Combinatorics
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Lecture 04
An infinite flock of Pigeon

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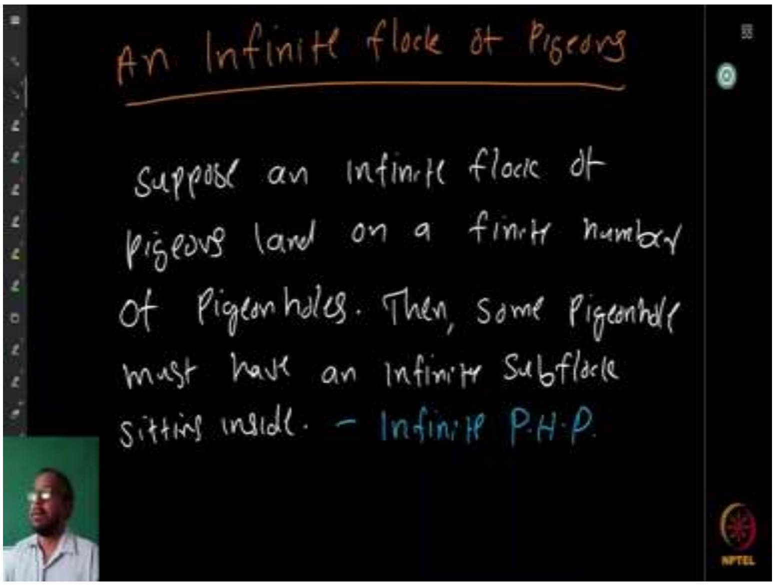

We have an infinitely many pigeons coming in. So, we have infinite flock of pigeons and they have to sit in a finite number of cages, finite number of pigeon holes. So, suppose an infinite flock of pigeons land on a finite number of pigeon holes, then some pigeonhole must have infinite sub flocks sitting inside. I mean, if every one of the finite cages has finitely many pigeons, then we know that there is only finitely many pigeons sitting everywhere together.

So, therefore, one of them must have infinite, this is also an obvious one. And this is called infinite pigeonhole principle. Now, this can be used to do some very amazing results. For example, you can use combinatorics to prove for example, theorems from analysis. So, let us look at some examples.

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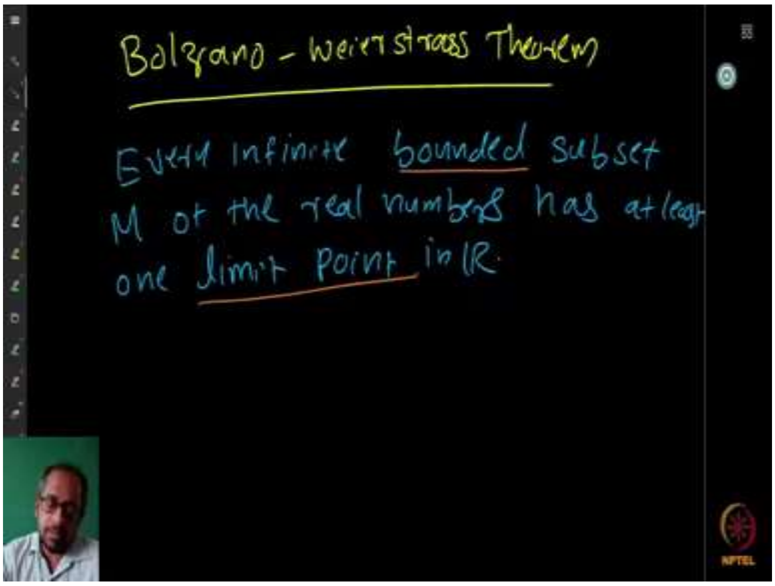

So, if some of these concepts are a little difficult for non-math students, they can really just browse through it and without looking its much details, but it will be still interesting and instructive to see how this kind of method can be applied, but I think the first part apparently, it was full for everybody, and then I will end up with one question, which I will not prove here and I will ask any math student who may be looking at this course to go through it and try to prove it themselves.

So we start with a very famous theorem called Bolzano–Weierstrass theorem, so what does the Bolzano–Weierstrass theorem say? It says that every infinite bounded subset M of the real numbers ℝ has at least one limit point in ℝ. Now, maybe you are not math you will not know what is bounded and what is limit points, they are very simple, I am going to explain.

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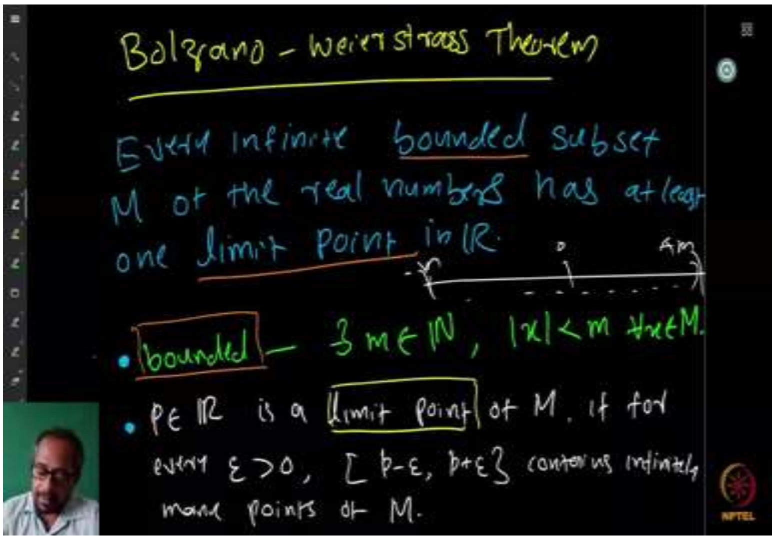

So, a set is bounded if you can find some positive integer m, such that the absolute value of any element of the set M, that we are looking at is strictly less than this number m. That is, there is some natural number small m such that the $|x| < m$, for every element $x \in M$.

So, in some sense it is saying that, you have this real line, then there is some numbers $+m$ and $-m$. So, the set M that we are considering is going to be sitting inside this interval $[-m, m]$.

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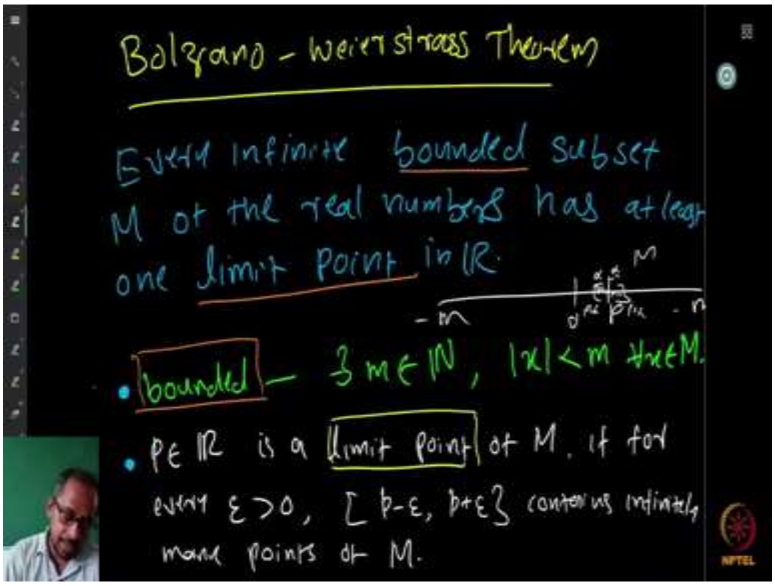

So, that is what is boundedness, that is always inside this $[-m, m]$, so the absolute value is strictly less than small m. Now, $p \in \mathbb{R}$ is a limit point of the set M, if for every $\epsilon > 0$, the interval $[p - \epsilon, p + \epsilon]$ contains infinitely many points of M.

The number of points of M that belongs to this interval $[p - \epsilon, p + \epsilon]$ must be infinite. In that case we say p is a limit point. Now ϵ can be made as small as you want it. So, if I give you $\epsilon = \left(\frac{1}{10}\right)^{100000}$ then you should still be able to find infinitely many points of M inside.

So, what the Bolzano–Weierstrass theorem says is that, if the infinite set is bounded then it has at least one limit point in $\mathbb{R}$.

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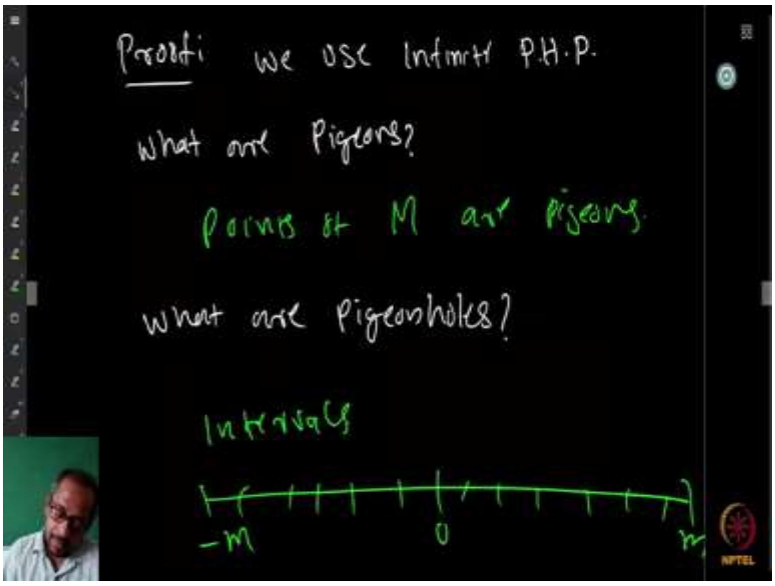
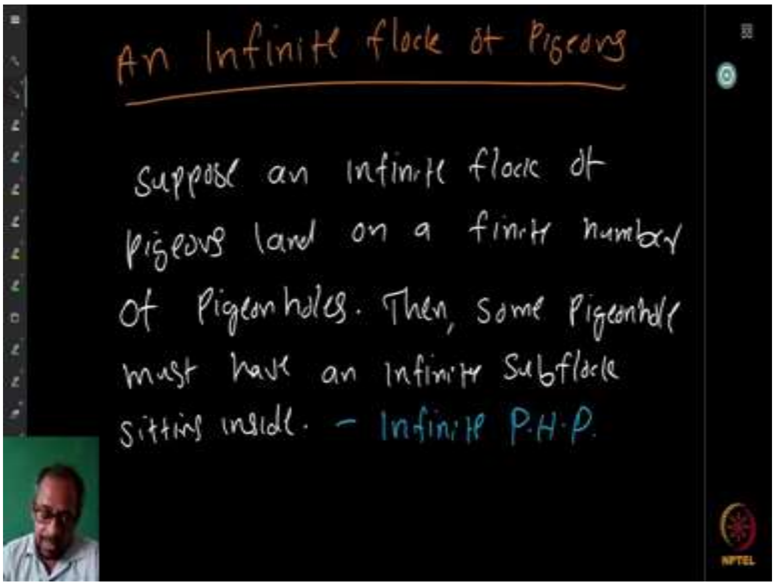
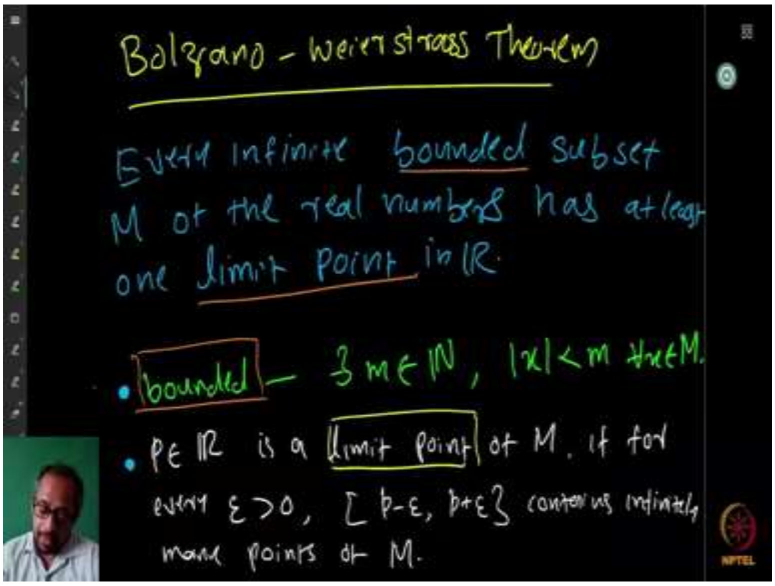

Proof: We use Infinite P.H.P.
What are Pigeons?
Points of M are pigeons.
What are pigeonholes?
Intervals
−M
0
M


An Infinite flock of Pigeons
Suppose an infinite flock of pigeons land on a finite number of Pigeonholes. Then, some Pigeonhole must have an infinite Subflock sitting inside. – Infinite P.H.P.


Bolzano – Weierstrass Theorem
Every infinite bounded subset M of the real numbers has at least one limit point in ℝ.
• bounded – ∃ m ∈ ℕ, |x| < m ∀x∈M.
• p ∈ ℝ is a limit point of M, if for every ε > 0, [p−ε, p+ε] contains infinitely many points of M.

So, the proof of Bolzano–Weierstrass theorem by using infinite Pigeonholes principle. So, what are the pigeons ? Well points of M are the pigeons that is easy to imagine, because we know that for infinite pigeonhole principle, we need infinitely many pigeons and then we need to find some pigeonholes of course, which are finitely many in number. But since we already have this infinite set M , points of M are the pigeons.

Now what are the pigeon holes. So, we are going to define the pigeonholes as the intervals because what we really wanted to show that, some interval contains infinitely many points, so basically the pigeonhole principle, when we apply is going to the us some interval contains infinitely many points, and that is what we precisely want, for any epsilon we need this, but this is what we want.

So, therefore, we look at the interval. So, we the use the fact that the set M is bounded, so therefore we can find this $-m$ and $+m$ such that the values of the points of M are strictly between the intervals $[-m, m]$. Every element are now going to be present inside this interval.

Now, in $[-m, m]$ there are finitely many intervals, $[-m, -m+1], [-m+1, -m+2], \ldots, [-1,0], [0,1], \ldots, [m-1, m]$. So there is at most $2m$ intervals here. When you take the $2m$ intervals, then we know that, because they are finite, by pigeonhole principle there must be some interval which contains infinitely many points of M.

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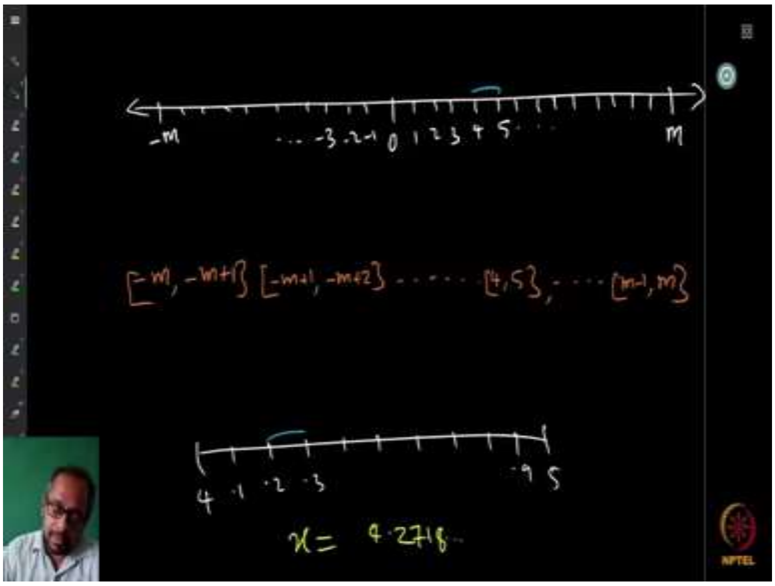

Now let us consider some interval let us say [4,5] contains infinitely many. There could be several intervals contains infinitely many. If there are more than one you pick one of them arbitrarily. So, we found out [4, 5] contains in infinitely many let us say. So, now what I am going to say is that this number is going to be the limit point that I am going to define is going to be 4 point something.

Then what I do is that, because [4, 5] contains infinitely many, I take the interval [4, 5] and then divide it into 10 subintervals like 4.1, 4.2, etc., 4.9 and 5. So take this 10 subintervals, again, since there are infinitely many in this interval, we know that there are infinitely many in [4,5] and there are only 10 finitely many pigeonholes, so therefore some interval must contain again infinitely many points, whichever interval maybe more than one, you would select one of them at random, then any of the intervals [4,4.1], [4.1,4.2], …, [4.9,5], definitely contains infinitely many points.

So, I will say that now the limit point is going to start with $x = 4.2$. Then take [4.2,4.3] and again subdivide, I will get something maybe the 7, 8 interval so I will select 7. Then I will take the 7, 8 interval and say that okay 1, 2 we will have this. So, I will take the next digit as 1, then in 1, 2 will contain 1.89 may contain infinitely many, so I will take this, so this way I keep on doing. So x can be 4.2718…

I can apply it as many times as I want depending on the accuracy whatever epsilon you give I do it as many times and then I will show that I can continue to this. So, this defines a real number, maybe it does not stop, but still it defines a real number and by the decimal expansion and then the number is a limit point.

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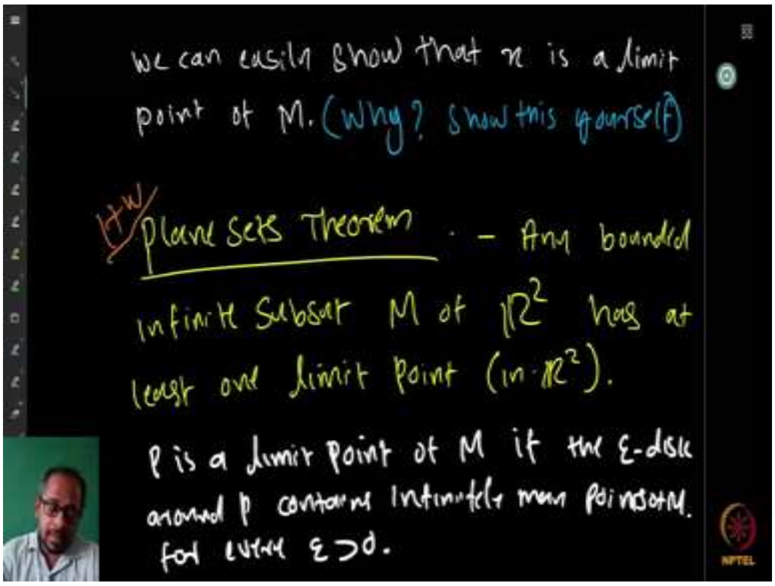

So, we can easily show that x is a limit point of M. So, I want you to think about why and show it to yourself, why precisely you can say that, the x that we have defined now is a limit point of M. So, this is the Bolzano–Weierstrass theorem.

Now as a homework I want you to do the following theorem, this is the generalization of Bolzano–Weierstrass theorem. The above was Bolzano–Weierstrass on the real line, now let us take the real plane, you have x axis and y axis. So, any bounded infinite subset M of $\mathbb{R}^2$ has at least one limit point in $\mathbb{R}^2$. So you can find some, some point in $\mathbb{R}^2$ with this property. Now, what is the limit point in $\mathbb{R}^2$, earlier we said that the interval $[p-\epsilon, p+\epsilon]$, now here you cannot say that. Here you take the point p and then you look at look a disc around it with radius epsilon.

So, I takean ϵ-disc around p and then say that this contains infinitely many points of M for every $\epsilon > 0$. You make it even smaller, it does not matter, you will still find infinitely many. But this way you can define what is called a Plane Sets theorem, this is again the generalization of Bolzano–Weierstrass to dimension 2. So, this is a home work for you.

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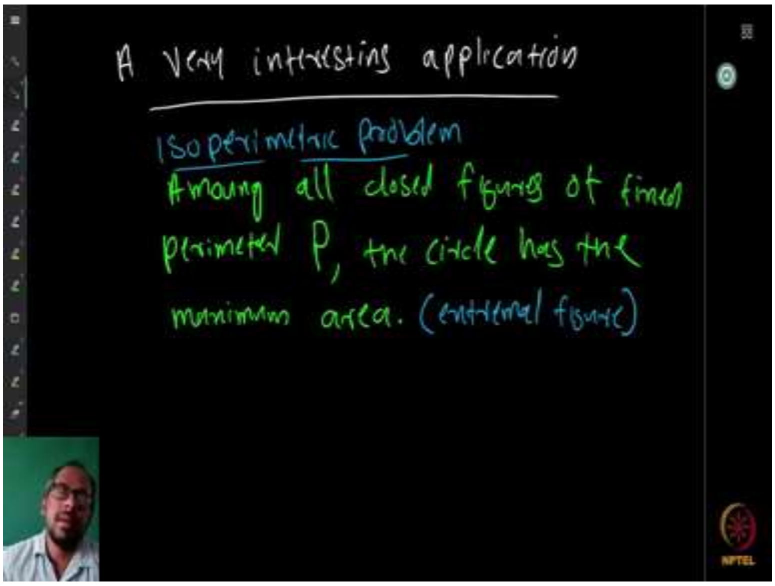

Now, I want to finish up this topic on pigeonhole principle with a very interesting application. This is a very beautiful application of some results that all of us have known from the school time itself, but most of us have never seen the proof of it. And this is well-known result. If you take all closed curves like, triangles or squares or hexagons, polygons or all kinds of shapes with the fixed perimeter, the length of the boundary is the same, then out of which some particular figure has the largest area and we know that from that school time, this is a circle, a circle maximizes the area and this is called the isoperimetric problem.

So, isoperimetric problem says that among all closed figures of fixed perimeter p, the circle has a maximum area or we call circle is a extremal figure. Now, we all know this, but how do we prove this? This result have been known for from 1000s of years, in fact, Greek people used to know this theorem, isoperimetric problem. And then, everybody in this like last 2000 years have probably heard about it, but how many proofs were there? Very few.

In fact, proof attempts were also like probably few, and one of the persons who tried to prove it was Jacob Steiner. So Jacob Steiner came up with a proof. And we are going to see this proof. But the proof had a small gap inside, in fact there is a big gap, let us say a small gap. And that gap was pointed out by a German mathematician, I think I forgot his name, sorry, I will try to look at it later. And then he pointed it out, and then they had some argument. Steiner said that okay, that is not really required. Then finally, other person convinced them that it is required.

But we will first look at the Steiner's beautiful argument which does not need any of the things that we looked at now. Something you can do from school time itself like almost something like that, maybe a little bit of calculus in it, but you can still do it and then we are going to see why that proof is not really complete and then how do you complete it and for that, you can use pigeonhole principle again. So, the isoperimetric problem.

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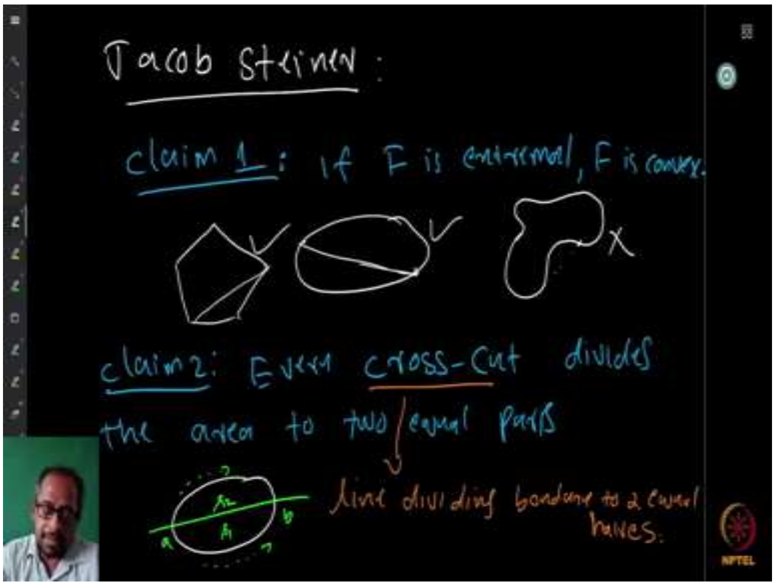

So what is the Steiner's proof? So I am going to give only hints about this, I am not going to really do the entire proof. I want you to write down the proof. So, Steiner's argument is following. If F is extremal then F is a convex figure, so what is the convex figure?

The figure is convex, if any two points that we take, the line segment joining them must lie within entirely that set.

On the other hand our polygons that we look at, they are convex, because you take any two points no matter which two you take, the line joining is within that. On the other hand, a figure like this is not convex because I can take this point and this point the line segment joining that is not inside the set, it is outside. So, this is not convex, this is convex. So the claim is that if you are talking about extremal figures, it must be convex, can you think of why? I want you to stop and think about it.

Now, suppose it is not convex, we are talking about figures with the same perimeter, the length of the boundary is the same. Now, if the figure is not convex, I want you to show that you can basically increase the area without increasing the boundary. To show that if the figure is not convex, you can increase the area without increasing the length of the boundary. So, you show that the claim 1; "if F is extremal, then F must be convex" holds.

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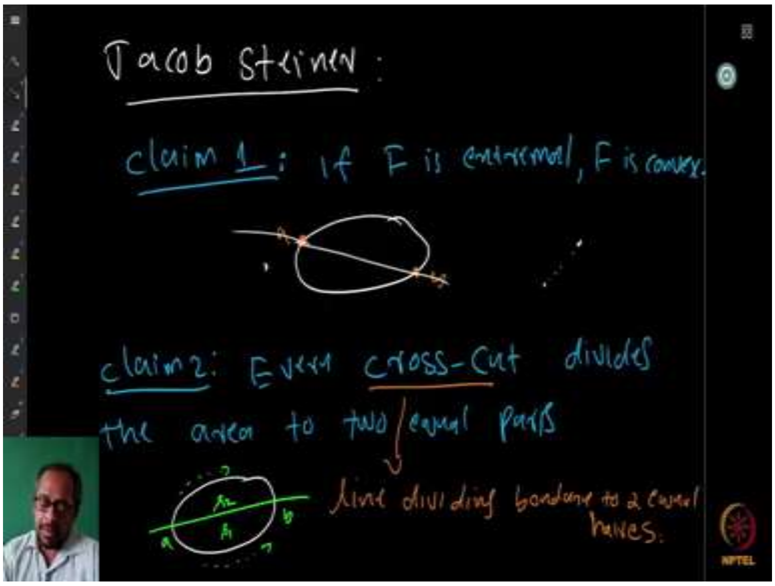

Now you take this boundary, now from the boundary you pick one point. Now in the boundary, what you do is that, you pick one point and then what you do is that you go along one direction. And go exactly halfway through the perimeter, perimeter is exactly half that you can do, you just move along one side till you reach exactly half and once you reach exactly half, what you do is that you select your second point, so these points let us say is a and b. You take a line connecting this a, b this I call as cross cut.

Now, once you divide the perimeter into exactly 2 halves, the claim is that the areas must be equal also. So if you take such a cross cut. Then, the area of this half and the area of this half, they are both equal. So every crosscut divides the area into 2 equal parts. Now basically, if strictly speaking, you can always divide the parameter into exactly 2, you need a result from calculus that we study known as intermediate value theorem, we will not go into details there.

So, every crosscut divides the area into 2 equal parts so, that is a claim 2. So, try to prove this claim 2, again using the fact that if that is not the case you can still without increasing the boundary you can still increase the area.

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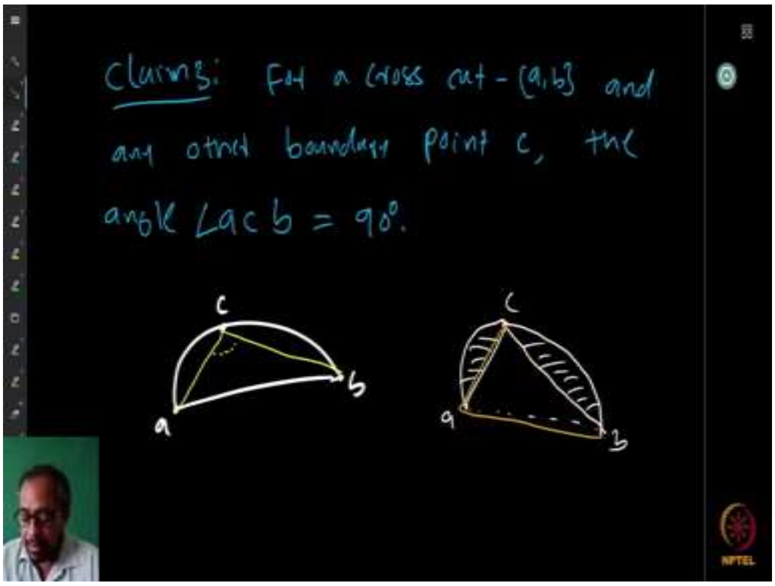

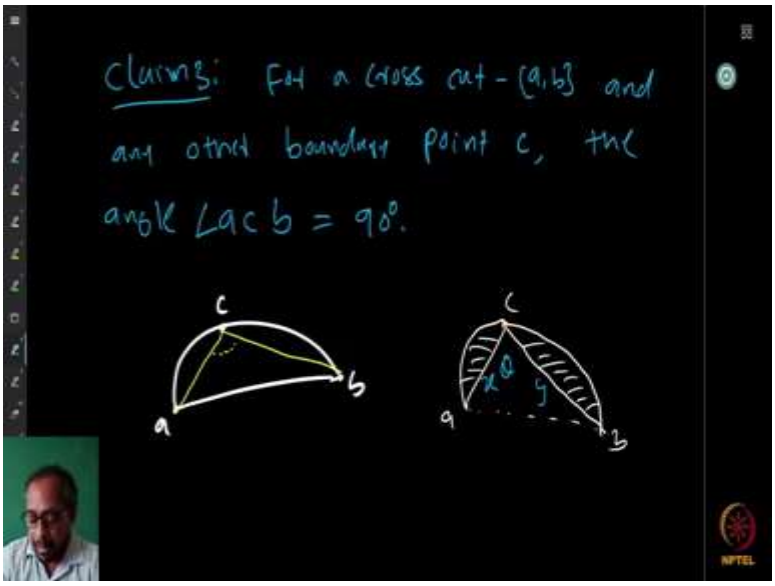

Now, claim 3, this is the most important claim for a crosscut [a, b]. So, crosscut [a, b] basically divides the perimeter into exact half. So, I can just take half of it because the other half has the same thing and same area also. So I just take one half. And now we take any third point let us say c on the boundary. So, you have [a, b] cross cut, a and b are on the boundary then you take the point c. Now, the angle acb is exactly 90 degree.

Now, why is this true? Can we prove that it must be 90 degree? I want you to think about this, but let me give you some idea. But yeah, you think about it sometime and then you will listen to my idea then you try to finish the proof. So here is the idea. Suppose not. So, we have some angle let us say $<$acb. So, I have this ac line segment and the cb line segment, at this point c on the boundary, I am going to put a hinge or nail there in the point c. Then I will assume that the entire thing that we are looking at is made of some thin paper and then you have cut this part and put this nail there.

So, I have cut this part out and this part out, so I have these things. So these 2 things are there, then what I do, then because there is this hinge, I am going to take this piece of paper, and I am going to rotate it, I am going to rotate it in this angle or this angle. Now what happens when I rotate this?

Well, this area that we are looking at, this area is not going to change because I am just taking this entire sheet of paper, the cut out part then moving it around, so this area does not change, this area does not change. And what happens to the boundary of the figure, well the boundary also does not change, because in the boundary, this length and this length is the same. I am not changing anything there. Even if I rotate it like this, the boundary still remains the same length.

Now once I do this what happens? Once I do this, all that changes is the triangle in between, right this triangle abc. This triangle changes because the angle between these 2 line segments ac and bc changes, of course the length of this ac and this bc does not change. So this length, and this length does not change, but the angle changes when I rotate.

Now suppose the angle between these 2 is theta (θ) and the length of ac is x and that of bc is y. So we know that the area of the triangle is determined by the length of this side ac this side bc and the angle between the sides ac and bc, which is θ. So we have $xy\ sin\ \theta$ and then half of that that will give you the area of the triangle. Now, this area is maximized when theta is equal 90 because $\sin\theta$ is maximized when $\theta = 90$. So, the angle must be equal to 90 for that area to be maximize.

Now this tells that if it is not 90, you can still maximize the entire area of the part that we are looking. So using that, one can complete this claim: for cross cut-[a, b] and for any other boundary point c, the angle acb must be equal to 90. So this is the claim 3.

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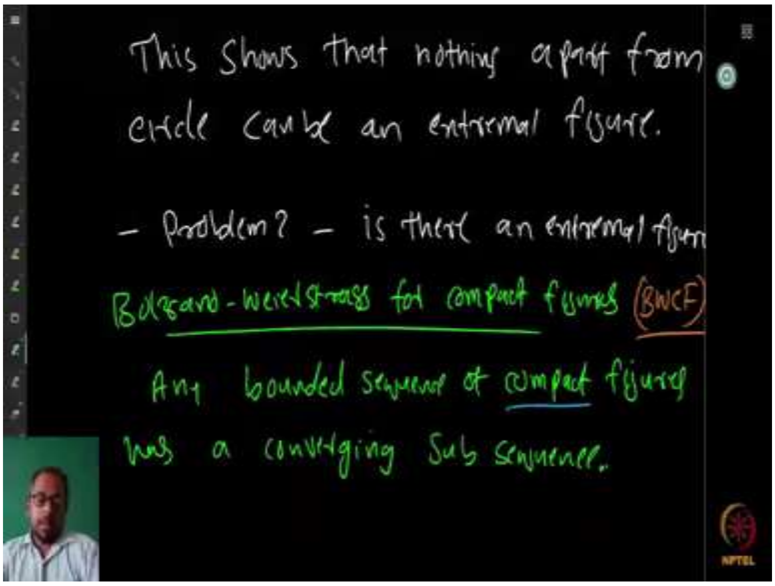

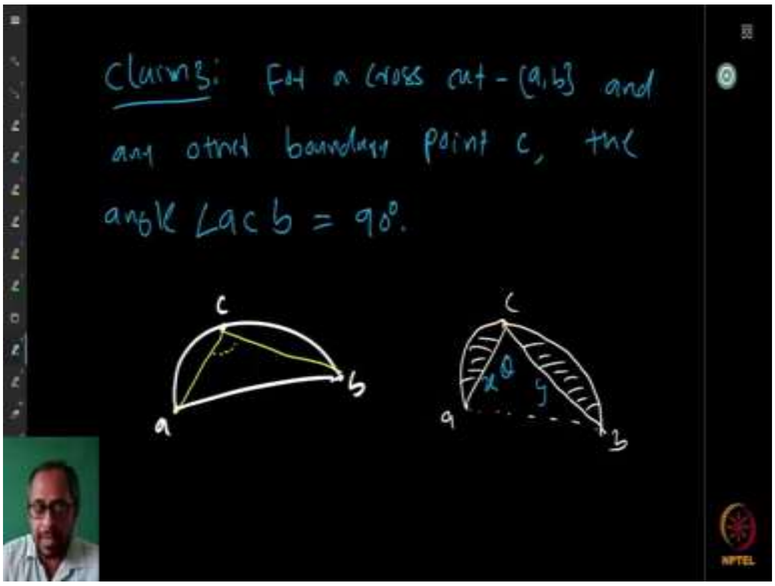

Now what does this show? This shows that nothing apart from circle can be an extremal figure. So, for this part I am using something which I have not really proved at this time that, if you look at the locus of the points or the only figure where it makes angle 90 on every point with respect to this cross cut, on the boundary is going to be the circle. Or that figure is this, and the boundary is a circle or semi-circle. Now, that result is something one can show without much difficulty, but it is not necessary for our current argument. So, let us take it for granted that such figures is the circle.

So, what this says is that nothing apart from circle can be an extremal figure. Now, the problem, see once you look at this you might think that okay, the proof is already done, we have found that you take the circle, your circle is better than everything else that is the extremal figure. But the fact is that it is not as simple as that. It will be kind of difficult to convince of this, but let us look at some weird analogy. Like suppose I ask you the following question, that if you take a needle, let me take a blank paper somewhere.

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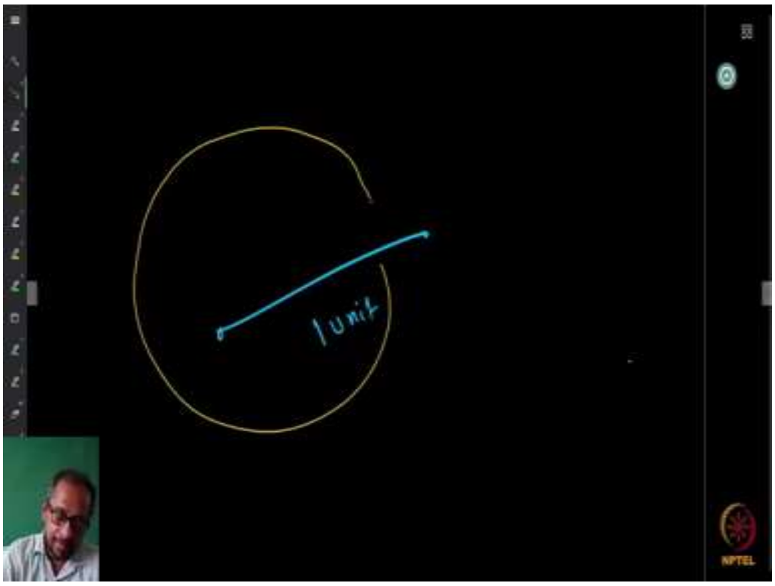

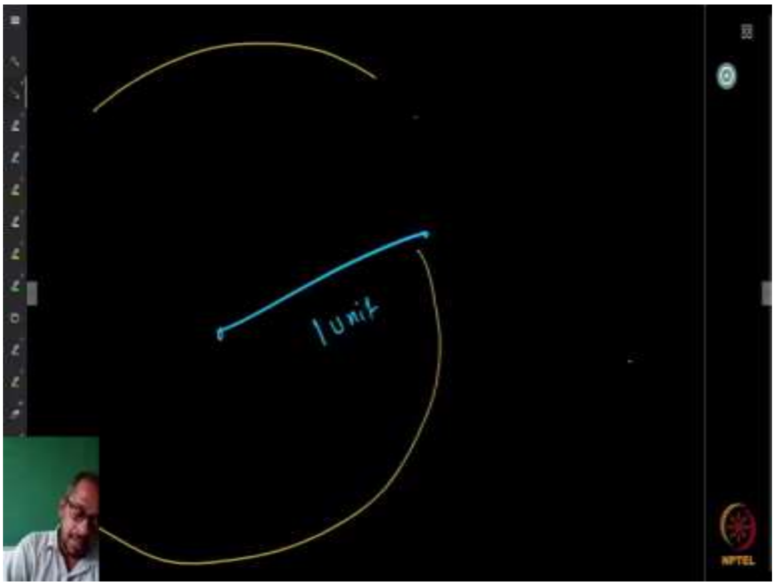

Suppose I take a needle, a needle is basically a line segment without any width, it is just a line segment of let us say unit length whatever it is, some unit 5 centimeter, 10 centimeter, 1 meter, 1 kilometer, it does not matter, some units, 1 unit line segment. What I want to do is that I want to place this line segment or in fact, I want to rotate this line segment in the plane. So, we are sitting inside this plane and I want to rotate the line segment 360 degree.

Now what is the smallest area in which you can do this, what is the smallest area in the plane, we want to find a set in the plane, in this set I want to place this line segment and rotate it. I want to rotate the line segment in the plane. So, by rotating I mean that I want to start from this initial position, I want to slowly change it I can I know I am allowed to move it is like left or right if you want I can just can shift it here or shift it here.

So, I shift it here a little bit and rotate it a little bit, then I shift it again back and then rotate a little bit that is okay, whichever way you want you can. But I want to slowly move this so that it covers every possible directions. So, the infinitely many directions, in this 360 degree, this guy should be going through all those directions before it reaches back, so 360 degree rotation must happen.

Now, what is the smallest area of a figure in which you can do this? Of course the obvious way is that you put a needle here and then rotate in this a big circle you can always do it, 360 degree you can rotate, and what is the area that you require for this that is πr^2, πr^2 is the area of the circle where r is unit so it is basically π. Within π area, whatever π unit square you can do this.

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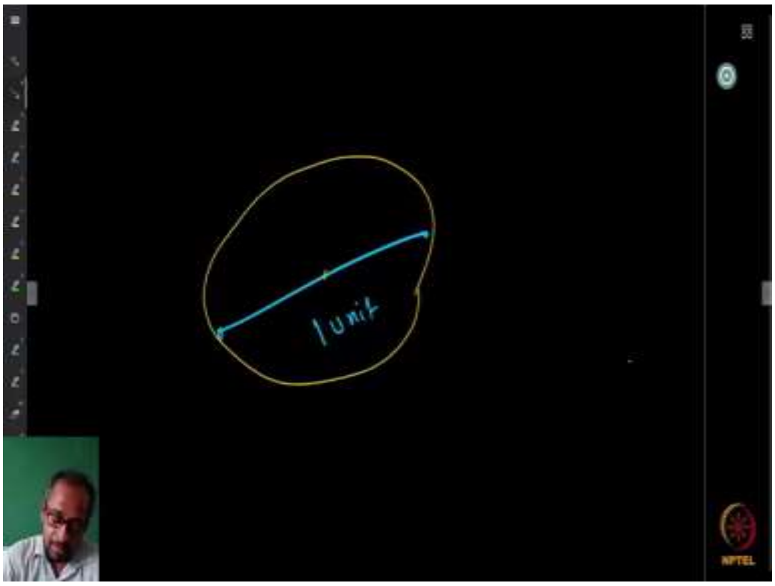

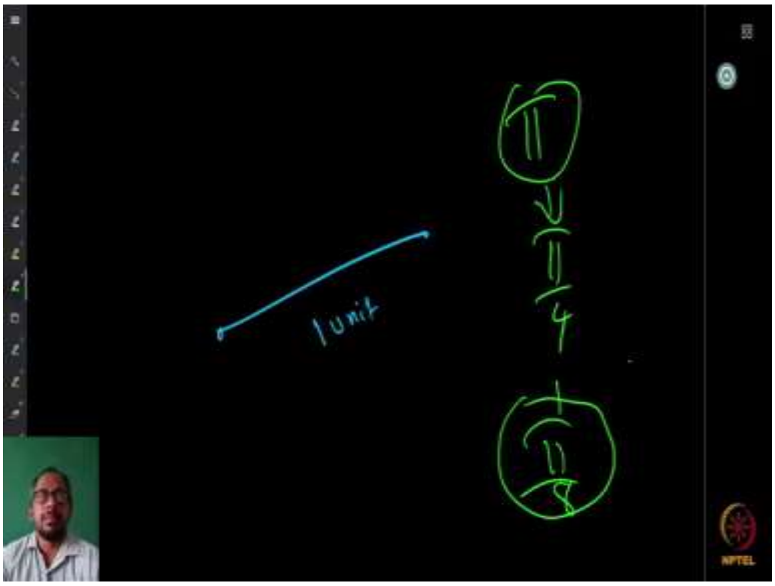
1 unit
π
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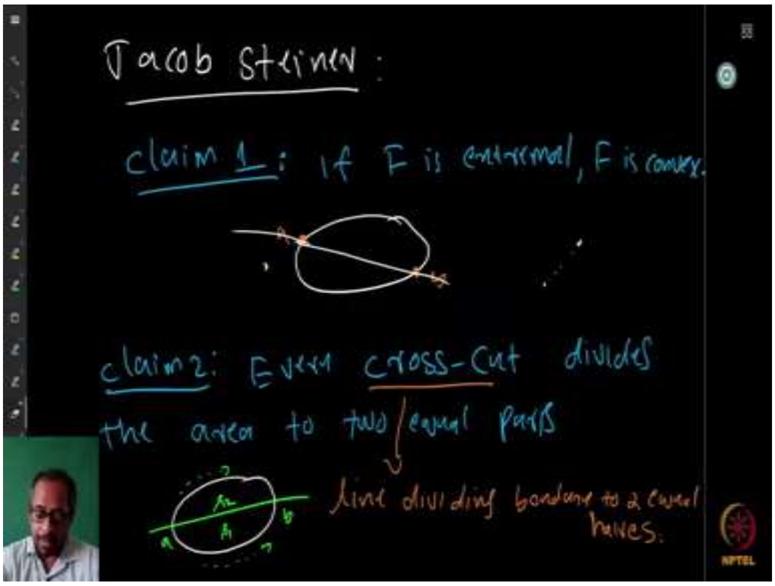
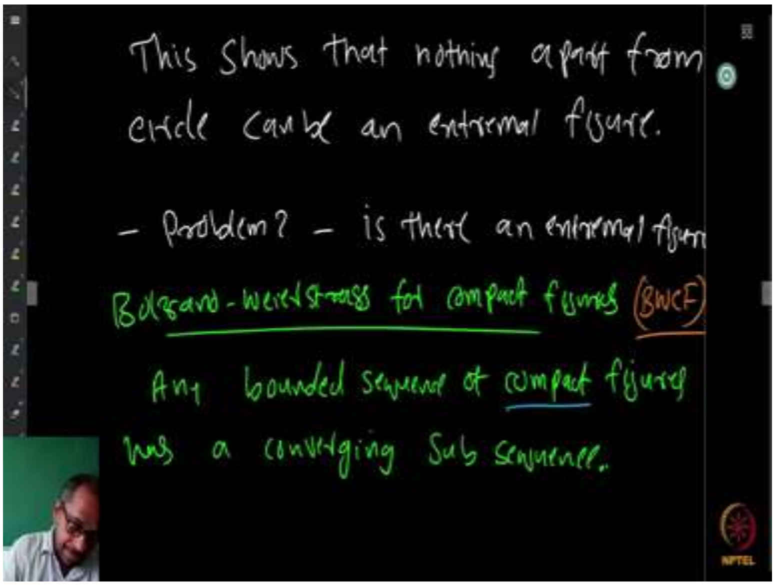
π/8


Jacob Steiner:
claim 1: If F is extremal, F is convex.
claim 2: Every cross-cut divides the area to two equal parts
line dividing boundary to 2 equal halves.


This shows that nothing apart from circle can be an extremal figure.
- Problem? - is there an extremal figure
Bolzano-Weierstrass for compact figures (BWCF)
Any bounded sequence of compact figures has a converging sub sequence.

Now, well of course, you can improve it further drastically instead of selecting this point I select the midpoint, I can just rotate it here this as the centre. So, then the radius divides by half so it will be $\frac{\pi}{4}$. Instead of π we started with the beginning, I can reduce it to $\frac{\pi}{4}$. If you are really smart, you can do an even more different way, you can make it $\frac{\pi}{8}$, can you think about how to do this $\frac{\pi}{8}$? So it is interesting question. Now, but apart from that, the question is that, can we even reduce it further? So what is the minimum area in which you can do this.

So people used to believe $\frac{\pi}{8}$ is the best possible and many proofs were there, many attempts were there to try to reduce it until 1 day a person called Besikovitch proved that there is no smallest area, what do I mean by there is no smallest area? When I say there is no smallest area what I mean is that you give me an area, I will say that 0.000001 unit square.

I can give you a figure in which you can rotate this. You give me an even smaller area 0.0000000000001 or like 10^{-20000} , no matter how small you give me and no matter how large your unit is going to be, like, this can be 1 light year length or 100 kilometer in length.

As far as you give me a line segment of that length, I can give you a set in which we can do this rotation. So which means that you can make the area go to as close to 0 as you want, you can of course never be 0. You cannot do rotation in a 0 area thing. So, therefore, it is never 0 but it can be as close to 0 as you want. So, which mean that there is no minimum area in which you can do this.

So, similarly, one question that one should really answer is that, is there actually an extremal figure? We said that nothing other than circle can be extremal, but is there an extremely figure? If there is an extremal figure we proved that assuming that there is an extremal right, this is all assuming there is an extremal figure.

These arguments are assuming that okay, suppose there is an extremal figure if F is extremal then F is convex. We assumed that there was an extremal figure and we said that if F is extremal, cross cut divides it into equal parts, but maybe there is no extremal figure. Maybe these properties are there for circle but still that only say that all these properties must be there.

But what is the guarantee that there is an extremal figure, maybe there is no extremal figure, there is no largest area, you can keep on doing like this, you can make it smaller like that you can keep on making it larger something, we do not know.

So, we need to prove that; and proving that requires another version of Bolzano–Weierstrass theorem. So this is called Bolzano–Weierstrass theorem for compact figures. Yeah so for compact figures BWCF, which says that any bounded sequence of compact figures has a converging subsequent. So this is for math students, other students need not really look into this, but just thinking about it may be interesting, but other than that, maths students can look at this, should look at this.

So any bounded sequence of compact figures has a convergent subsequence. So, what is a compact figure? We already said what is what is bounded. Now, if you have a set which is bounded and we will say what is the limit point. So, if all the limit points of a set belongs to that set itself then it is called closed. Now if you have a set which is bounded as well as closed then it is compact.

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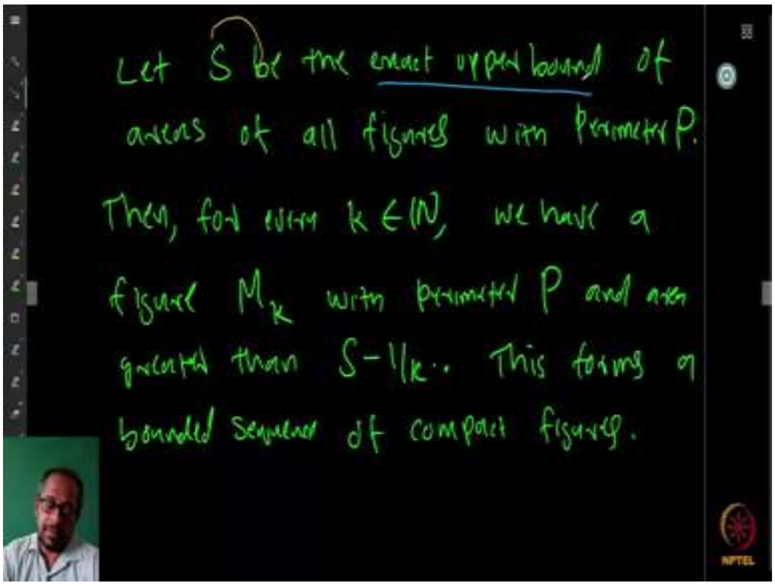

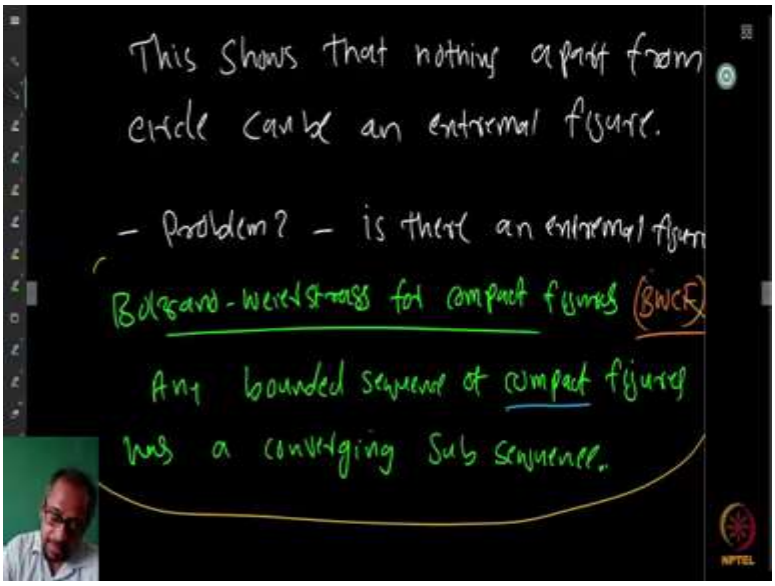

So, suppose we proved Bolzano–Weierstrass theorem, any bounded sequence of compact figure has a convergent subsequent.

Once you have this property, we can assume there is some figure which is the exact upper bound of areas of all figures with perimeter p, this is by using the property of real numbers existence of exact upper bound. I will mention what is exactly upper bound and how to prove this in fact, you can prove it using again pigeonhole principle if you want.

But, yeah, assuming that this is done already that we can say that S be the exact upper bound then for every natural number k, we can find a figure M_k with parameter p and area greater than $S - \frac{1}{k}$ because S is exact upper bound, in the neighborhood it should contain at least one point by the definition.

And so, therefore, if the exact amount of all figure of perimeter p, the S is the area then $S - \frac{1}{k}$ for any k you should be able to find some figure which is close to that, that is a property of exact upper bound. Because the area is bounded and the perimeter is fixed, these figures will form a bounded sequence of compact figures and therefore, we can apply the Bolzano–Weierstrass theorem so therefore, it has a limit point.

Now, if it has a limit point then the areas basically are keep on increasing to S, and this sequence of areas are converging to S, so if there is a limit point for this that only possibility is S only. Now, you cannot have a different value as the area, limit point of the area S. So, these things one can math students can easily figure out.

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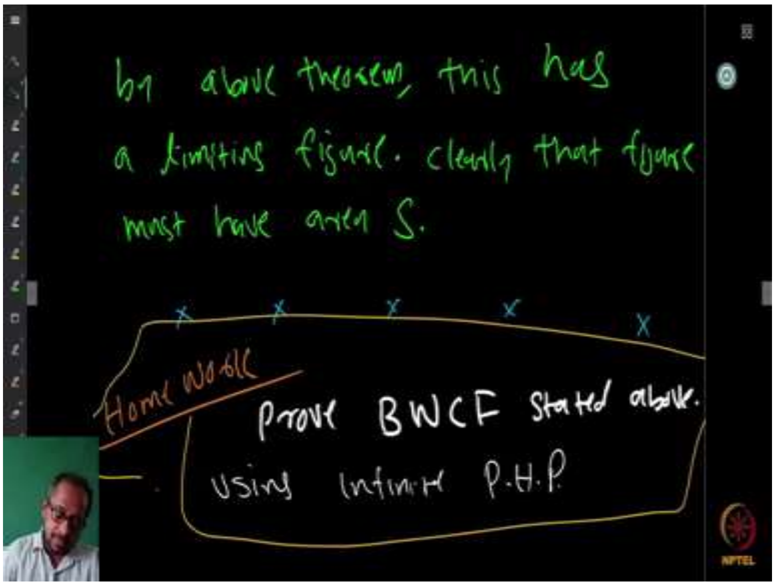

Therefore, by the above theorem, this has a limiting figure. So, clearly the figure must have area S. So therefore, by Bolzano–Weierstrass theorem for compact figures, we can show that there is a limiting figure and since, we already know that the only possible limiting figure is circle, and therefore we have a maximum there. Therefore, we have, circle obtains a maximum area.

Now, I want you to prove Bolzano–Weierstrass theorem for compact figures, this is result in analysis of course, using the pigeonhole principle infinite form. So, this is for just for math students, other students are welcome to try if you want, adventures students, but you need some concepts which mostly only math students see.

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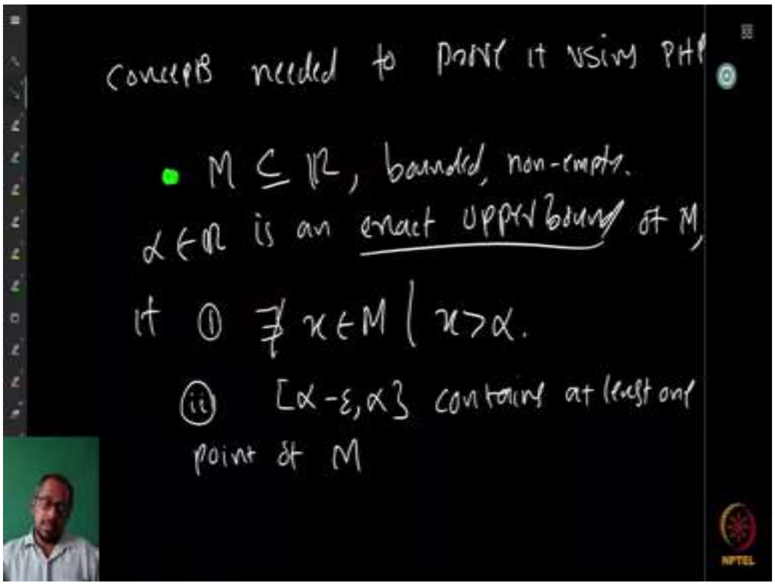

So, here are the concepts needed to prove it using the pigeonhole principle. So, one is the following:

Let $M \subseteq \mathbb{R}$ is bounded non-empty set, then $\alpha \in \mathbb{R}$ is an exact upper bound of M, if

(1) There is no larger element in M. That is, there is no such $x \in \mathbb{R}$ such that $x > \alpha$. (So, α is an upper bound for all the elements, so alpha is on the right hand side.)

(2) The interval $[\alpha - \epsilon, \alpha]$ contains at least one point of M for every ϵ.

Now, I am saying that it is at least 1 point of M, because this point of M may be disconnected and sitting outside as a single point on the right hand point, that is still a point with this property, these are exact upper bound. And you do not need infinitely many points for it to be an exact upper bound. But still you can show the existence of an exact upper bound using the infinite PHP if you want.

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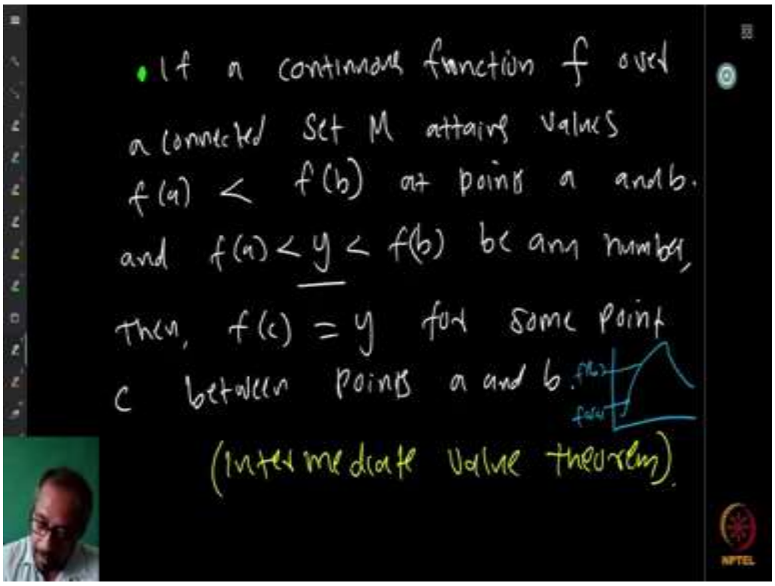

Then second condition, if a continuous function. So, again, I am not going to define a continuous function formula here, math students already know that, other students can assume that it is a smooth growing function in the sense that there are no breaks in the values. So, basically, like if you start from a value, it keeps on smoothly increasing till it reaches another value.

So, if a continuous function f over a connected set M attains 2 values $f(a)$ and $f(b)$, where $f(a) < f(b)$ at corresponding points a and b. And $f(a)$ and $f(b)$, contains some number y in between. Then, $f(c) = y$ for some point c between points a and b. This is called the intermediate value theorem.

So you have these 2 axes and then you have this continuous function, then it attains some values let us say $f(a)$ and some value $f(b)$. Now, you take any number between $f(a)$ and $f(b)$, let us say y. Then there is some point c between this a and b such that $f(c) = y$. This kind of obvious once you see it in a visual manner, but you need to prove it, this is called the intermediate value theorem, something one can show easily.

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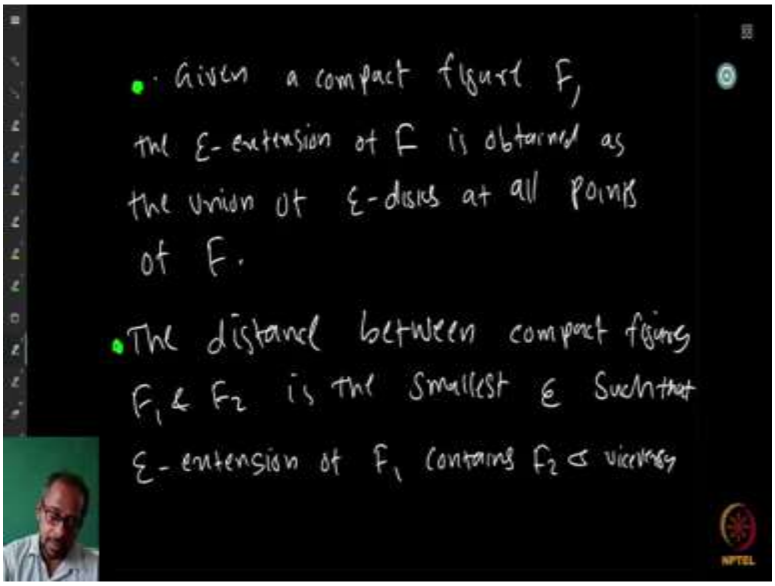

And this is a result that you might require if you want to formally prove this. Then given a compact figure F, the ϵ- extension of F is obtained as the union of ϵ- disc of all points of F.

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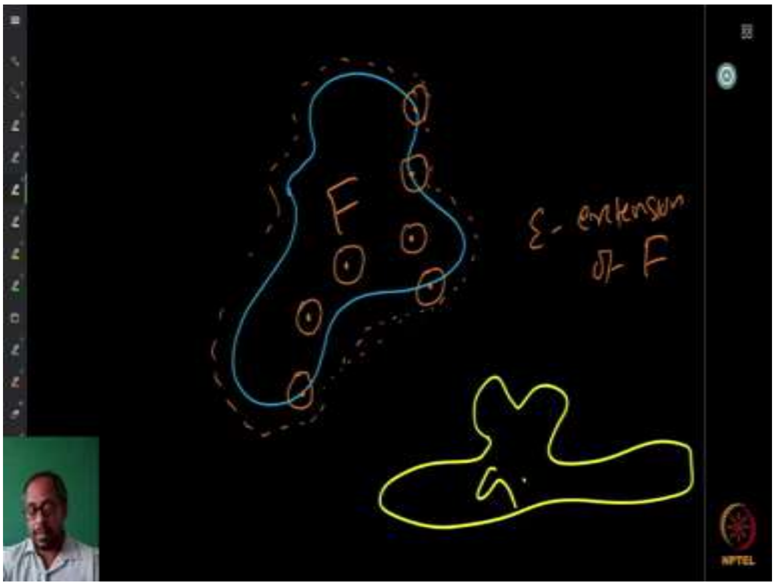

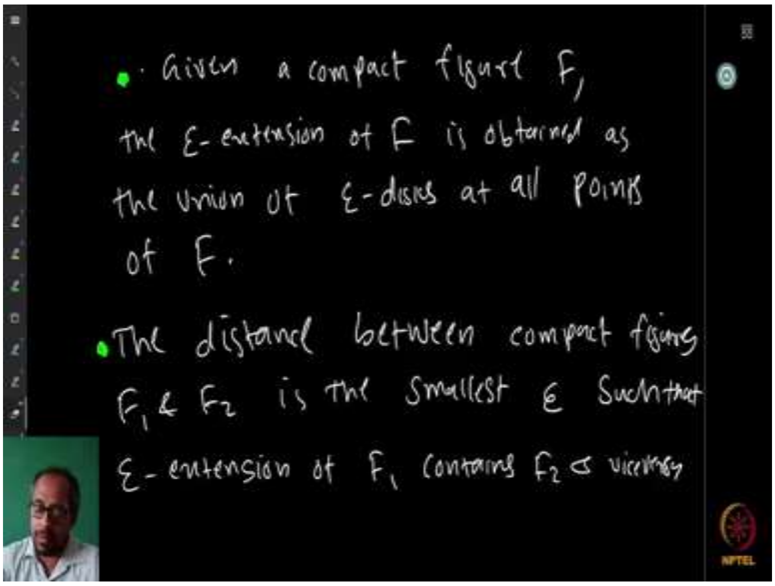

So, here is an example, so I take let us say some compact figure. Then what I do is that fixing some ϵ, I will say that okay I take an ϵ-disc around each point of the set, every point I am going to keep an ϵ-disc. Then this union of these ϵ-disc gives me another set, this is called the ϵ-extension of the figure F okay. So, we have the ϵ- extension of the figure F.

Now, once you have ϵ- extension, I can define the distance between 2 figures, what is the distance between 2 figures? The distance is defined as follows. You take the epsilon extension of F and suppose you have another figure let us say G, another compact figure G. Now the distance between F and G is you take the ϵ- extension of F and find the smallest epsilon such that you can put G inside F. But do you want your ϵ to be such that, if you take ϵ- extension of G, you should be able to put F also inside G , the extension.

So, the smallest ϵ such that ϵ-extension of F_1 contains F_2, and ϵ-extension of F_2 contains F_1 . That is called the distance between the compact figures F_1 and F_2. So these are the points or notions that you might require to formally prove the Bolzano–Weierstrass theorem for compact figures, and again using the infinite pigeonhole principle. So, try to prove this and then that will tell you, there is an extremal figure for the isoperimetric problem, then we will prove that circle has the maximum area among all these figures.

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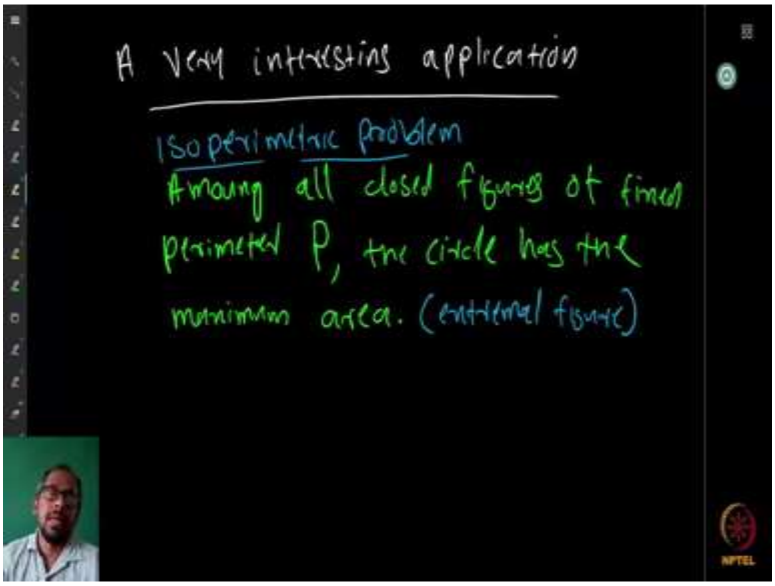

So, I think, that would be a very nice question, because it is a very classic question that all of us have studied in school, and probably not seen a proof of it. So try to look at this and try to prove this. So, I think we finished all the topics that we wanted to cover in pigeonhole principle and then, there are many questions that you should solve in the given textbook. And using this you will get more experience in solving. So with that, we stop for today and then we will continue in the next class.