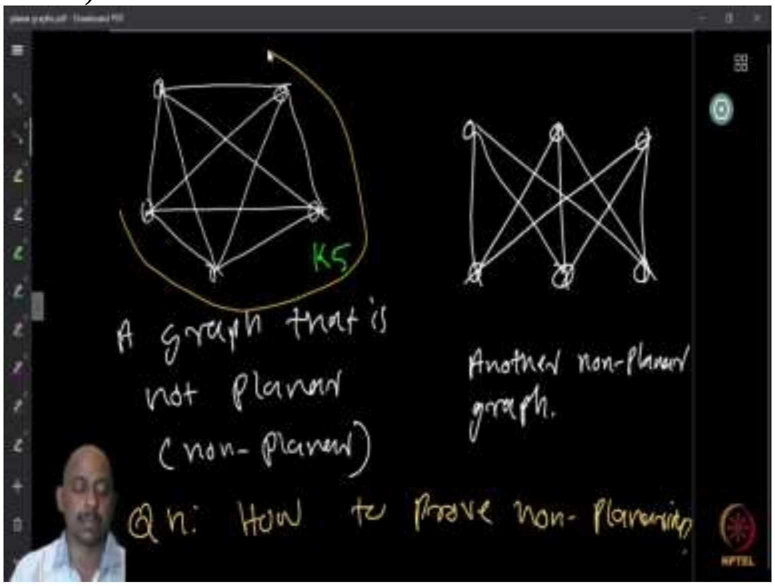

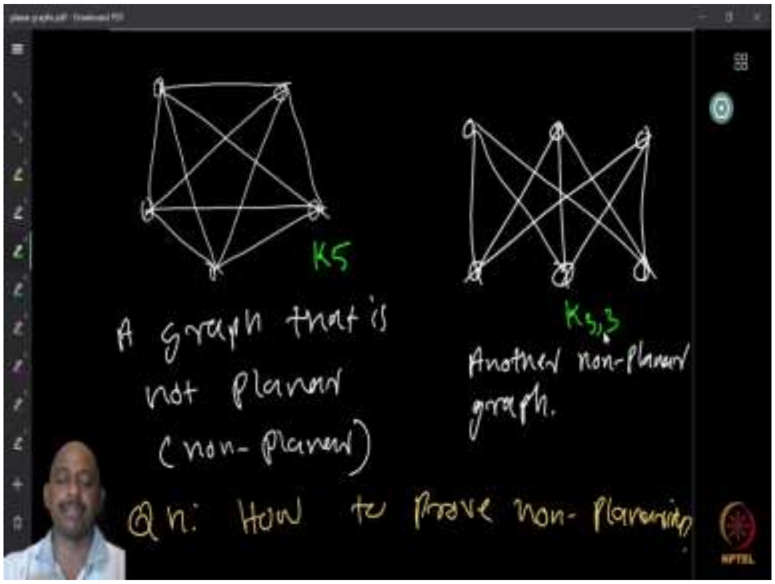

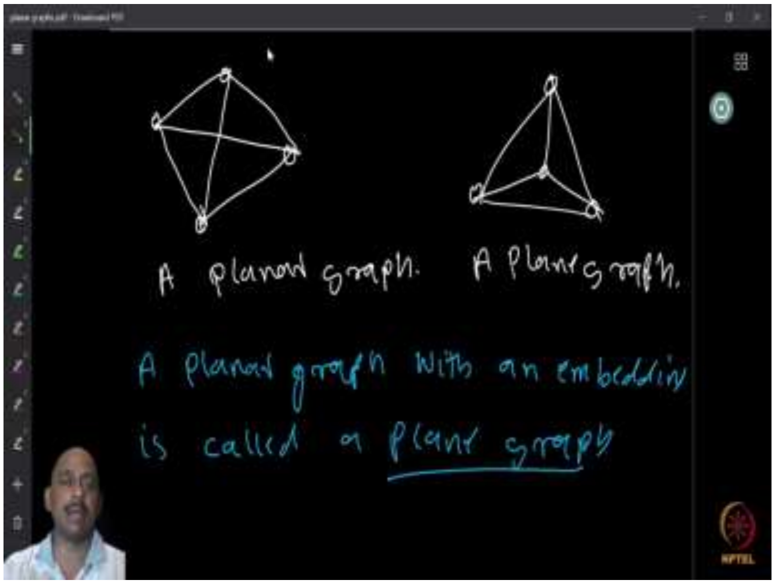

Here I give you an example of a graph, which I claim that it is not planar graph, that you cannot find an embedding on the plane where no edges cross each other. So, this is the first graph is the complete graph on five vertices, which is K_5. And I claim that this graph is not planar. I welcome you to think, why this is not planar or of how do you prove a graph is not planar.

Because, to show a graph is planar all you have to do is to come up with an embedding, where edges does not cross each other. To show a graph is planar is easy, in the sense that you just need to come up with one embedding. But now, if I ask you to prove a graph is not planar, then how will you do it? If you look at one embedding and say that, here that edges cross then I can say that that is not a proof.

Because as I showed you before, you have this graph K_4, where we saw that there is an embedding, where the edges cross but the graph is still planar. So, showing me ten different drawings and saying that, none of these drawings are planar embeddings is not a proof that the graph itself is not planar. So, think about this. In this example, we saw that you have the graph, complete graph on four vertices, but it has it also had a nice plane embedding.

Now, can you come up with an argument to show that the graph is not planar? Here is another non-planar graph which is complete bipartite graph, $K_{3,3}$. So, this complete bipartite graph $K_{3,3}$ is also not planar, that is my claim. Again, try to draw this graph and make the embedding so that it is possible, try to make it planar or try to minimize the number of edge crossing and see where you are going to miss out.

So, if it is possible to do this, then it would be nice. And then another way to show is to think about how to show this is non-planar? And I suggest that you can also try to use the ideas of Jordan Curve Theorem. How do you prove this using Jordan Curve Theorem? So, that would be a nice, nice exercise to think about. We will come to this later.

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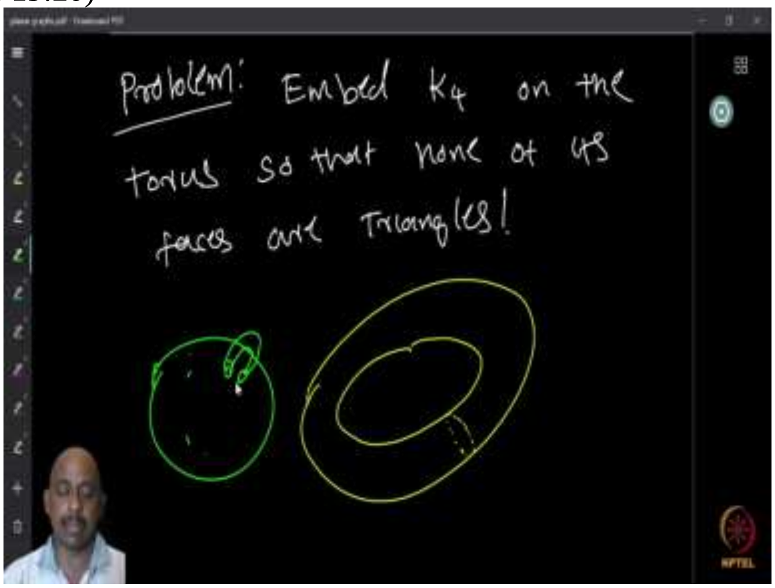

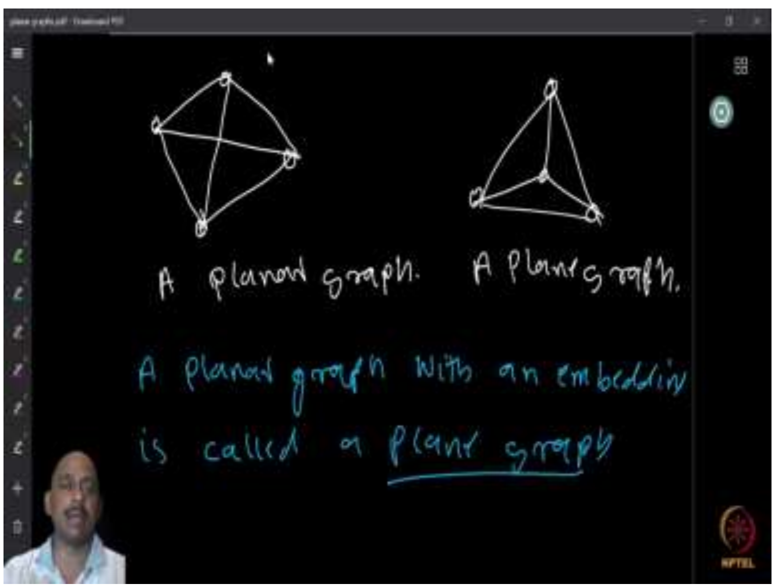

Now you have a homework question that, try to embed the complete graph on the torus. So, that none of it faces are triangle. This is just a intellectual exercise. Try to see how you can come up with this. So, what is a torus? Let me try to draw it. So, this is a drawing of a torus that just to show that this is basically, this is basically a cylinder that is connected together at the endpoints so that, you get a medhuvada shaped object or a doughnut shaped object. So, this is basically a surface with a genus, more than the genus of the sphere. So, like, if you take a sphere, you take a sphere and then you attach a handle like this. Then this topologically is equivalent to the torus. So, basically the number of handles tells you the genus.

And the genus of the torus is one. And you can have higher genus and when you increase the genus you will see that you will be able to put more and more edges without the need for crossing. Because the surface becomes more complicated, you can do more stuff like this. Now, the interesting part about this exercise is that if you, if you look at the embedding of the planar graph, the K_4 on the plane, the complete graph on four vertices, on the plane, all its faces are triangles.

You look at any of the faces, each phase is basically a three cycle, the boundary of any phase is a three cycle. Even the outer face, the boundary is a three cycle. But on the other hand what I want you to do is to find an embedding numbering on the torus for this graph, so that no face is a triangle. And this is probably not immediate, but if you think about it, you will be able to come up with an answer.

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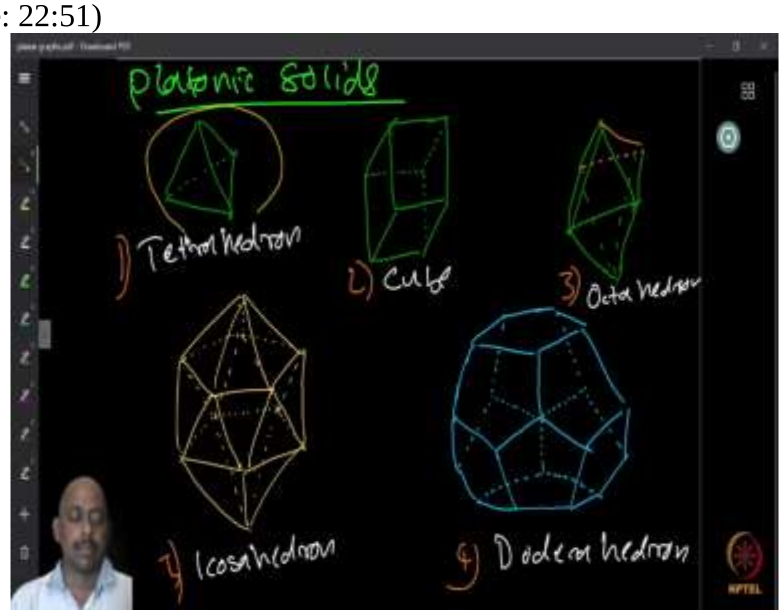

Now, if you studied school geometry, you might have come across these objects. These are the platonic solids, there are five regular polyhedron which are called platonic solids. And these are the tetrahedron, the first one. The tetrahedron has four vertices and four faces and then six edges you can verify that there are six edges, four vertices and four edges, no, four vertices and four faces.

Now, the reason for the graph, the vertices of the graph, edges of the graph and faces of the graph are called to be vertices, edges and faces also has something to do with these shapes. So, you have these platonic solids, first one is a tetrahedron, second one is cube. Which we are all familiar with. It has 8 vertices, 12 edges and 6 faces. Then you have the octahedron.

And each phase of the cube is basically a square. And you have the octahedron, where you have a triangular surface again, you have 8 faces, then you have 12 edges and you have 6 vertices. Then you have the icosahedron, which has how many faces? So 20 faces. It has 12 vertices. And then you have a how many edges you have? 30 edges and 20 faces. So, 20 faces. Now, then you have the dodecahedron. Which has, pentagonal faces and then it has 12 faces and then it has 30 edges.

So, these are the platonic solids. Now, if you look at this more carefully, you will see that the number of faces of the cube is the number of vertices of the octahedron, and the number of phases of the octahedron is the number of vertices of the cube are also the same. Similarly, if you look at the icosahedron and dodecahedron, again the number of phases in one becomes the number of vertices in the other and vice versa.

And then tetrahedron stands alone and but if you can try to find a connection between the cube and the tetrahedron or the icosahedron and the dodecahedron, the same relation holds for tetrahedron with itself. So, this is a kind of what is called duality, we will not go into duality in this course, but it may be nice to think about this. But, we will see something very nice about this. And there is a relation between the vertices, edges and faces, which was proved by Euler, Leonard Euler. And that has been generalized to planar graph also. So, we will see what is the relation.

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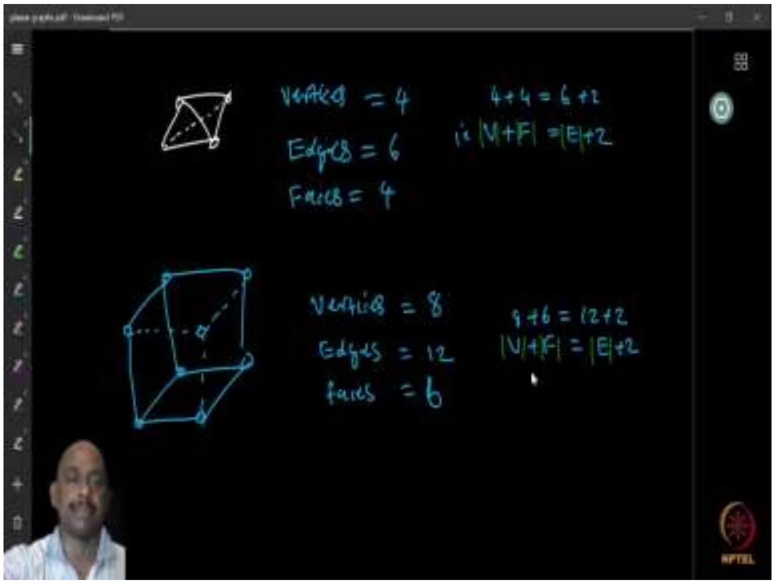

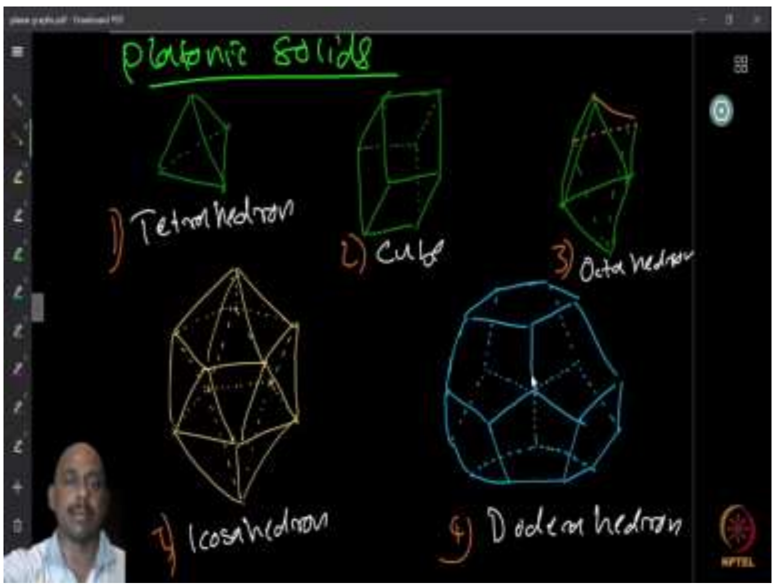

So, let us look at the tetrahedron face. So, it has four vertices, 6 edges and 4 faces. Let us observe that the number of vertices, which is 4, plus number of faces which is 4 which is equal to number of edges 6 plus 2. Now, if you take the cube also you will see that number of vertices which is 8 plus number of faces, which is 6 which is equal to 14, which is 12 plus 2 which is the number of edges plus 2. Similarly, you will see that same thing holds for the, the octahedron icosahedron and dodrahedron.

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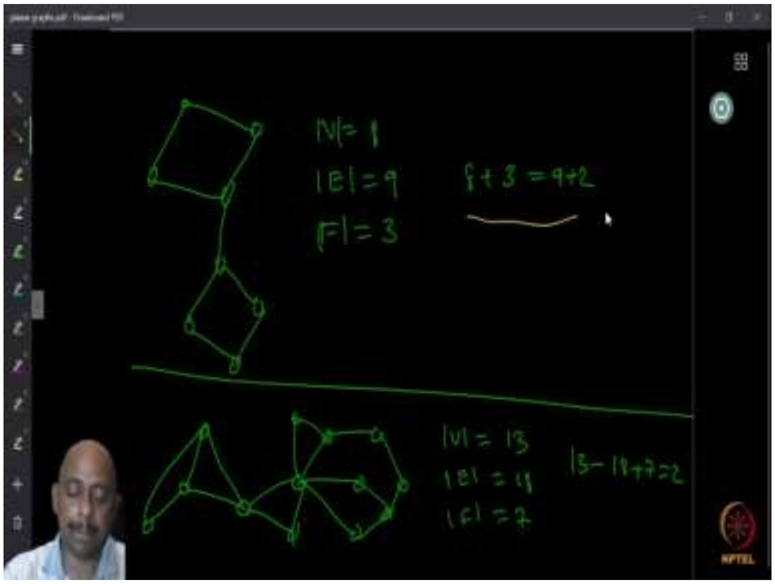

So, verify this, then I take any graph. So, take a planar graph, look at the embedding, and look at the number of faces. If you take any graph like this, it has eight vertices, then it has nine edges, and exactly 3 faces. And the 8 plus 3, the number of edges plus 3 is actually equal to the number of vertices plus number of faces that will be equal to number of edges plus 2. Take any other planar graph for example, the one below you have 13 vertices, 18 edges and seven faces. And again, 13 plus 7 is equal to 18 plus 2, so this is easy to see.