Combinatronics
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Lecture 40
System of Distinctive Representative

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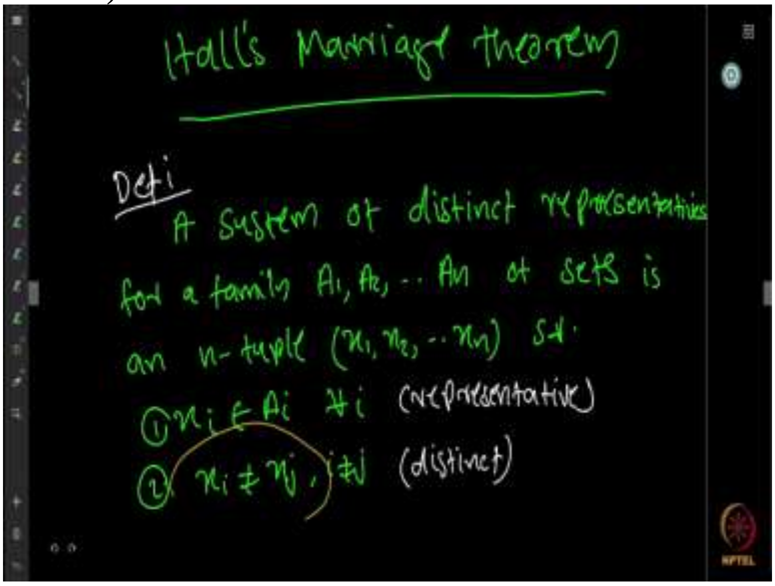

And one of them is this theorem in combinatorics called Hall's Marriage Theorem. So, what does Hall's Marriage Theorem say? Before defining the theorem let me, define what is called a system of distinct representatives. So, it is like this, the idea of Hall was that, suppose you have let us say a set of men and women. And let us say that women have preferences for men.

Each lady is now asked to give a list of men she is willing to marry. And then what you do is that, you try to see whether you can find a matching for these people, so that, each men can get married to one of the ladies who prefer him. So, if you can find at least one such matching, we can say that all of them can get married.

So, every lady has a preference whom she is willing to marry, otherwise she is not going to marry. And men of course, does not have any such preference as far as they can get some but if married, he is happy. So, finding such matching, that is why it is called the Marriage Theorem. But there is a much general statement in set theory, which we are going to state.

So, this is called a system of distinct representatives. Given a family of sets let us say $A_1, A_2, \dots, A_n$. A system of distinct representatives for this family is an n-tuple of elements $(x_1, \dots, x_n)$ such that, each element x_i is a member of A_i. So, I have to pick x_i from A_i. So, I want to find representatives of each of the sets. So, I have to pick x_i's from the corresponding sets.

And they must all be distinct, I cannot pick an element who is going to represent two different sets. So, x_i must be different from x_j whenever i ≠ j. So, I want to pick up elements such that, they represent the sets. And they are distinct. So, for each set I need a representative. Now, when is it possible to find such representatives for a set system? We will assume that all these sets are basically defined on a universal set, let us say U.

So, of course they could have intersections, I mean A_1 and A_2 all have intersections. Otherwise, we can clearly select if they are not empty, you can just pick elements one from one, one from each of the sets. You can find representatives, but when they have intersections this may not always be possible. So, Hall's condition, Hall's theorem gives a condition, when this is actually possible. And it is very interesting because one of the obvious conditions is also a sufficient condition.

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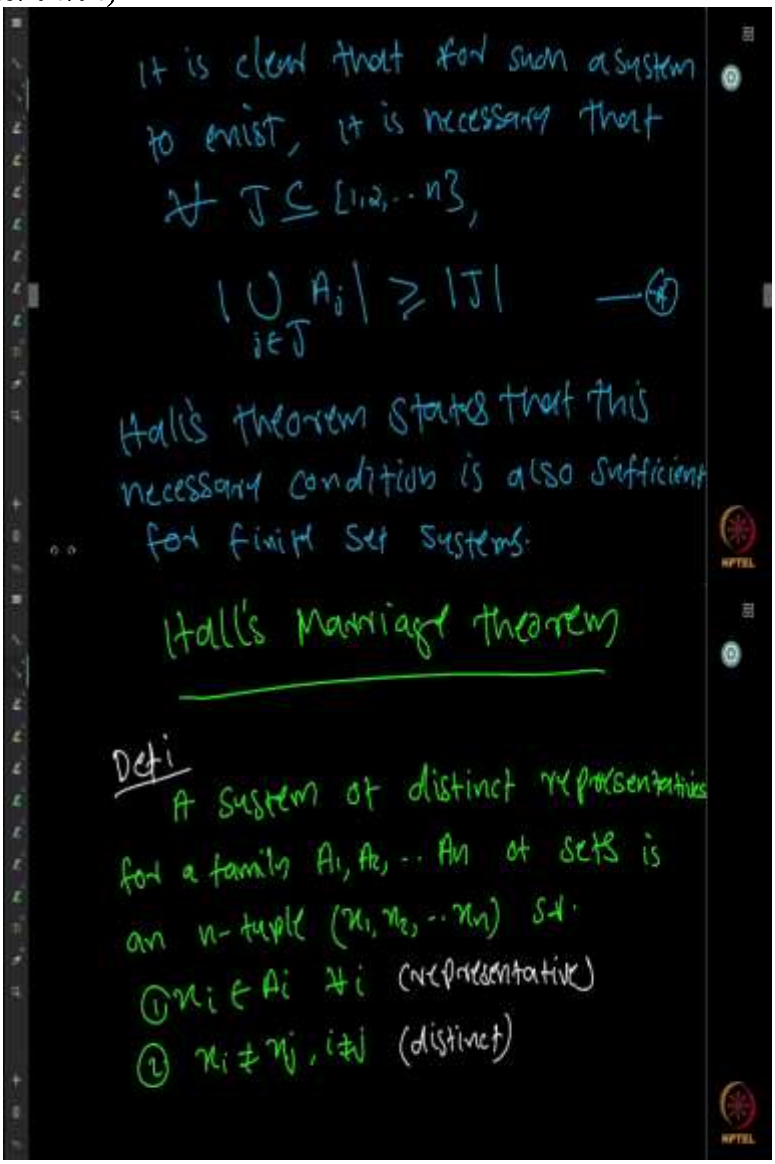

The obvious necessary condition is also sufficient condition. So, what is the obvious necessary condition? That if I want to find such a representatives, then for every union, of the set. So, if I, the union of sets, $A_1,\ A_2, \dots, A_k,$ for some k, or some subset of this sets I take the union, if the union has less, so union of k sets has less than k elements in common.

That can happen, right? For example, each set has let us say, two elements or something like that. And then, they have many intersections, like 1 and 2 are use here, 2 and 3 are use there, 3 and 4 are use there, 1 and 4 are use here, 2 and 4 are used somewhere else in different sets. Then the union of this six sets might have only 4 or 5 elements. In that case, you can definitely not find a representative for each set.

Because the total number of elements in the universe is less, than the number of sets you cannot find distinct representatives. So, the necessary condition that is obvious is that, for every indexing set J, J $\subseteq \{1,2, \dots, n\}$, $\left|\bigcup_{j\in J} A_j\right| \geq |\, J\,|$

So, if this is not true, we cannot find a representative, even for some arbitrary subset. So, therefore, this must be true for every possible subset J of $\{1,2, \dots, n\}$. If A_1 to A_n are the sets we are considering, then for every subset of J of $\{1,2, \dots, n\}$, we should have the union A_j over elements in J must have cardinality at least the number of elements in J.

So, this is the obvious necessary condition. But Hall's theorem states that, if this property is true for every subset, then we can find a system of representatives for finite set systems. So, if the set system is finite, that we are looking at only finite sets, then we can do this. If we are looking at infinite sets, it need not be true. So, finite set systems, when every sets that we are considering A_1 to A_n are all finite sets, then we can clearly find a system of distinct representatives according to Hall's Theorem.

So, this is what we want to prove. So, now what we are going to prove this is using a version of the same theorem in terms of graphs, we will, prove the graph version of the theorem. And then using that graph version we will see that, it is actually identical to this version, we can prove the same here.

Now, before proving the graph version, let me introduce a couple of other notions. So, this will be useful to prove our later equivalences between several of the theorems that we are proving. So, what we are going to now do is to use Menger's theorem to prove one theorem and then use that theorem to prove Hall’s theorem. And to just show the equivalence of these results in some sense.

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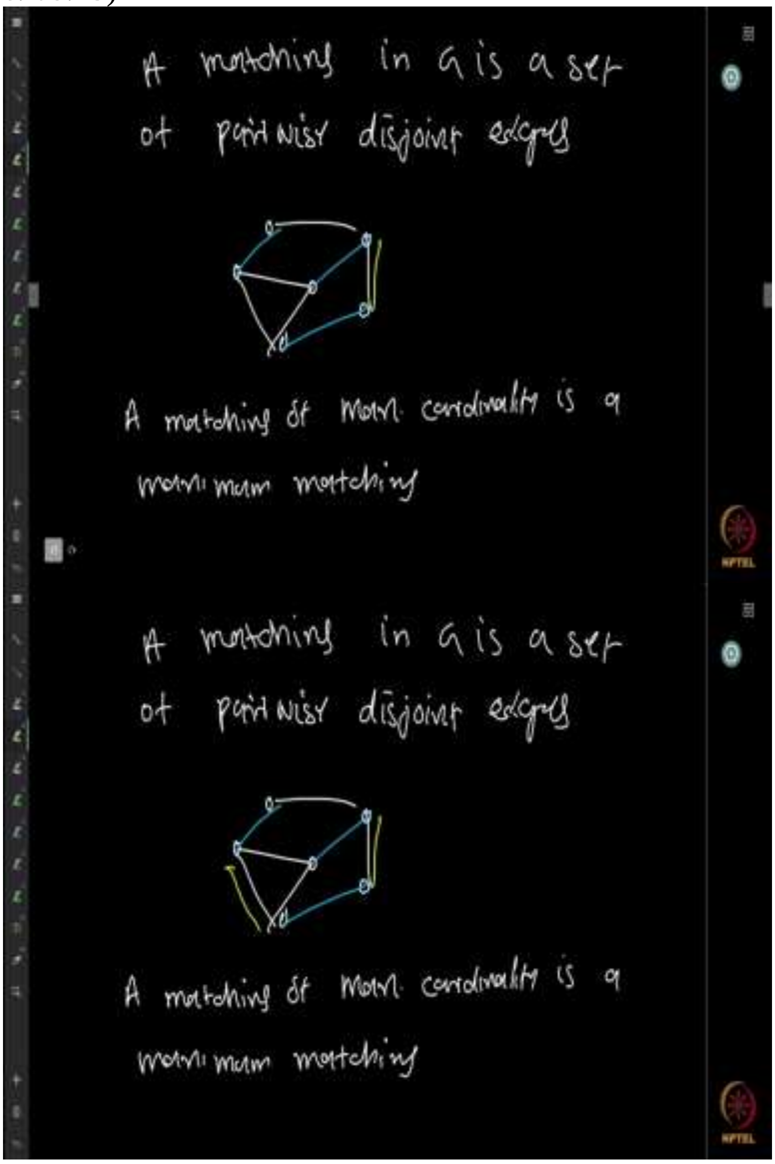

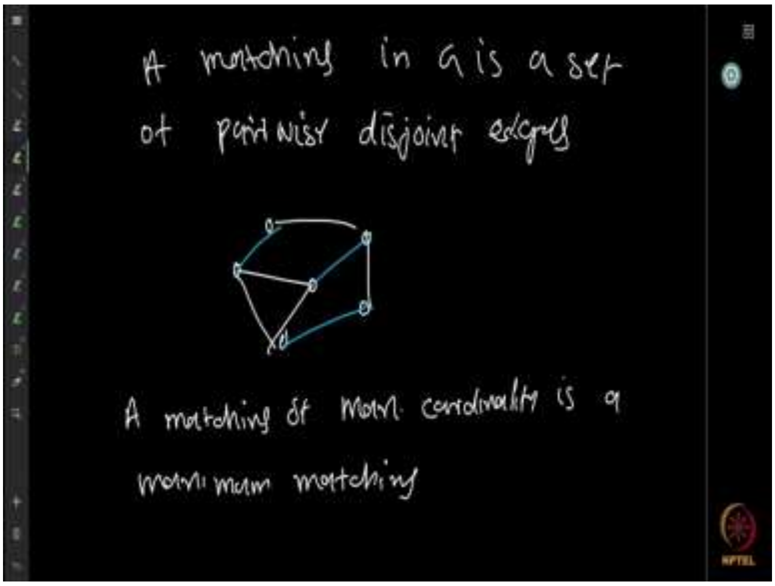

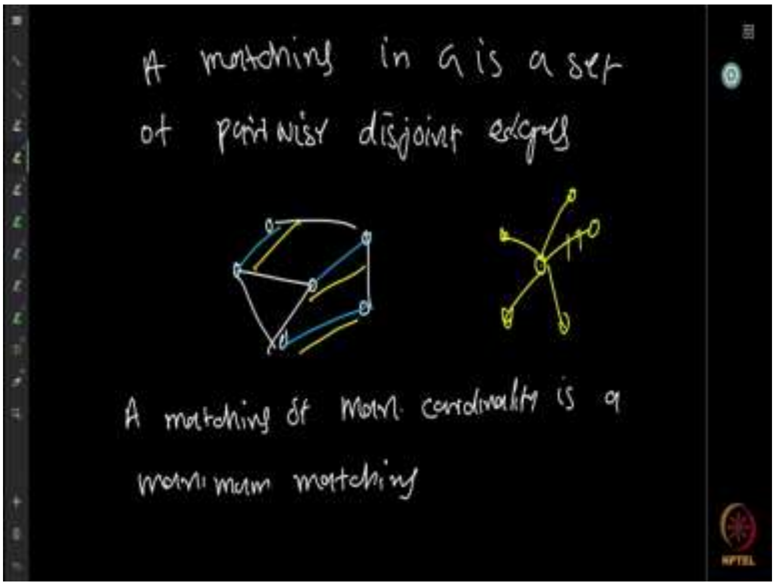

So, what we want is the following. Given a graph G, a matching is a set of pairwise disjoint edges. For example, this set of three edges, the blue edges form a matching in the graph. Because these edges do not have any intersection for any pair of them. So, they are pairwise disjoint and they are a subset of edges. Therefore, it is a matching in the graph.

Now, you can also talk about some other matching for example, this edge if I pick this edge and then of course, I cannot pick this edge because they have intersection. So, instead of picking that, I can pick something which does not intersect. For example, this itself. So, these yellow edges form a matching or I can pick for example, this and this, that is the matching. Or this and this is another matching.

So, we can find matchings like these. Or I can pick this, this and then this is another matching. So, a matching is just a set of pairwise disjoint edges. Now, a matching of maximum cardinality is called a maximum matching. One can prove the maximum matching in this graph is 3. Because every matching uses two edges. Of course, you cannot have more than $\lfloor \frac{n}{2} \rfloor$ edges in any matching.

That is immediately clear. The matching of maximum cardinality is a maximum matching in the graph. But of course, you can have graphs with much smaller number of matching. For example, if I take this graph, it has many vertices and edges. But the matching cannot have more than one edge, because if I pick any one edge, so let us say this edge then I cannot pick anything else. Because everything else is going to intersect with this vertex. So, this is the maximum cardinality of matching is one here.

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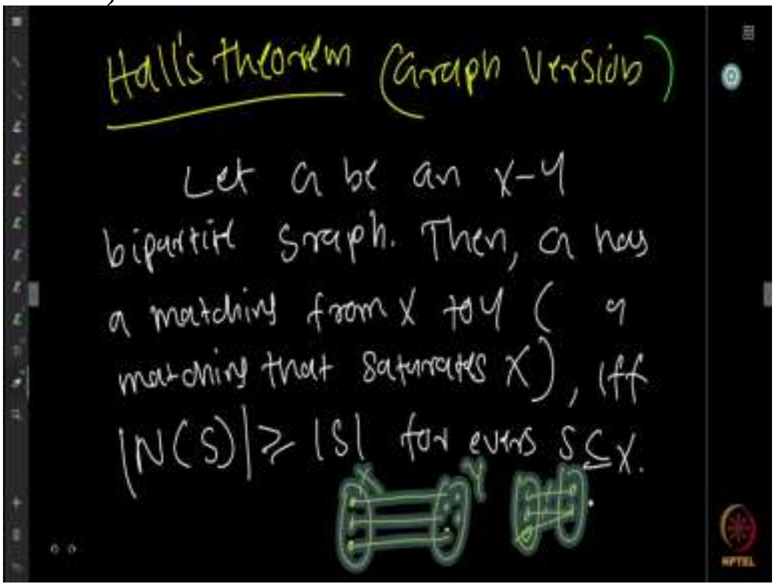

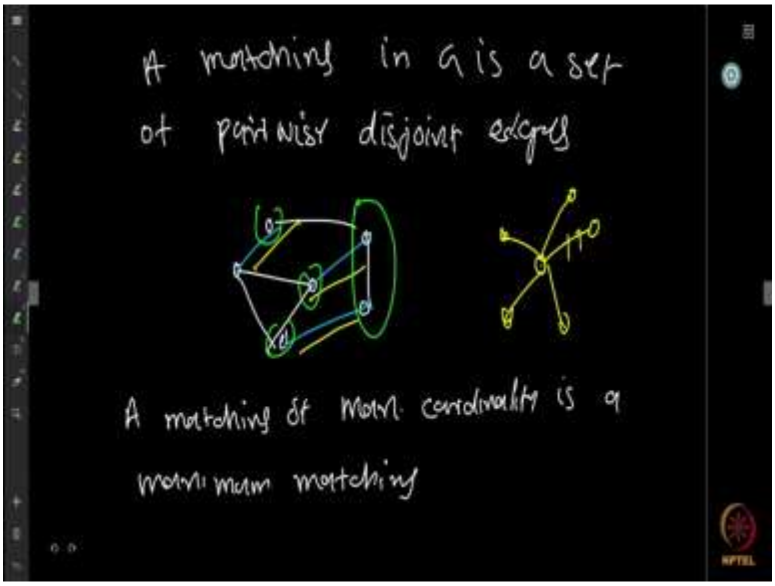

Now, the graph version of Hall's theorem is the following. So, let G be a bipartite graph, which we call an X-Y bipartite graph. Because one side of the set of vertices is called X. And the other independent that is called Y. So, consider an X-Y bipartite graph G. Then G has a matching from X to Y. So, matching from X to Y is that matchings that saturates X. What, do I mean by that?

A matching where all the vertices of X have some edge incident to it in the matching. So, some matching edge must be incident to every vertex. Then such a matching is called a matching that saturates, the set X. Of course, there could be some vertices in Y which are not mapped. So, such a matching is matching which saturates X. On the other hand, you can also have matching which saturates Y. For example like this and then this and this are matching edges which saturates Y but not X.

Because X has more vertices and therefore, I cannot saturate X. So, this is possible. A matching which saturates X, if and only for every subset S of X $|S| \leq |N(S)|$. That is the cardinality of S is less than or equal to the cardinality of the neighborhood of S. What is the neighborhood of S? Neighborhood of S is the set of all vertices, which are adjacent to the elements of the vertices in S.

For example, a neighborhood of this set is basically, what is the neighborhood of this set? Let us say neighborhood of the set is the set of vertices which are neighbors to this, like these guys. So, what we are saying is that, if for every subset S of X, we can show that the cardinality of S is less than or equal to cardinality of its neighborhood. There are more number of neighbors, at least as many neighbors as the elements of the set itself. Then you can find a matching which saturates X.

In other words if, if Y is the set of ladies and X is the set of men, and the edges are representing the preferences of the ladies. So, every lady say that I prefer these guy. So, I can define a bipartite graph as follows.

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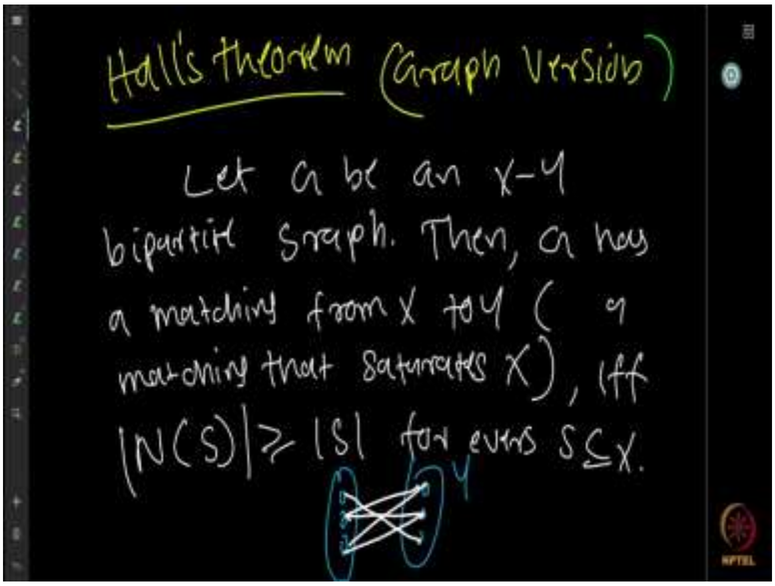

So, this is the set Y, with the ladies and then the men. And then each lady says that, I am willing to marry these two guy. And this lady says I am willing to marry this guy, this lady says that I am willing to marry this guy. And maybe also this guy and something like this is given. Then you are asked, can you find a matching which saturates X? Which means that can you find a matching for each of the men? Some lady who prefers him, if that is possible or not.

So, Hall's theorem says that, if for every subset, you can find at least that many ladies in the neighborhood, that many ladies preferring that subset of men, then we can find a matching. Now, again, to prove this we are going to use a slightly related concept. Of course, we can directly do this, we can prove this without using any other theorem.

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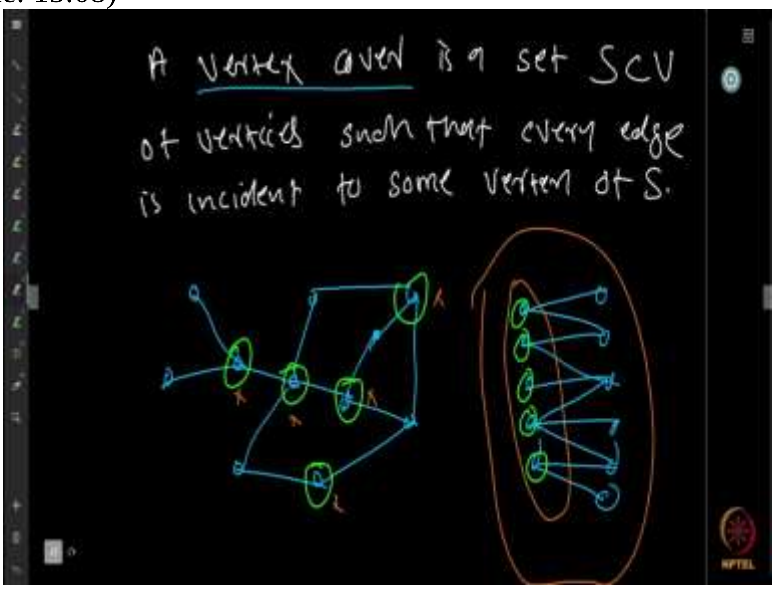

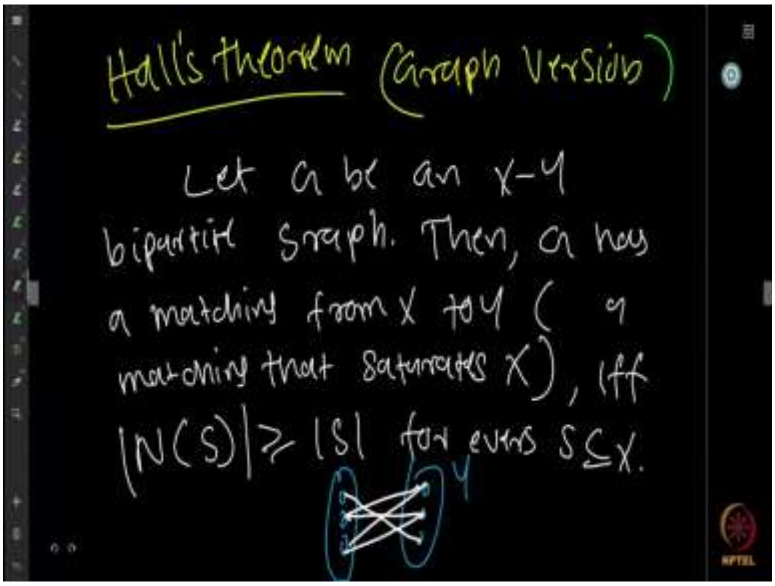

But we are going to prove it using an application of Menger's theorem that we just learned. So, we are going to define what is called a vertex cover. May be you should stop and think about the previous definition, because we have defined many things today, just think for a few minutes and then come back to this.

So, a vertex cover is a subset of vertices, S subset of V such that every edge in the graph is incident to some vertex of S. It is actually a vertex cover of the edges. So, vertex cover is basically you cover all the edges of the graph by the vertices. Which means that if we remove this subset of vertices then all the edges in the graph must be destroyed. So, you will get an empty graph, you will get vertices, maybe other vertices will be there, but there will not be any edge.

Every edge must be incident to one of these vertices. Such a subset is called a vertex cover. Here is a graph and its vertex cover. What is this? So, this is the graph G and, we have this cover. So, this vertex, this vertex, this vertex, this vertex and this vertex they cover all the edges because if I take any edge, look at the any of the blue edges, all the edges that are blue is incident to at least one of these vertices.

This is incident to both edges, that is okay, this is incident to both edges. But it is, every edge is incident to at least one of these. So, then it is such a subset of vertices is called a vertex cover. Here is another example there is a bipartite graph. And of course, one of the independent sets in the bipartition is going to give you a cover. Because every edge must be going from this side to that side.

So, we can easily find a cover in this case. But it need not be always possible to find the smallest cover easily like this, but you can always find a cover immediately like this, you just pick one such subset of vertices. Now, such subsets of vertices are called a vertex covers. Now, our question is that, can you relate between the covers of a bipartite graph and the matchings? So, can you find a relation between the coverings and matchings? So, can you think of some relation between the coverings and matchings. So, think about this.

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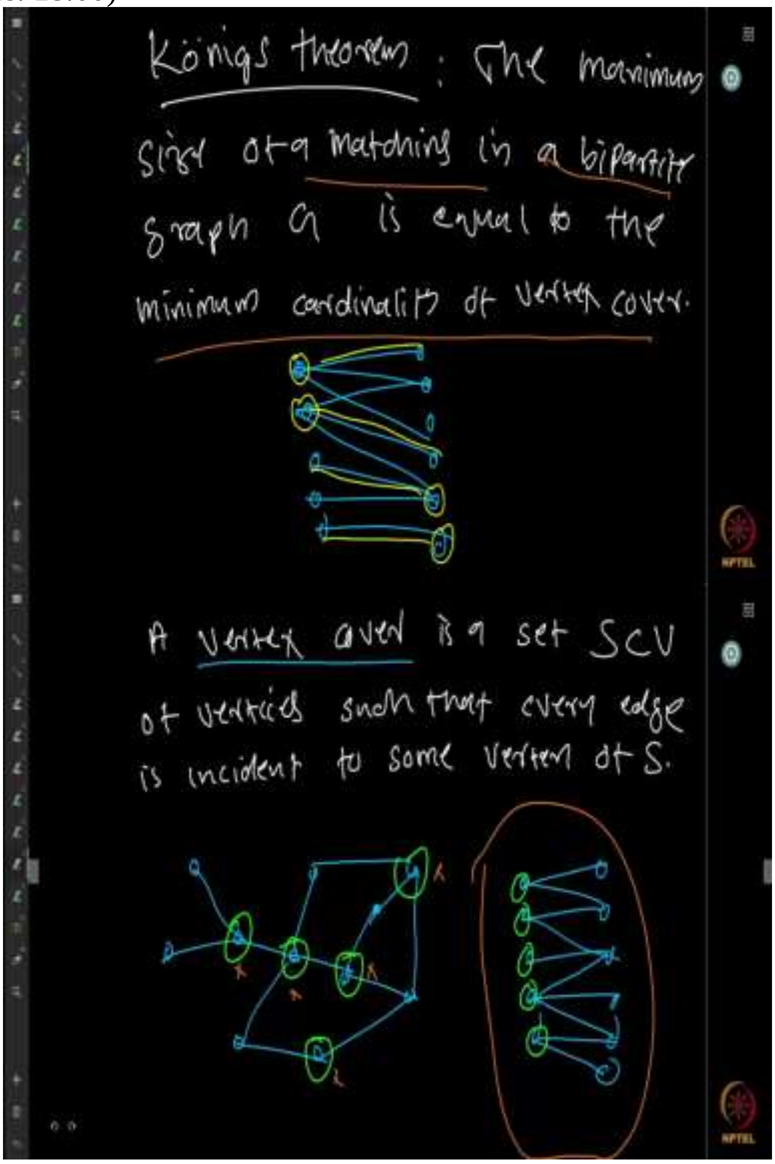

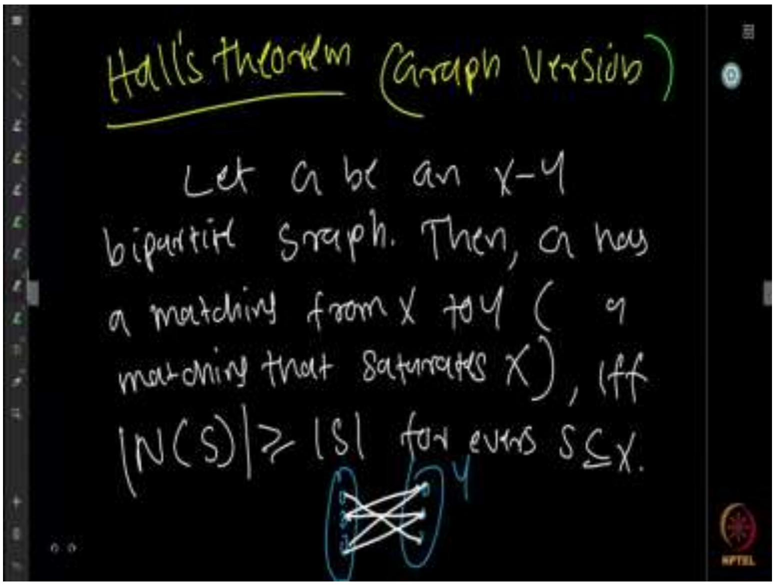

So, the theorem by mathematician called Konig says the following. The maximum size of a matching in a bipartite graph is equal to the minimum cardinality of a vertex cover. So, the maximum size of a matching in a bipartite graph is actually equal to the minimum cardinality of a vertex cover. So, the selection of vertices, which will cover all the edges in our bipartite graph will give you the cardinality of the largest matching that you can find.

Now, you can see how this relates with the cover and matching and how it relates to Hall's theorem. The maximum size of a matching in a bipartite graph is equal to the minimum cardinality vertex cover. For example, if you look at this bipartite graph, can you find a cover of the edges of the minimum cardinality and show that it is actually the largest matching that you can find out?

For example, I say that this one and this one, this one and this one, this defines a cover of all the edges. Because all the edges are incident to these four vertices. So therefore, you will not be able to find a matching of size larger than this. Now, similarly you can always find a matching of this size. That is why, it is an interesting result. For example, I can just pick this guy, then this one, this one and this one, so this is the matching of size four.

But you will not be able to find a larger matching. Now, one direction you can immediately see and we have to prove the other directions. So, Konig's theorem we are going to prove using Menger's theorem. So, how are you going to prove? Of course, you can prove Konig's theorem without using the Menger's theorem, but let us try to apply Menger's theorem to prove that the maximum size of matching is actually equal to the minimum cardinality of vertex cover. So, how do you connect the Menger's theorem to this? So, that is a question. You think about this for a minute, then we will prove it.

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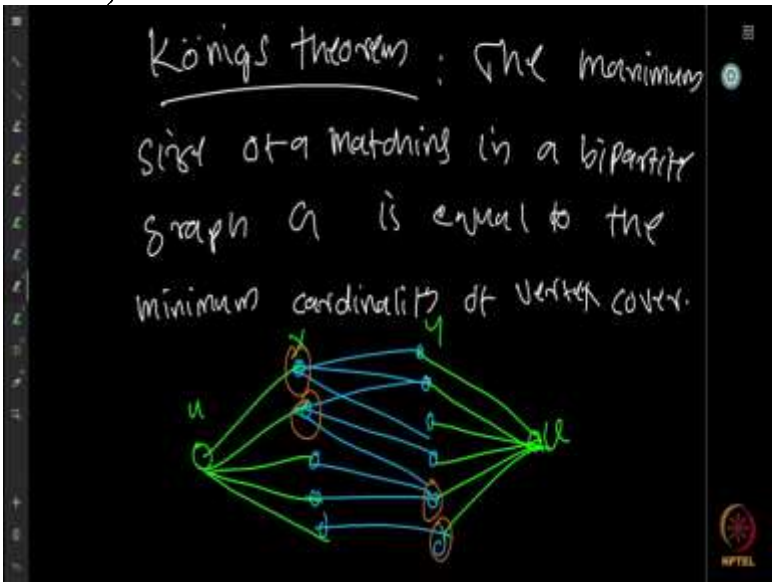

To prove Konig's theorem, I am going to introduce two new vertices. So, I am going to introduce a vertex here call it u, which is adjacent to all the vertices of this set, on this side which is called X. Then, I am going to make another vertex, which I call v. And this v is adjacent to all the vertices of Y.

I make a new graph now the bipartite graph by adding these new vertices u and v. Now, u and v are not adjacent in the graph. Then what the Menger's theorem says is that minimum cardinality separating set from u to v is precisely the number of vertex disjoint u to v paths, in this graph. Now, suppose you can find a separating set from u to v, suppose you find a separating set, u-v separating set.

All the path from u to v must be destroyed by deleting these vertices. Suppose you find such a separating set, now this u-v separating set must delete all the edges, all the edges of this blue graph. Because if it does not delete all the edges of the blue graph, there must be some blue edge? Which means that I should be able to go from u to that vertex using the blue edge go to some vertex of y and then from that vertex y directly to v. That path must exist if the separating sets does not cover all this. Therefore, any minimum u-v separating set must necessarily be a cover of the bipartite graph.

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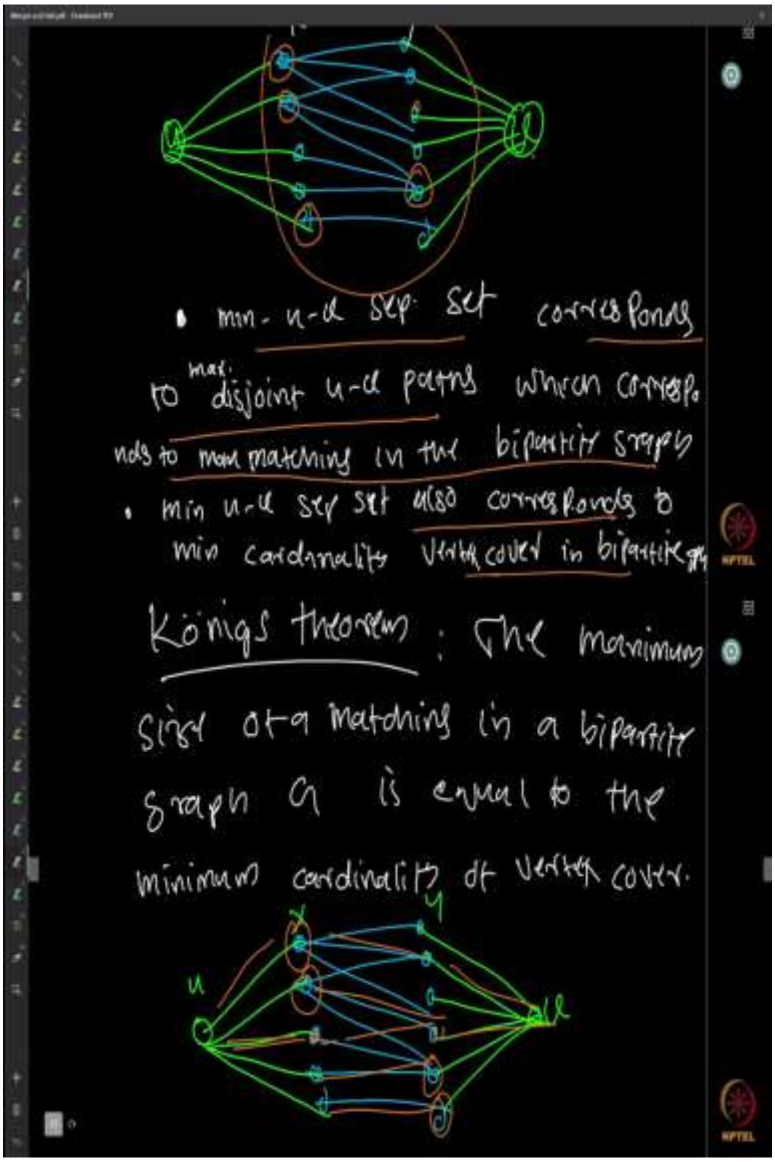

So, a minimum u-v is separating set correspond to a maximum number of disjoint u-v paths, by Menger’s theorem. But this correspond to maximum matching in the bipartite graph. Because, maximum number of disjoint uv paths must use necessarily distinct edges from this the bipartite graph, the blue edges any, any such path from u to, u to v must use edges of the bipartite graph. And they must all be a disjoint, they must form a matching because there must be all, they are all internally disjoint.

So, therefore, all such edges will give a matching in the bipartite graph. This paths will define on matching, that is the maximum cardinality matching. Because, if there is one more edge which is, there is a matching edge, which was not covered that will give a new path. Because this vertex and this vertex, these edges are present because we did not use any of these vertices. So, once I selected the separating set, that must give me exactly the, the cardinality or the maximum matching in the bipartite graph. Therefore, a minimum u-v separating set correspond to maximum number of disjoint u-v path which correspond to maximum matching in the bipartite graph. But the minimum separating set also correspond to minimum cardinality vertex cover. Because, if all the edges are not covered, there will be some remaining edges, blue edges, that blue edge is going to create one more path.

So, that edge is not covered, means that, that is going to give one more path which means that we did not select all the separating vertices. So, therefore, therefore, this correspond to vertex cover in the bipartite graph. So, therefore, we get a connection between the vertex cover and the size of the maximum matching. So, the minimum vertex cover is actually equal to the maximum bipartite matching.

Because any separating set must use vertices of X and Y only, the blue part. Because u and v cannot be deleted in the separating set. So, using Menger's theorem we just proved the Konig's theorem. Now, Konig's theorem gives a connection between the minimum vertex covers and the maximum matchings. Now, using the Konig's theorem we will prove Hall's theorem.

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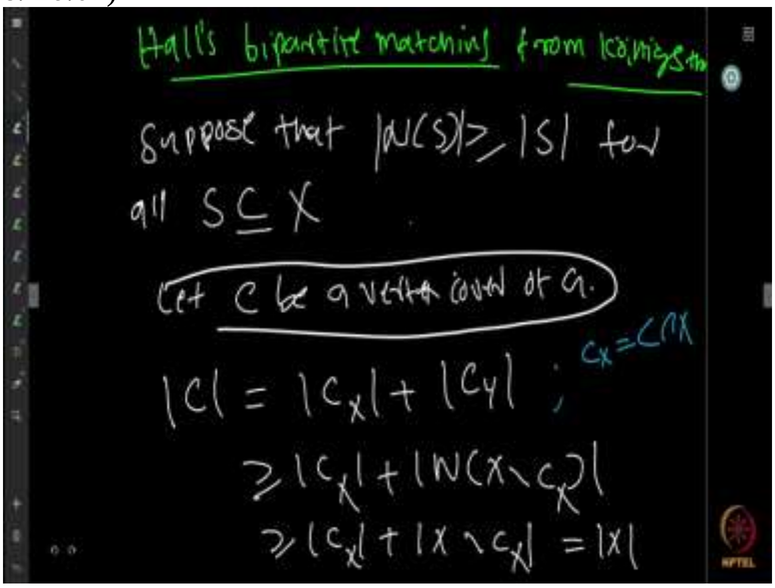

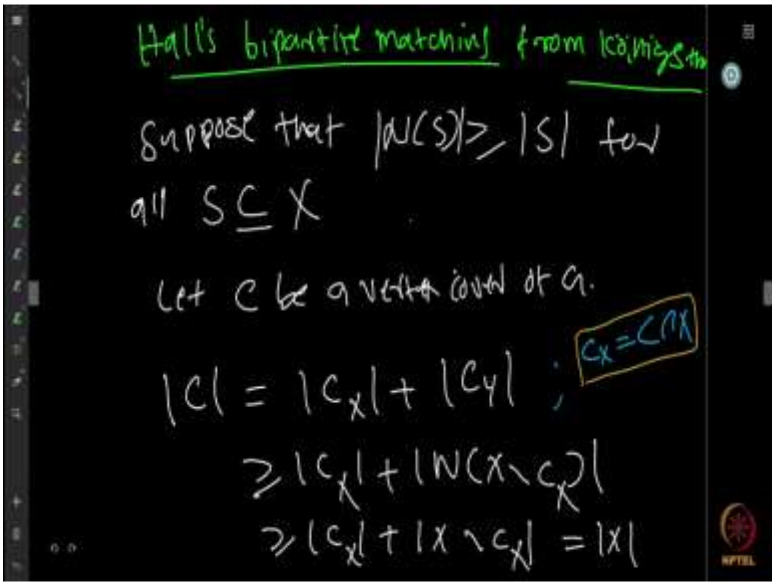

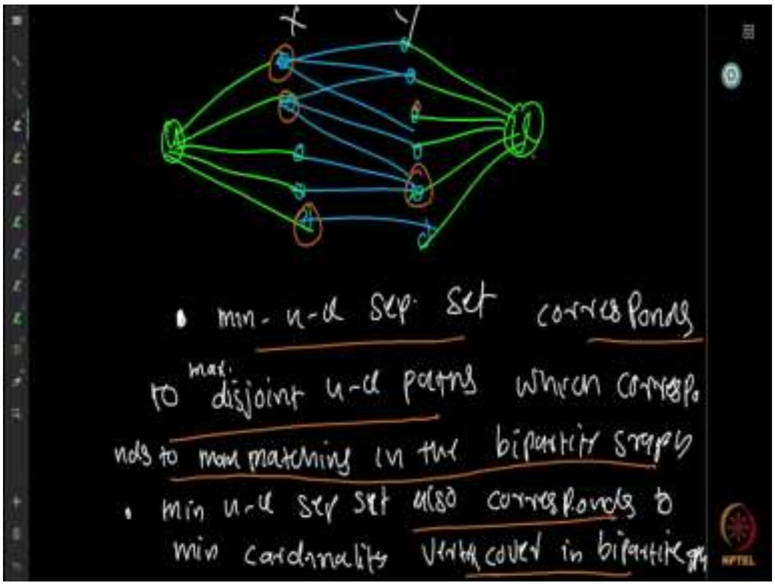

Hall's theorem is the statement that if the Hall's matching condition is true then, there is such a matching which saturates one of the sides. So, we are going to prove that. We are going to now prove Hall's theorem by using Konigs' theorem. So, how are we going to do that? Suppose that, neighborhood of a set S has at least as many elements as in S, for every subset of X.

So, XY-bipartite graph we are considering. And we are saying that for this XY- bipartite graph every subset of X has at least that many neighbors in the Y. Now, let us start with a vertex cover of the graph G, we call it C. Now, let me define C_X as C ∩ X. The vertices in the cover which appear on the side X. And there are some vertices which appear on the side Y also. So, C_Y is the set, C ∩ Y.

Now, the cardinality of the cover C is basically the cardinality of the vertices in X plus the cardinality vertices in Y. The cover vertices in X plus the cover vertices in Y. Now, this is at least the cover vertices in X plus the cardinality of the neighbors of X\ C_X.

Because, if you look at X\ C_X, it is a subset of vertices in X. Now, this X, these guys has let us say that if we have some neighbors in Y. So, X\ C_X has some neighbors in Y. Now, because the edges going from this X\ C_X to Y, all these neighbours must be covered by some vertices.

Which means that those vertices which go from these neighbors must be picked up by the vertices in C_Y. Because vertices in C_X cannot pick up the edges which are incident to this element. Because the vertices in C_X are only the edges from which are only going to Y. So, the edges which are starting from this vertex, will not be picked up by this. Therefore, C_Y must be at least all the elements in the neighborhood of X\ C_X, they must be covered by C_Y.

But by Hall's condition, neighborhood of X\$C_X$ the cardinality is greater than or equal to the cardinality of X\$C_X$. Because every subset of X\$C_X$ is a subset of X, its neighborhood has strictly larger cardinality. So, it has at least that many cardinality. So, this is greater than or equal to cardinality of the C_X plus cardinality of X\$C_X$.

But what is the cardinality of C_X plus cardinality of X\C_X? That is cardinality of X itself.

$$|C| = |C_X| + |C_Y|$$
$$\geq |C_X| + |N(X\backslash C_X)|$$
$$\geq |C_X| + |X\backslash C_X| = |X|.$$

So, what we proved is in fact cardinality of C is greater than or equal to cardinality of X. Which means that, the covering has at least these many vertices in the graph. Now, from this Hall's theorem follows, because, we started with any arbitrary cover. So, in particular it could have been a minimum cover.

So, if I started with a minimum vertex cover, minimum vertex cover has at least these many vertices. But now what we proved earlier that by Konig's theorem, the cardinality of minimum vertex cover is actually the cardinality of the maximum matching. Now, in the bipartite graph, if the maximum matching has these many edges, they must use all the vertices of X. Because every edge must go from X to Y and every matching is an independent edge.

So, it must use all the vertices of X. So, that proves the graph version of Hall's theorem that there is a bipartite matching, which saturated the side X if for every subset of X its neighborhood has at least as many elements as in the subset. So, we applied Menger's theorem to prove Konig's theorem for the matching in bipartite graphs, which relates the vertex cover and matchings. And then we use the Konig's theorem to prove Hall's theorem.

Because Hall's theorem's condition gives a direct way to look at the vertex cover. And that translates to bipartite matching in the Konig's theorem. So, therefore, we get a bipartite matching theorem of Hall's using our Menger's theorem. Of course, you can take it as an exercise to prove Hall's theorem without using either of the theorems or using directly Menger's theorem, all this can be done. But we just give a proof using Menger's theorem to Konig's theorem and Konig's theorem to Hall's theorem. Now, maybe you can use Hall's theorem to prove Konigs theorem or Menger's theorem. These are, these can be good exercises to work out.

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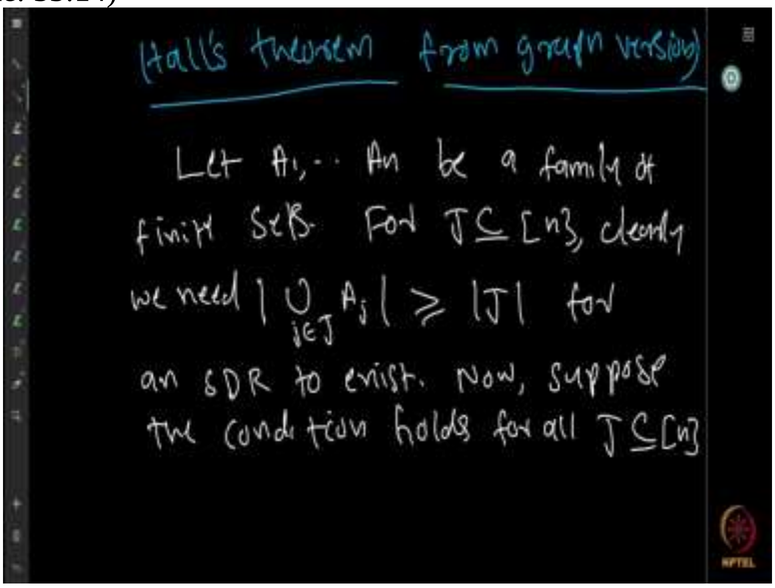

Now, using the bipartite version or the graph version of Hall's theorem, we are going to prove the set theory version of Hall's theorem. So, the general Hall's theorem from the graph version. How do you prove this? So, let $A_1,\dots, A_n$ be a family of finite sets. This is our theorem statement, for every sub collection of these families, the cardinality of the union is at least the number of sets involved in the union. So, for every J ⊂ [n], $|\cup_{j\in J} A_j| \geq |J|$.

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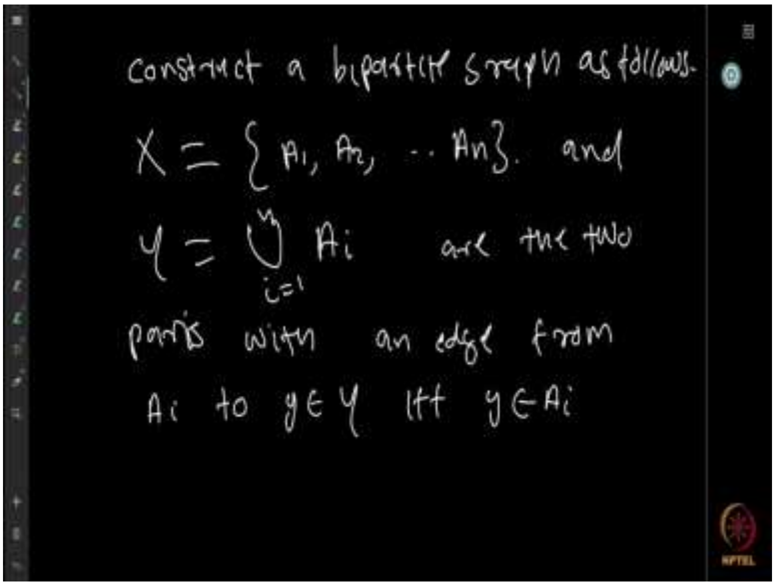

Now, suppose this is true for every such subset, then we can of course construct a bipartite graph from the given family always. Given a family $A_1, \ldots, A_n$ we form a bipartite graph by taking the elements of A_i containing the union. And this union has the set Y and then each of the sets has the elements of this set X.

So, one side you just write down the sets in the family $A_1, \ldots, A_n$ as the vertices. And on the other side, you take the universe from which all these elements are drawn, the union over all A_i as the elements of Y. So, these are the vertices. Now, two parts X and Y, there is an edge from A_i to an element y ∈ Y, if and only if y is an element of A_i.

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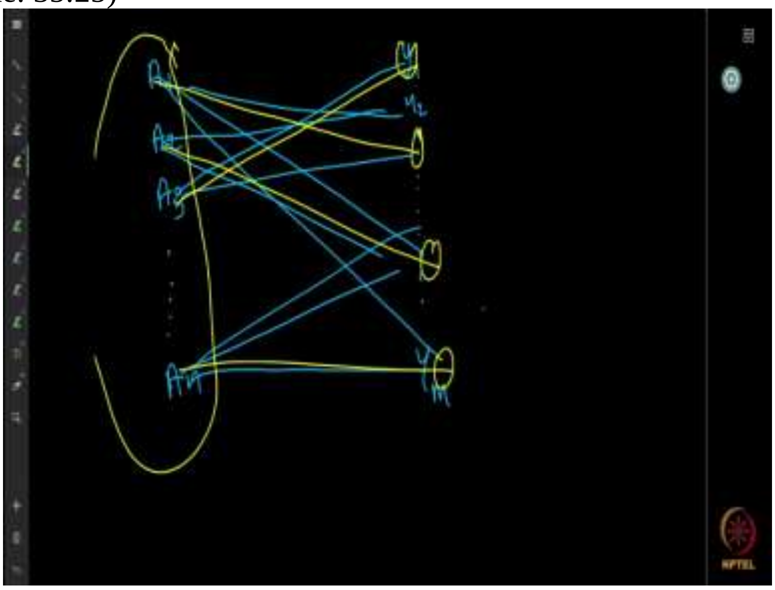

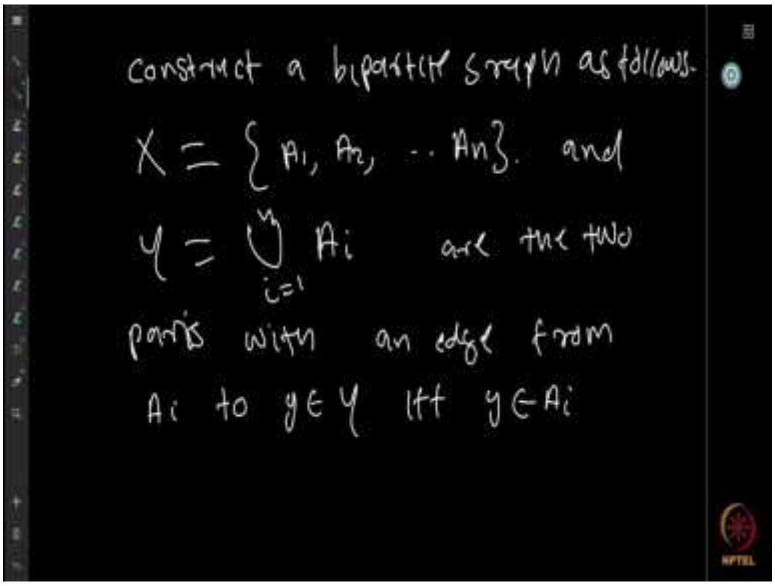

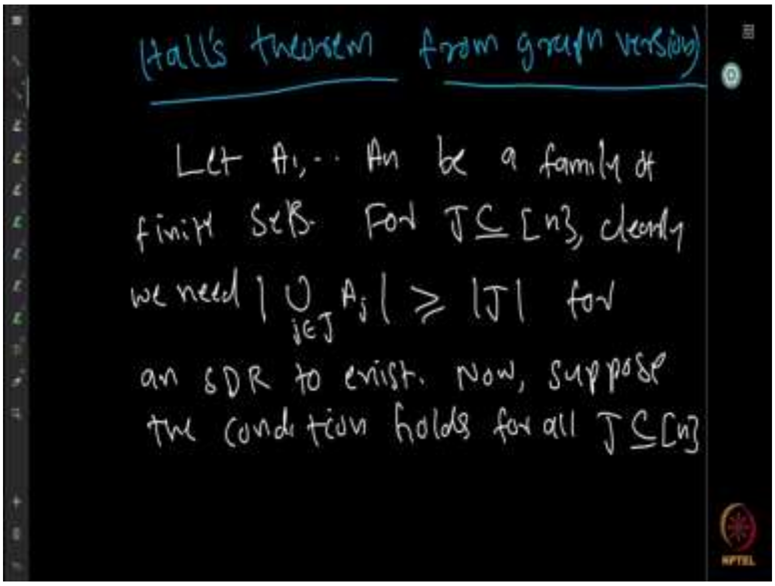

So, this is basically kind of an incidence graph, the bipartite graph which says that A_3 is basically connected to all its elements in the universe, A_4 is connected to all its elements, A_n is connected to all its elements. So, this is the incidence graph which is the bipartite graph. Now, what is the system of distinct representatives? Well, a system of distinct representatives is precisely a choice of elements here, such that each element belongs to a corresponding set.

So, what is that translate to? That translate to basically a matching, in the bipartite graph, because each element must have a corresponding I mean, each set must have a corresponding element which it belongs to and that will only happen if there is an edge in this graph, which can be picked as part of a matching edge. And therefore, if there is a matching which saturates $A_1, \ldots, A_n$ (which saturates the side X), then of course, we know that you have a system of distinct representatives.

Because this matching will give you precisely the representatives as the elements match in the set side Y. So, all you have to just do is to see whether the matching condition holds. But the matching condition is precisely the neighborhood of any subset must have at least that many here. But that is precisely what we are saying here, what is the neighborhood of a set?

It is basically the union of all the elements in that sets. But the condition is true according to our assumption and therefore the matching exists. So, that is the matching which saturates the side X or in other words there is a matching which saturates A_1 *to* A_n which says that there is a system of distinct representatives. And that is true if and only if this condition is true. So therefore, we get the general Hall's theorem, or system of distinct representatives from the bipartite version of Hall's theorem.

So, we proved several theorems today, Menger's theorem, then Konig's theorem, and then Hall's theorem, both the graph version and the set theory version. Now, we just finish this lecture with a few observations. And what I want to point out is that these three theorems of course, we saw that they are kind of equivalent, one can be obtained from the other, I did not proved all the directions, but you can take it as a homework.

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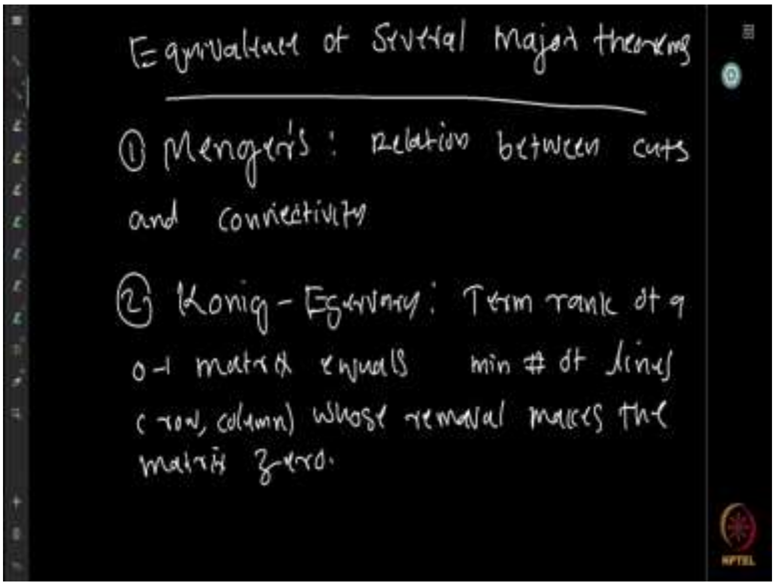

But they are also equivalent to many other theorems. I am going to show the equivalent of seven major theorems in combinatorics, which are all equivalent to Menger's theorem for example. So, one of them is Menger's theorem. Menger's theorem gives a relation between the cuts and connectivity, the separating sets and the number of disjoint paths.

Then another theorem, which we did not prove is Konig Egervary theorem, which says that the term rank of a 0-1 matrix is equal to the minimum number of lines whose removal makes the matrix a zero matrix. So, what is the term rank? Term rank is the, the number of non-zero entries that you need to pick up, in the 0-1 matrix number of 1's that you need to pick up such that no two of them are in the same, same row or column.

So, the such 1's how many maximum number of such 1's that you can pick up, that is the term rank. Now, the minimum number of lines, so line is basically a row or column. So, in a matrix, any row or a column is called a line. Now, the minimum number of lines that you need to remove to make the matrix, remaining matrix is just as zero matrix, that is the minimum number of such lines. And that number of such lines is called the term, is also equal to the term rank of the 0-1 matrix, that is the Konig's Egervary theorem. This has a graph version, you can think about what could be the graph version.

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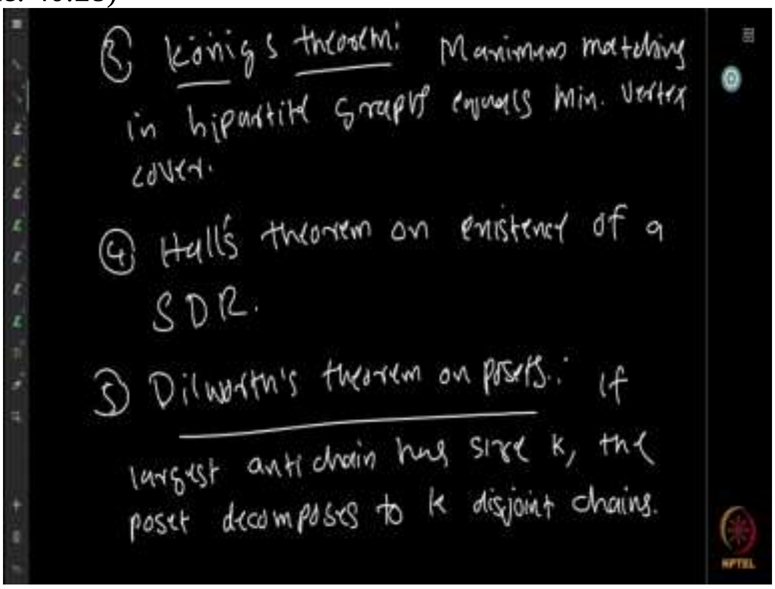

Then our Konig's theorem that the maximum matching in bipartite graph equals the minimum vertex cover. So, this is Konig's theorem. Then the Hall's theorem, which we just proved on the existence of a system of distinct representatives. Then you have Dilworth's theorem on partially ordered sets. What does this theorem say? It says that, if the largest anti chain in a poset has size k, then the poset can be decomposed into k disjoint chains. So, if you remember chains are the totally ordered subsets. So, you can basically partition the poset into k disjoint chains, if k is the largest number of antichain. So, that is a Dilworth's theorem.

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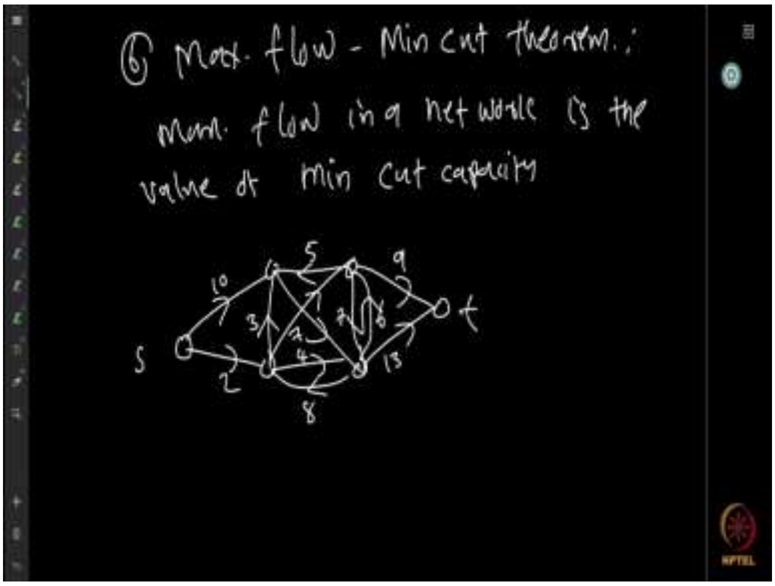

Then there is max flow-min cut theorem. So, max flow-min cut theorem is a theorem in the flow theory. Flow theory is basically a study about the transportation networks. Like you have directed graphs where you have capacities for each of the paths. Each path can take at most some amount of information or load. So, it could be like network, data network or it could be transportation network, communication network etc.

Where you have specific directions and, bandwidths which are the capacities. Then of course, the flow is basically the movement of information or data or objects which cannot be more than the capacity. So, then you can define this network as a directed graph with the associated weights of the edges, which are the capacities. And then the max flow-min cut theorem says that if you have a source where everything is starting, it is basically a factory which produces things like ‘s’.

And there is a terminal which is the outlet maybe. So, you want to transport things from the source to the terminal. And the numbers given here are the capacities of the lines, the transportation. There is a bridge here which cannot carry more than two tons, there is a bridge here which cannot carry more than ten tons. This will not permit trucks of size larger than nine to go through this, things like that.

So, these are the capacities and then the flow is basically like, how much of the load can be transported at the same time, something like that. I am not being precise here. One can make it precise. And then you can talk about the maximum amount of flow from s to t, that can happen. And this can be equal to I mean, this will be equal to the minimum cut which separates s to t and the sum of the capacities.

So, the sum of the capacities in a minimum cut is basically equal to the maximum flow that can happen. So, flow has some additional things like for example, whatever enters a node must also leave the node except for the starting and ending node. So, it is basically whatever going in, so, we are going to associate some numbers here to define the flow. Some value is going from here, then that must also go out of this.

For example, now, if, this guy only allows to go 7 from here to here, then I cannot send 10 here. Because if, 10 comes here the 10 must go also out. So, the incoming and outgoing must be balanced in a flow. So, whatever all these conditions there, we can show that the maximum flow is precisely equal to the minimum cut of the capacities. So, this is also identical to know these theorems that we have discussed before Menger's Theorem etc.

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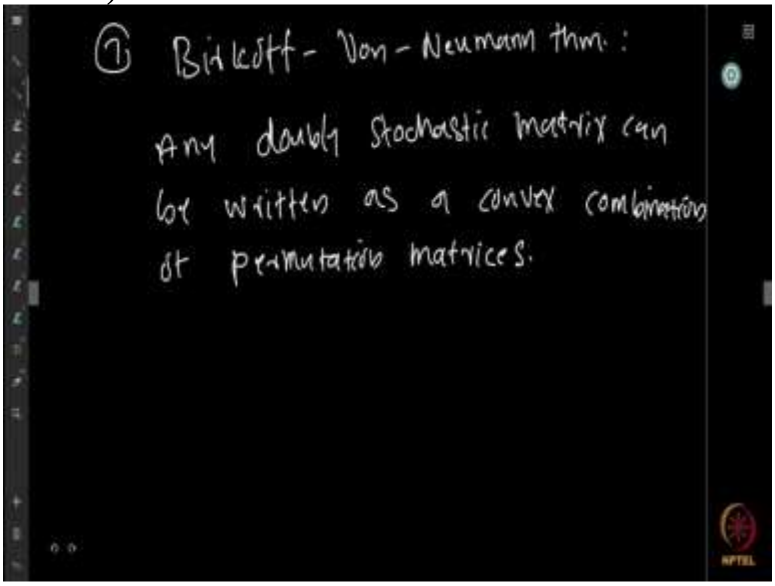

And then there is also another interesting theorem, which is the Birkoff-Von-Neumann theorem, which says that any doubly stochastic matrix can be written as a convex combination of permutation matrices. So, doubly stochastic mean that it has non-negative real values, and the sum of the rows and columns are all 1.

This matrix can be written as a combination of permutation matrices. All these theorems are kind of equivalent. So, this is what we wanted to say. So, I will end the lecture and we will stop the graph theory lectures for the time being. And we will look at other topics like probabilistic method and maybe species of combinatorial structures in the coming classes. So, we will see you in the next class.