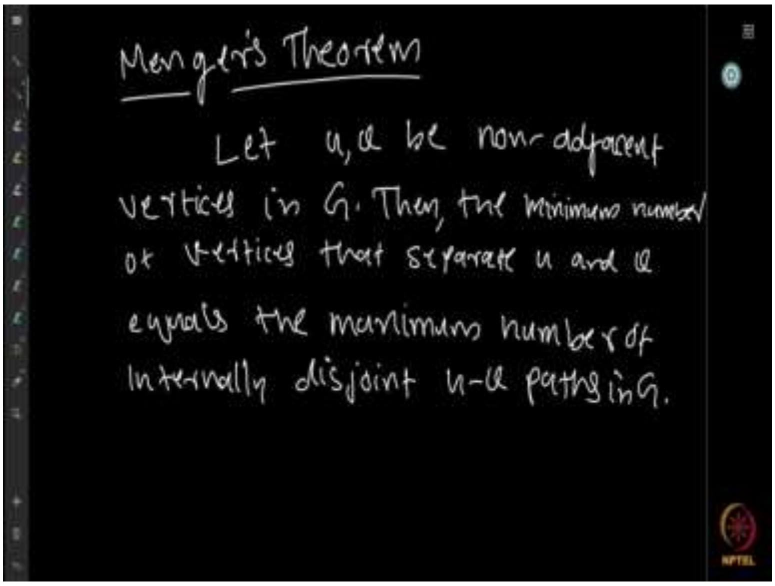
Combinatorics
Professor. Dr. Narayanan N
Department of Mathematics
Indian Institute of Technology, Madras
Menger's Theorem

(Refer Slide Time: 0:15)



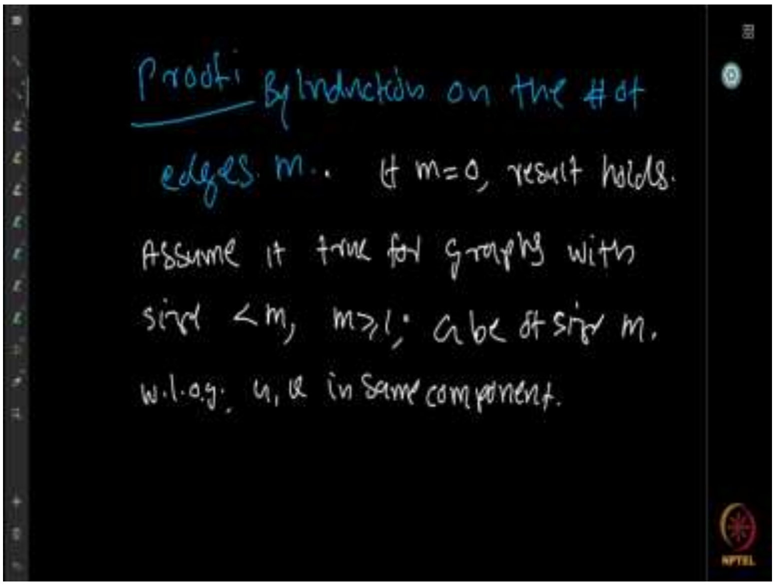
Now, here comes one of the most important theorems that we are going to learn in graph theory, which is called Menger's Theorem. So, at the end of this lecture you will see that this theorem is equivalent to many important theorems in combinatorics. In fact at least 7 of them we will list today and each of the seven theorems are basically equivalent to each other. So, if you prove one of them, from assuming that you can prove the others easily, that is the idea. So, they are kind of equivalent results.

This is very important theorem and is the Menger's theorm of connectivity. Here is the statement of the theorem. Let u and v be non-adjacent vertices in the graph G. Then the minimum number of vertices that separates u and v is equal to the maximum number of internally disjoint u-v paths in the graph. So, the theorem says that there is a close connection between the cardinality of the separating sets and the internally vertex disjoint paths.

So, you take u and v look at all the paths and find a subset of vertices which will disconnect u and v. So, if you remove these vertices, then there is no u-v path in the remaining graph then it is a separator. Find the minimum cardinality separator. That minimum separating set is actually equal to the maximum number of internally disjoint u-v paths in G and one direction you should be able to see very clearly because if every path is going to use the separating set vertices then you just use at least one vertex from the separating set to go from u to v.

So, you definitely cannot have more than these many internally disjoint paths that is clear because every part contains one of the vertices and therefore each path have at least one, of course we can have more maybe! but then you cannot have more than the minimum separating set that many disjoint paths, so that is clear. But what is not immediately clear is that why there should always be that many internally disjoint u-v paths. So, this is what we are going to prove today and the proof is also very interesting. So, you should pay attention to it okay.

(Refer Slide Time: 3:28)

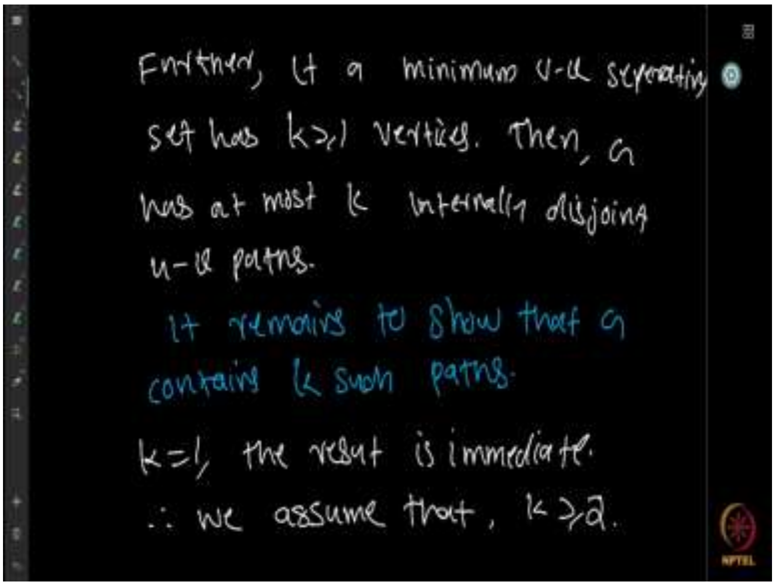


There are several proofs, but we are going to give a specific proof by induction on the number of edges. So, let the graph contain m edges. G is the graph it has m edges and if m is equal to 0, then this result holds right trivially, because there is no edges, so, there is no u-v path. u and v are non-adjacent vertices, there is no u-v path and m is also 0. So, separating set is also empty set. The empty set separates because there is no path.

Therefore, the result is trivially true. So, now we can assume so, this is the base case. So, we can assume that the result holds for graph with size strictly less than m, where m is at least one. Now, let G a graph of size m. Now, without loss of generality, we can assume u and v are to be in the same component. Why is this? Because, if u and v are in different components, then there is no u-v paths and you do not need any vertex to separate u and v because they are already separated.

So, that is empty set again. And therefore, we can assume that u and v belong to the same component. So, we assume that the graph has size m and u and v belong to the same component in the graph.

(Refer Slide Time: 5:07)

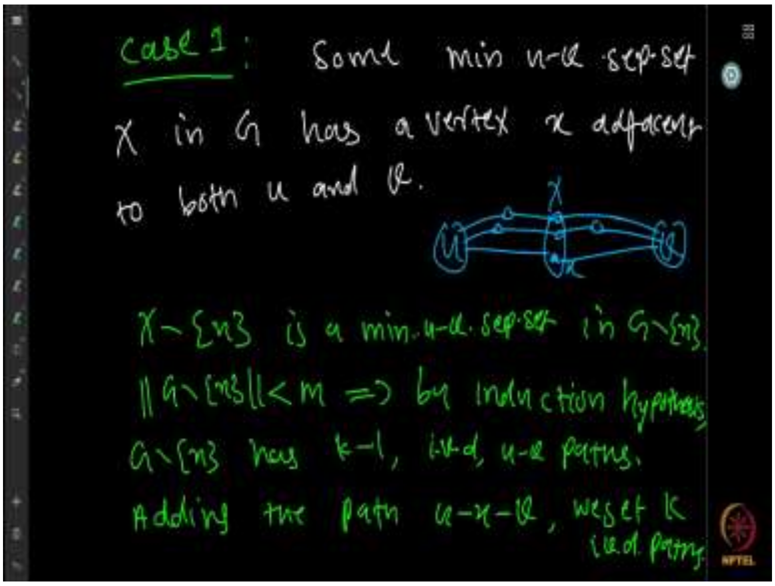


Now, we can assume something further. If you have a minimum u-v separating set, having let us say k at least one vertices (k greater than 1) because zero is already done. Then definitely G has at most k internally disjoint u-v paths. This we have mentioned before because every path must use at least one of the vertices. So it cannot have more than k ivd (internally vertex disjoint) u-v paths. So, what remains to prove is to show that we can actually find k such paths.

Now, if k is equal to one again the result is immediate because if k is equal to one what it says is that, there is a cut vertex between u and v. u and v we can be separated by a single vertex means that there is a cut vertex that separates u and v.

Now, if there is a cut vertex between u and v, then every path must go through that vertex. So, since every path must go through that vertex you cannot have more than one disjoint path, but because the u and v are connected, there is at least one path. Therefore, we get one path and that is the maximum that we can have. So, that also holds. So, k is equal to one is also immediate. Without loss of generality, we assume that k at least 2 because we have done this. So, we will assume that k is at least 2 for the remaining part of the proof.

(Refer Slide Time: 6:47)



So, we divide the proof into 3 cases. The definition of these cases are very interesting because it makes the proof very easy. Let us see how we define this case. The first assumption is that, you have this component in which you have u and v. u and v are connected and you have a separating set let us say X in the graph G. Now, the assumption is that for some separating set of the minimum cardinality, X is a minimum separating set for u and v and for at least one of the separating sets, minimum separating set has a property that there is a vertex which is adjacent to u as well as v. The assumption is that, there is some minimum separating set where a vertex in the set is adjacent to both u and v. So, there is an edge from u to x and also an edge from x to v. If this happens in the graph, this is the first case, suppose there is such a minimum separating set in that case, what we do is the following.

So, what we do is we remove the vertex x from the graph G and you look at the graph G\{x}. So, in G\{x} of course, this edges right u to x and x to v will also go and then you get a smaller graph with less number of edges. And in this graph, we can see that X minus the vertex x (X-{x})is a separating set for u and v because once you remove the entire set X, u and v are disconnected. I can do this in two steps I first remove the vertex x and then remove the remaining vertices in X.

So, X – {x} is a minimum cardinality u-v separating set in the graph G\{x} because if there is even as a smaller one, I can remove that and remove x so that it gets separated. So, therefore, we will assume that is not the case, I mean, therefore, it is not the case.

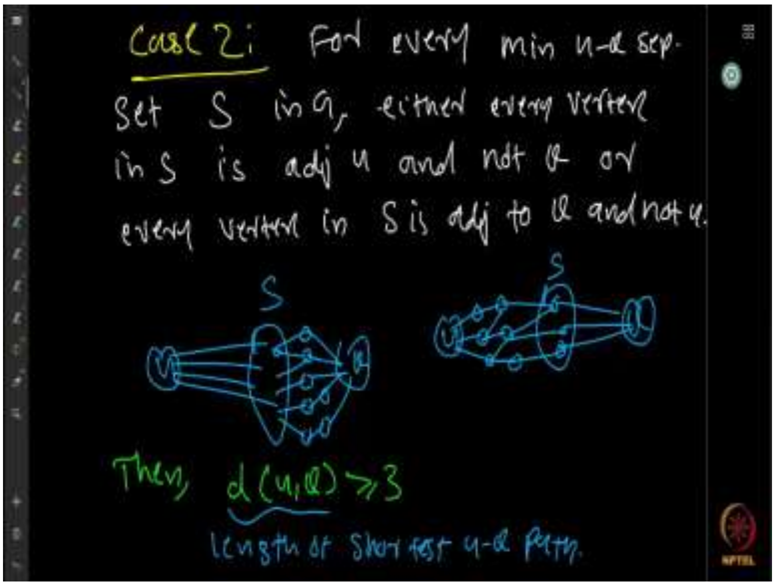
But the point is that if G\{x} has at least two edges less, so, I can use induction because the number of edges is less and I am inducting on the size of the graph, which is the number of edges. So, if the size of G\{x} is strictly less than m, this implies that we can use induction hypothesis. So, what is the induction hypothesis? If the number of edges is strictly less, then the result holds.

This was cardinality k then you can find k-1 internally vertex disjoint u-v path from u to v in the graph G\{x}. So, in the graph G\{x}, we can find internally what vertices disjoint u-v path from u to v. There are k-1 such paths. So, once you get this many paths by induction. What you do is to just add the path u to x, x to v because u and v are destined to x.

So, u-x-v is a path which was not present in the graph G\{x}. So, this path is going to be internally vertex disjoint from all other paths, from the all other paths that we obtained by the induction hypothesis.

So, adding this path I get k - 1 + 1 that is k paths. So, I get k paths in the graph which solve the case for this case. So, this case is okay. If the graph contains a separating set with a property that one vertex is adjacent to both u and v, then we can easily use induction. So, that is the case 1.

(Refer Slide Time: 11:44)



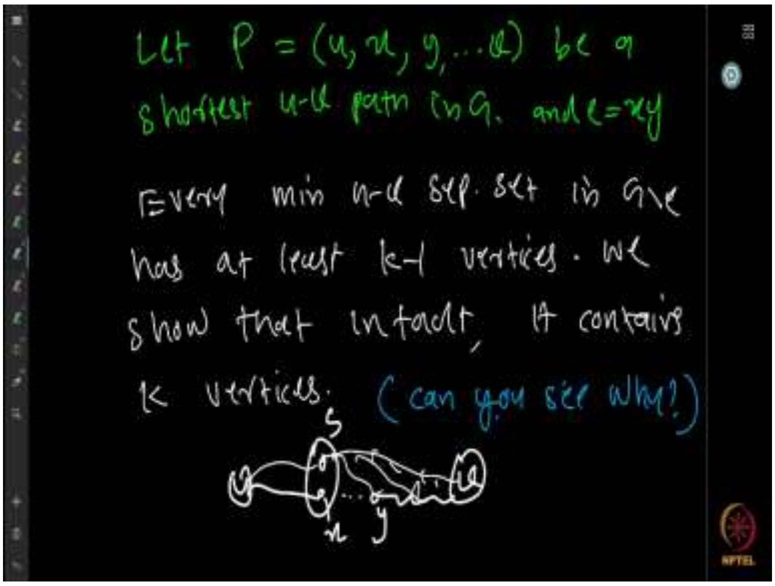
Once we finish with this case 1, we will assume case 2 has the following structure. So, in this case we assume that for every minimum u-v separating set in the graph G, either every vertex in the separating set S is adjacent to u and not to v because if there is one which is adjacent to both u and v, then we are in the previous case.

So, we will assume that every minimum u-v separating set has a property that all its vertices are either adjacent to u but none of the vertices are adjacent to v, or all the vertices are adjacent to v and none of them are adjacent to u. So, every minimum separating set has this property. This is the second case. Maybe this does not happen, but we will assume that suppose this happens, if one of these cases.

What can we see from this structure? If every vertex in the separating set is adjacent to u but none of them are adjacent to v, then the u-v path of course goes through the vertex set S. So, every u-v path goes to a vertex in S, then from there it also takes some path to v. But, since there is no direct edge, there is at least one vertex in between when I go from S to v. So, S to v, we should contain at least one vertex.

Which means that the length of the shortest u-v path from u to v (let us say), must contain at least 3 edges because u to S I should need one edge. S to v, I need at least two edges. So, the length of the shorter u-v path which is the distance between u and v is at least 3, because every path must go through S and then some path of length two from S to v. So, distance is at least 3. We will observe that if the case 2 happens, then the distance between u and v is at least 3. Now, what we are going to do is the following.

(Refer Slide Time: 14:50)



So, since you know that there is a path of length at least three, we will take such a shortest path. So, you take a shortest u-v path, it can be 3, it can be 4 whatever it is, but the shortest one that you can find in the graph from u to v whatever path that you can find. So, consider P to be the shortest u-v path.

So, it starts from u go to some vertex x in the set S by the separating set, then from x it goes to an immediate neighbour y, which is not the vertex v of course because there is no direct edge from x to v and then from y there is some path to v. It could be an edge or a longer path. So, you have the shortest path which goes from u to x, x to y and y to v. Now xy is an edge because that is how we defined the vertex y. So, the edge e is equal to xy.

Now, what you will do is that in this graph, I removed the edge e I do not remove the vertices I just removed the edge e. So, suppose you have something like this. So, let me draw it for you. So, you have u then you have some vertex small x let us say in the set S and then some y and then some path to the vertex v. So, you have this u-x, x-y. So, I am going to remove basically this edge xy. The xy edge is removed in the graph of course the other edges whatever it is are going to be there. I just removed the edge xy.

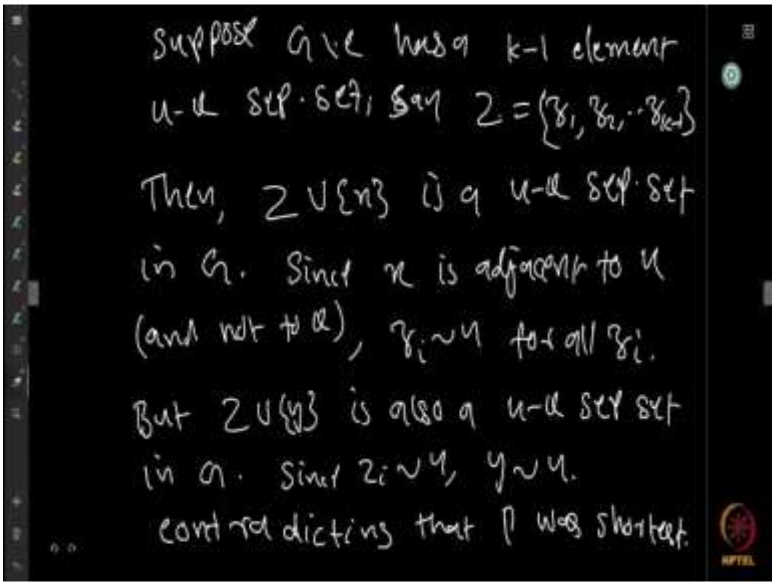
Now, once I removed the edge xy what happens to the minimum u-v separating set, see I know that S is the minimum separating set which means that removing the vertices of S will make the graph disconnected and that is the smallest set. Now in this, if I remove one of the vertices, if S has k vertices and I remove the vertex x of course, I know that the remaining set has k minus one cardinality. But if I remove x, I also removed the edge xy because x is a vertex incident with the edge xy.

So, removing the vertex x will also remove xy. If I remove only the edge xy the number of the minimum separating set cannot decrease by more than 1, that is what I wanted to say. Because just removing x itself will remove xy. So, in G\xy, the minimum cardinality u-v separating set cannot be less than k-1 because I can just remove x which will be a subgraph of G\xy and then that has separating set size actually equal to k minus 1.

Now, the claim is that of course in a typical case once you remove an edge, it can also decrease the cardinality of the separating set but we are going to say that in this particular case, it will not decrease the minimum separating set in G\xy is actually again has cardinality equal to k. So, we are going to prove this but I want you to think about why in G\xy you cannot have separating set of size k - 1.

Think and try to prove it yourself if we are in case 2, that is every minimum u-v separating set S in G has either all the vertices in S are adjacent to u and not adjacent to v or all the vertices are adjacent to v and not adjacent to u. In this particular case, if I remove this edge xy in this shortest u-v path, then it cannot decrease the cardinality of the separating set. So, think about this.

(Refer Slide Time: 19:33)



How do we prove this? Suppose for the contrary that there is a k - 1 element separating set in that graph G \ e. We call the set as let us say $Z = \{z_1, z_2, z_3, \ldots, z_{k-1}\}$. Now, what we know is that Z ∪ {x} is a u-v separating set for the graph G because once you remove x, you will get a subgraph of G\xy.

So, if I remove all the elements of Z, then you get a separating set in the graph G because this is a separating set in G\xy and G\{x}. G\{x} is a subgraph of G\ xy. Now, since Z is a separate u-v separator for G\xy, it is also a separator for G\{x} because this one is a subgraph.

But now we know that the minimum cardinality of a separating set is k in the graph. So, I have removed one vertex which is x and then I have removed one vertex x and then I removed the k-1 vertices in Z. So, it has cardinality k, therefore, it is a minimum separating set also. So, Z ∪ {x} is a minimum u-v separating set in the graph G but we know that, by the choice of x, x is adjacent to u but not adjacent to v. So, that was our assumption. x is basically adjacent to u and because of the property of S, it is definitely not adjacent to v.

Now, x is adjacent to u and not v, each of the z_i's must also be adjacent to u, because Z ∪ {x} is a minimum separating set and we assume that in the case, every vertex in the separating set is either adjacent to u and not to v, or adjacent to v and not to u but one of them is already adjacent to u therefore, everything else must also adjacent to u. So, z_1,…, z_{k-1} is also adjacent to u but not adjacent to v.

Now, this we can do because we assume that k is at least 2. Because, when k is at least 2, we do not hold to the induction trap. We know that this set is non-empty, this set $\{z_1, \ldots, z_{k-1}\}$is non-empty and therefore, we can do the following. So, if I look at Z ∪ {y}, Z ∪ {y} is also a u-v separating set because once I remove y, again I remove the edge xy.

So, G\{y} as we just noted earlier G\{x} similarly G\{y} is also a subgraph of G\xy. Now because this is a subgraph of G\xy, if I remove z_1 to z_{k-1} which is the separating set for G\xy, then I get a u-v separator for the graph G also. So, we see that Z ∪ {y} is also a separating set for the graph G.

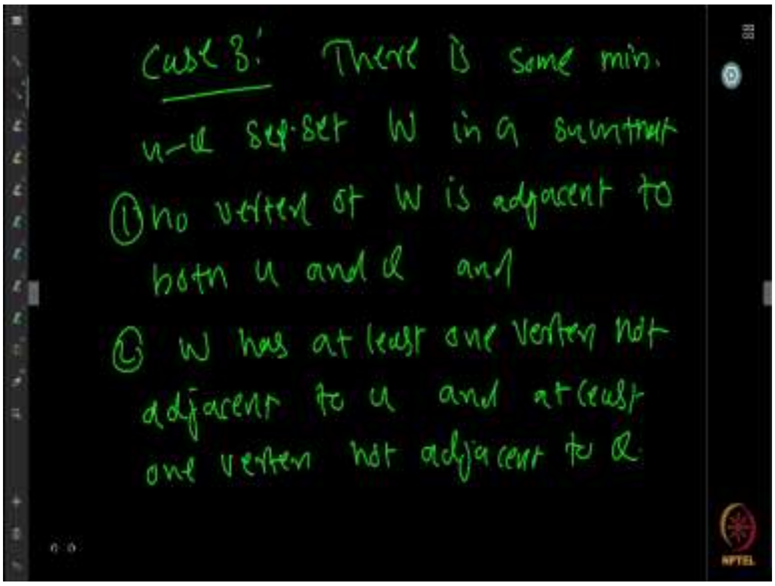
Now, Z ∪ {y} is a separating set now, $z_i's$ are adjacent to u because we proved in the previous case, we are not changed the z_i. But because of the property of the case, y also must be adjacent to u because we said that every vertex in any minimum separating set is also adjacent to u. Therefore, there must be an x from y to u also.

Now, this contradicts the fact that p was a shortest u-v path because we said that we started with a shortest u-v path which means that we go from u to x to y to v was of the shortest length. But now if y is adjacent to u, I can actually go from u to y and then continue this path which gives me even smaller path but by assumption P was the shortest path.

So, this is a contradiction and therefore, we cannot have a separating set of size k - 1 in this case. So, G\xy cannot have a separating set of size k - 1. But G\xy has a separating set of size k only. So, every separating set in G\xy has cardinality k. But what we achieved by doing this is that we got a graph with smaller size because G\xy has strictly less number of edges and therefore, we can assume induction.

So, by induction hypothesis this graph G\xy has k internally vertex disjoint path from u to v. But all these paths are internally disjoint paths in the graph G because you know xy is an edge whether I can choose to put in the path or not I can just throw it away. So, all the paths in G\xy are also paths in G. Therefore, by induction I get k internally vertex disjoint path in the graph G\xy and therefore, in the graph G. So, case 1 and case 2 are done.

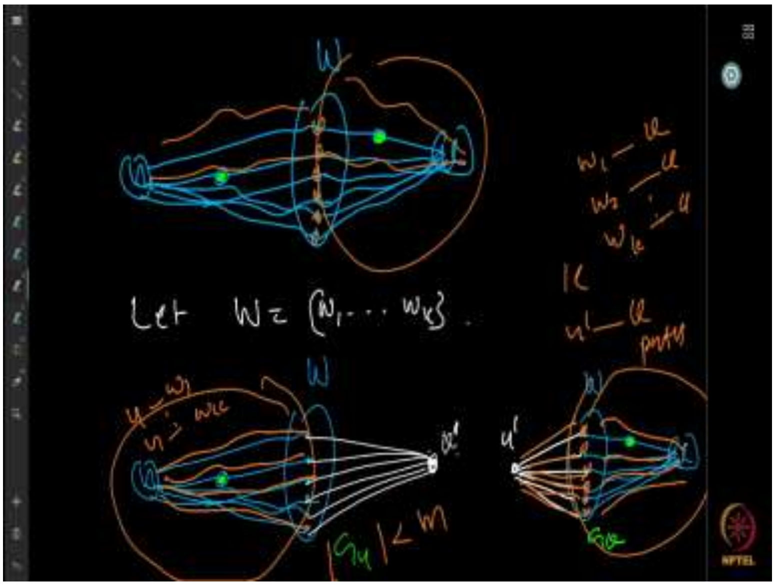
(Refer Slide Time: 26:28)



All it remains is our case 3 and since we already assumed that case 1 and case 2 are not there so, we can assume that you cannot find a separating set which is having a vertex which is adjacent to u and v and you cannot find a separating set, you do not have the property that every separating set has either every vertex is adjacent to u or adjacent to v. So, that is not the case because these two cases are not there, we will assume the following.

We can find at least one minimum u-v separating set that is a W such that no vertex of W is adjacent to both u and v. And that is the first case first case does not happen because in that case we will apply the first case and W has at least one vertex not adjacent to u because if every vertex of W is adjacent to u then again so, it will fall into the first case, if every separate set had this property, but because at least one separating set does not have that property, we can say that there is some W where at least one vertex not adjacent to u and at least one vertex which is not adjacent to v.

(Refer Slide Time: 27:58)



So, which means that we are in a case like this. So, we have u and v and W is a minimum u-v separating set and at least one vertex in W is not adjacent to u and at least one vertex in W is not adjacent to v. Now, we are in the third case. So, if this is the case, let us list our separating vertices, let us say $w_1, \dots, w_k$. What I am going to do is the following. So, since W is separating set, every path from u to v must cross W.

I cannot go from u to v without using some vertex of W. So, I am going to define a subgraph, by starting from the vertex u traveling paths and going through the vertices of W and the first time I hit a vertex in W in a path I just stop and I mark that path, add that path. Then I look at another path. So, I look at all the paths from u, which starts in u and ends in W where I stop the path the first time I crossed W. I do not want to go further after I reached W, I do not want to go any further. So, every paths starts from u and reaches W for the first time.

All such paths I take, so this forms a sub graph of the graph G. So, all the u to W paths, starting from u and ending in W where I do not repeat vertices of W. I take this subgraph and then I add a new vertex called v′. What I do is that I make v′ adjacent to exactly all the vertices of W, w_1 to w_k.

So, I take this subgraph u to W paths together with this new vertex that I am going to add that I call the graph G_u. G_u is not a subgraph of graph, but it is a graph obtained by adding v′ to this subgraph of G by the following way. Similarly, I look at all the paths starting from W and going to v without repeating the vertices of W.

So, all the path starting from here and going to v or I can just look at the path from v to W without repeating the vertices. So, collect all sets W-v paths and that subgraph together with the new vertex u′, which is adjacent to all the vertices of W. This defines the new graphG_v. So, I have the graph G_u and I have the graph G_v.

Now, the interesting property of G_u is that the number of edges in G_u is strictly less than the number of edges in G because we assume that there is at least one paths from some vertex to v. Since this vertex is not adjacent to v, you need to take at least two edges to go here. So, if in this graph, I have added exactly k edges from v′ to W but here I have at least k +1 edges because v to every W there is a connection and at least one is not a direct edge.

So, I need at least k + 1. So, the number of edges in this graph is strictly less than the number of edges in this graph. But of course, W is a u-v separating set, minimum u-v separating set because we did not remove anything in this part and to separate W, I need to remove each of them because again if I reach any other vertices in W, I can go to v′.

So, u-v′ separator W is a minimum separator for G_u but now for a smaller graph of cardinality strictly less than m, cardinality of G_u is strictly less than m, I can use induction. Using induction, I can find k internally vertex disjoint u-v′ path in the graph G_u. Similarly, I can find k internally vertex disjoint u′-v paths in the graph G_v.

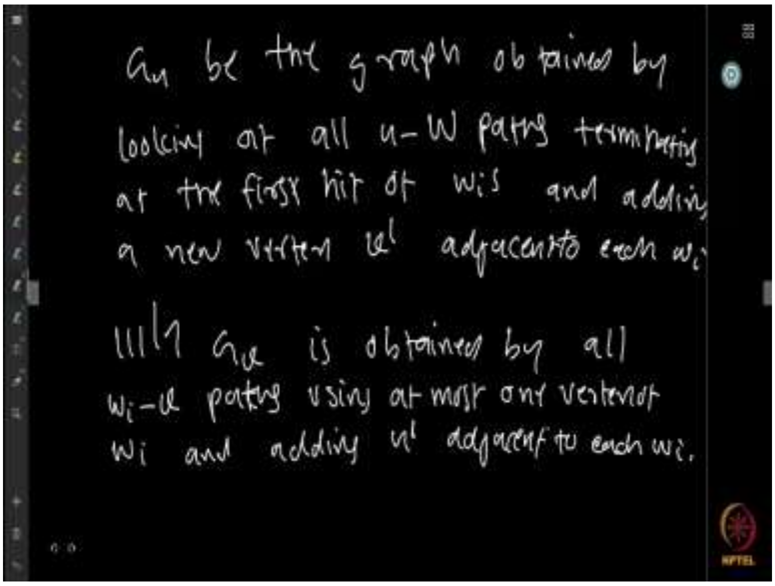
So, I have k, u′-v path which are internally vertex disjoint and I have k, u-v′ path which are also internally vertex disjoint. Now, it is just a matter observing that if there are such paths, those paths must use precisely one vertex of W because there are k such vertices in this graph, one vertex of W. So, these edges must be precisely the edges used to define this k path because there are only k edges here and each of them has to be used exactly once. So, therefore, what we get is k ivd path.

So, there is path from, w_1 to v, w_2 to v, w_k to v. So, these paths are in G_v similarly, I have a u to w_1 paths, u to w_k paths because these paths must be going through u to w_1 and v′, u to w_2 and v′, u to w_k and v′, u′ to w_1 and v etcetera. But now, these paths are all internally vertex disjoint similarly, these paths are also internally vertex disjoint. So, what I do is that I take the u to w_1 paths and join with the w_1 to v paths.

So, u to w_1 path in the graph G and in this, from this part I take the w1 to v paths. Now, that cannot be intersecting u to w_2 paths and w_2 to this path because we were very careful in selecting the subgraph from u to v that we do not repeat the vertices of W. So, therefore, once I reach W from here to hear there is no vertex or edge which is used on this path.

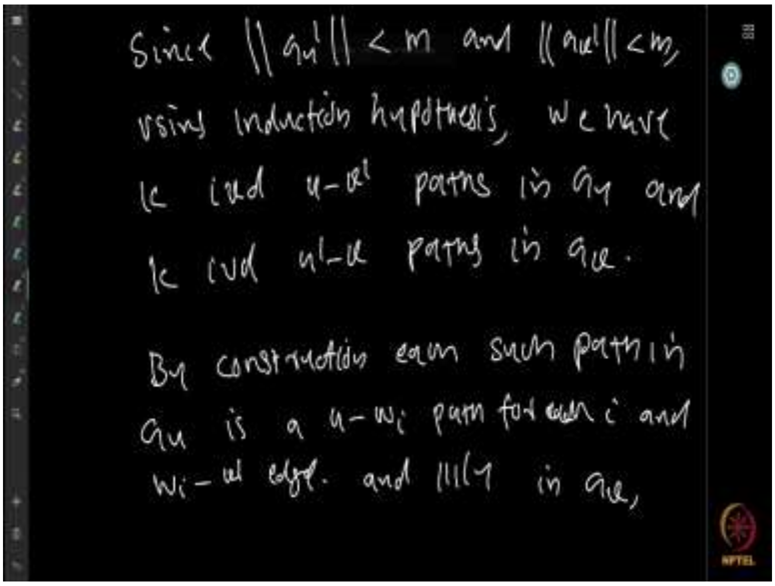
So, these parts of the graphs are basically disjoint except for the vertices of W. So, I get internally vertex disjoint path u to v through w1 through w_2 through w_k. So, I have constructed k paths from u to v in the graph which are all internally vertex disjoint. So, that is the proof of case three and therefore, we finished all the cases

(Refer Slide Time: 36:14)



So, G_u is the graph obtained by looking at all the u-W paths terminating at the first hit of w_i and adding a new vertex v′ adjacent to v that is the case I am just describing in words. Similarly, G_v is obtained by all w_i −v paths using at most one vertex of the w_i and adding u′ adjacent to each of the w_i's.

(Refer Slide Time: 36:35)



And since sizes of this graph G_u and G_v, are strictly less than m, using induction hypothesis we have k internally vertex disjoint u-v′ paths in G_u and k internally vertex disjoint in u′- v paths in G_v. By the construction of this graph G_u and G_v, each such path in G_u is u - w_i paths and w_i -v′ edge.

Similarly, in G_v we have this case and combining these paths you get the internally vertex disjoint paths in the graph. So, that is Manger's theorem. What we have proved is that if u and v are non-adjacent vertices in the graph G, then the minimum number of vertices that can separate u and v is equal to the maximum number of internally disjoint u to v paths in the graph. This is a very, very important theorem and we will use it now to prove some other results, some other important theorems.