Combinatorics
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Product of Graphs

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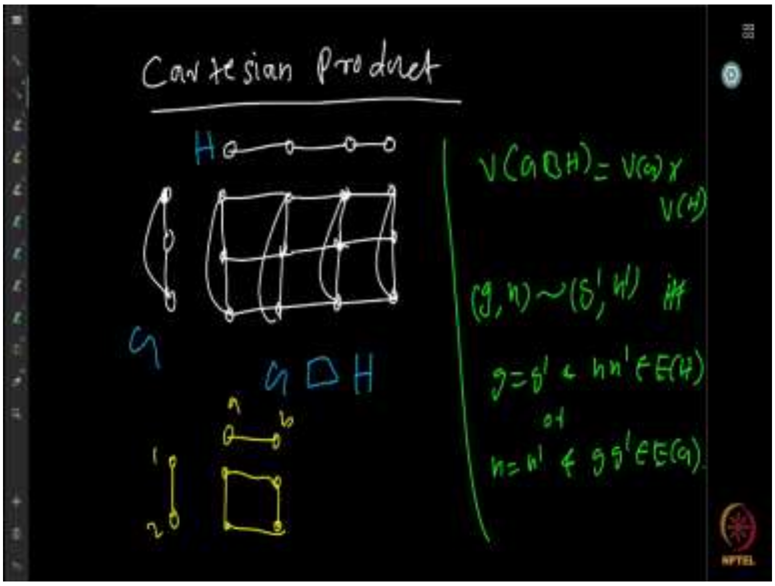

Welcome back to this course on Combinatorics. So, we continue with our topic in graph theory. Today, we will talk about some very important results. Before going into that, let me introduce a couple of product notions in graphs. So, the first product that we are going to see is called the Cartesian Product.

So, given two graphs, let us say G and H, we want to look at the product of the graph G and H and this product is defined on the Cartesian Product of the vertex sets of G and H. So, the vertex set of the product graph is the cartesian product $V(G) \times V(H)$. So, this is the ordered pairs from vertices of G and vertices of H, where the first component is from vertex set of G and the second component is from the vertex set of H. So, that is the vertex set of the product graph.

Now, how are the edges defined? There is an edge from let us say a vertex (g, h) in the product to (g′, h′) if and only if either g = g′ and hh′ is an edge in the graph H or h = h′ and gg′ is an edge in the graph G. So, if either of this happens, you have an edge from (g, h) to (g′, h′).

So, if you look at this product, we will see that the if suppose your graph G is just one edge and your graph H is also this one edge. So, what happens if there is an edge in G and there is an edge an H. So, what happens is that, in the product graph, this will be the corresponding points.

So, this is let us say {1, 2} and {a, b}, then you have d {(1,a), (1,b), (2,a), (2,b)}, so, the edges are going to be like this.

Basically, you will get all this four edges. So, this is why it looks like a box and the symbol for the product is basically a square, a box '$\square$'. This is the symbol that we use the box symbol and that box tells you how to define the product I mean like, whenever you have an edge in the graph G and an edge in the graph H, the corresponding endpoints so, in the product you can take and then you know how to put edges. That will tell you how to do it for the entire graph and that is the reason this symbol is used to represent the product. The cartesian product of G and H is defined like this.

So, here is an example of a graph on three vertex cycle G and four vertex path H and the product graph. So, if you look at the product, you can see how this is. So, you have several copies of G you can see right and you can see also see several copies of H if you look at it like this. So, these are basically copies of H. So, if you think about this product, take this definition and look at how this edges are going to come in the product, you will see that this product can be thought of as obtained in the following way.

You take the graph G and you replace every vertex of G with a copy of H. So, here you take the graph G and replace every vertex with a copy of H like this. Now, the corresponding vertices in H we will have an edge if and only if there is an edge in the graph G. So, between this copy and this copy right there is an edge, there is an edge from G, from this vertex to this vertex. So, the corresponding copies, the vertices corresponding to this, they will have an edge between them.

And you can also see that like, taking the graph H replace every vertex of H with a copy of G and then do the same thing right, put an edge whenever there is an edge in the H. Basically, it is a symmetric product. So, you can see that $G \square H$ is isomorphic to $H \square G$. So, this is something that you can prove in as a homework.

So, that is the cartesian product. So, now, given such a product, we can ask several questions because all the parameters that you have studied in graph theory we can try to ask like what happens to this parameter when I take the product of two graphs. So, given the parameter for the G and the same parameter for H. What will be the parameter for the product graph?

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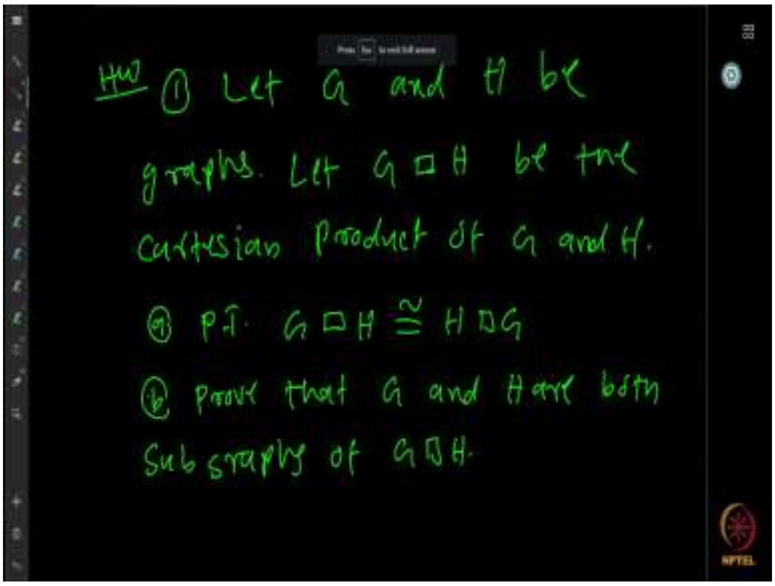

Now, here are some basic questions that I want you to try to work out as homework questions. So, here is the first question. Let G and H be graphs and G□H be the cartesian product of G and H. Now, prove that G□H is isomorphic to H□G. The second question ask you to prove that if G and H are to show that G and H are both sub graphs of the product graph G□H. We observe this in an intuitive fashion, but we have to prove it formally from the definition right. So, here is a definition of the product.

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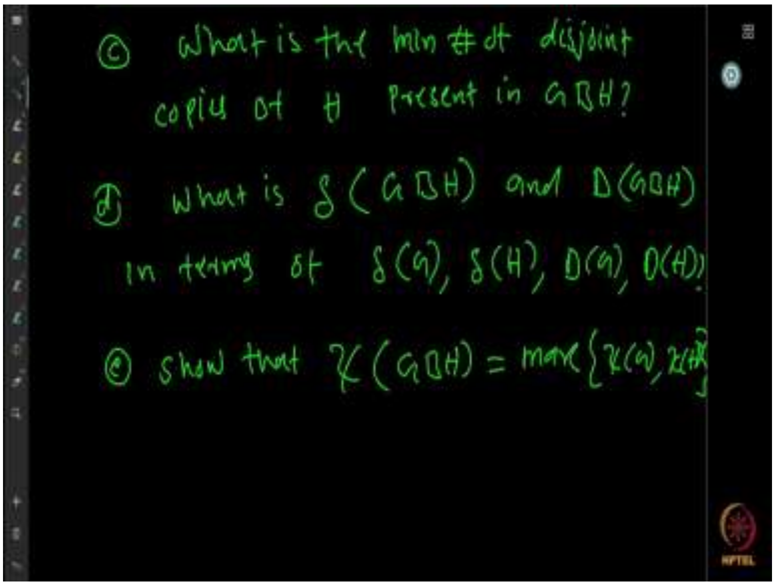

And the third question is that suppose, you are given this product graph, what is the minimum number of disjoint copies of H present in the product. Again, it should be clear from the picture like we gave, but you have to prove it formally. Then the fourth question is that the small delta (δ) of a graph is the minimum degree, as you remember and capital delta (Δ) is the maximum degree of the graph. Now, you want to find these parameters for the product in terms of the parameters of the component graphs.

So, G and H are the components. So, can you say these two parameters in terms of the δ(G), δ(H), and Δ(G) and Δ(H). And finally, you can show that the chromatic number of the product graph is actually equal to the maximum of the chromatic number of the constituent graphs.(chromatic number is the minimum number of colours that suffices to colour the vertices of a graph such that adjacent vertices does not get the same colour.)

You take the chromatic number of G and chromatic number of H, what is the maximum with that many colours you can colour the product graph. These are kind of immediate questions that you can ask and they are not very difficult to prove. I would like you to work out this as your homework questions.

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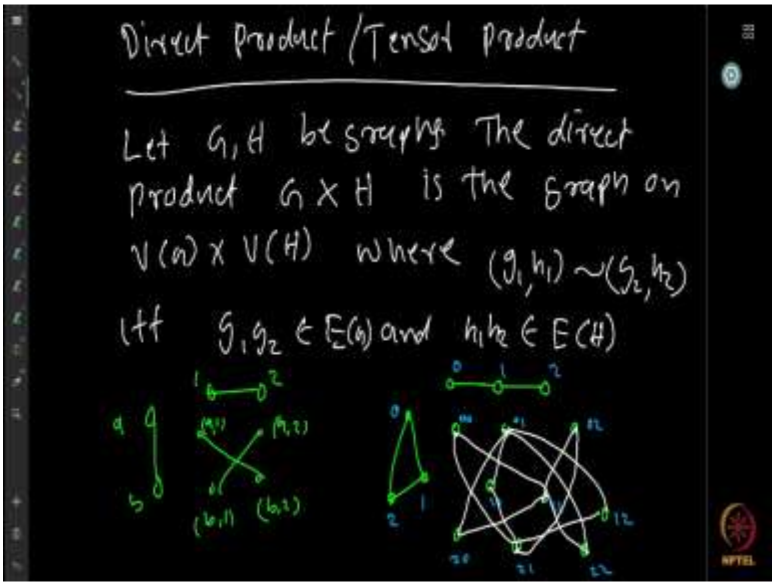

Once you have the Cartesian product, we can also define a few more other products. So, I will define one more product which we call the Direct Product or Tensor Product. So, given two graphs, let us say G and H, the direct product G $\times H$ is again defined on the cartesian product of the vertices, V(G) $\times$ V(H) and the adjacency is defined as follows. So, if (g_1, h_1) and (g_2, h_2) are vertices in the product graph, then, there is an edge from (g_1, h_1) to (g_2, h_2) if and only if $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$.

So, if you take the product of two edges for example, as we looked at another case, you will see that so, you have this edge here and the edge here, in the product you have this cross edges. You do not have the other four edges that they had for the direct product I mean for the cartesian products but you have the these two edges the cross-edges and this you can see is very different from the previous product that we looked at and the symbols that we use again denotes how they edges are defined.

So, if you have an edge here, an edge here what happened to the edge in the product is given by the symbol of the product itself. Now, here is an example of the product of a cycle and a path of length three. So, you will see the graph looks like this. So, just look at the definition and see how these edges 00 to11, 20 to 11 and 20 to 01, 01 to 12 all these edges are there. So, one can feel that you know like you do not see directly the copies of the graph H and G here, now you can ask whether they are present and in what cases they may be present or may not be present. Such questions one can ask.

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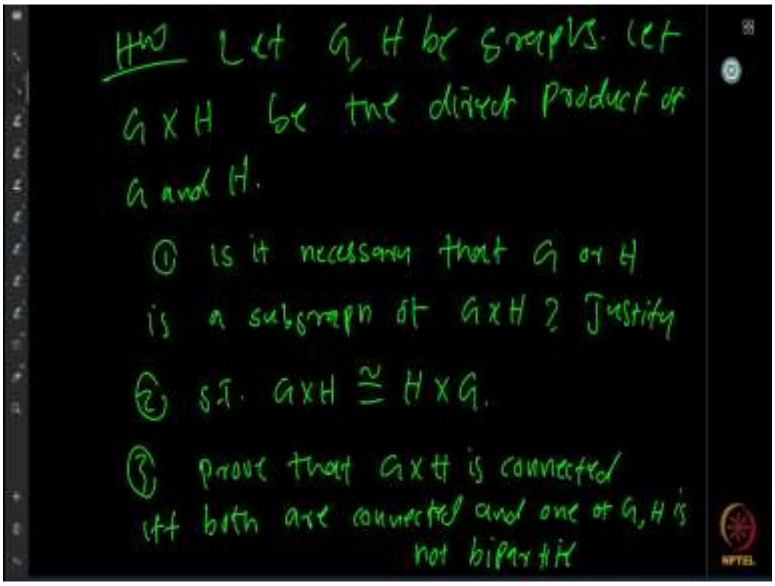

So, here are some questions for you. So, if G × H is the direct product, then first question is that is it necessary that G or H is a sub graph of the product graph. So, if it is yes or no you want to give a justification. Then the second part is to show that the product is basically like if you take G × H is isomorphic to H × G and also to prove that the product is connected if and only if both components are connected and one of the components is not bipartite. If both are bipartite, then the graph is not connected. So, this is something that you should prove.

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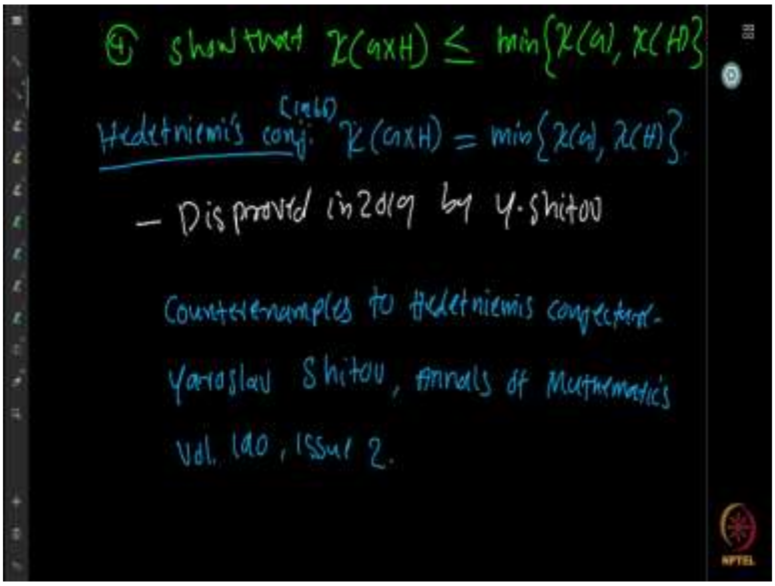

And, then the fourth question is to show that the chromatic number of the product graph is at most the minimum of the chromatic number of G and chromatic number of H. So, in the earlier cartesian product, we actually proved that they chromatic number of the product is actually the maximum of the chromatic numbers of G and H. Here it is saying that it is at most minimum of the chromatic number of G and H. Now so, one of the reasons I introduced this product is, this conjecture it has been open for several decades before it was proved in last year 2019 by a very young Russian mathematician called Shitov.

So, the conjecture due to a famous mathematician called Hedetniemi states the following that the chromatic number of the direct product of G and H is actually equal to the minimum of the chromatic number of G and chromatic number of H. So, I asked you to prove that it is at most a minimum. Now, the question is that is the minimum always required. So, if the chromatic number of G and H are given, can you say that in the product you always need the minimum of these two.

So, a Hedetniemi conducted that this is actually equal to the minimum. It cannot be strictly less, but it was disproved two years before. So, this paper is a very short paper actually, the content of the paper is just one and a half pages. But it appeared in analysis of mathematics because of its very high importance. And there have been several attempts to prove this in this several by 6 decades and there were many progresses and each of them tried to prove the conjecture proof for certain classes of graphs etc. And but it is false, the conjecture is in general, it is false.

It could be true for several subclasses but you can find counter examples of the conjecture. So, this is a very interesting paper, very short paper and very cryptically written and I recommend interested students to take a look at this paper and try to read it and see how much you can follow and you know the proofs is not really very difficult. It is just written in a very short manner.

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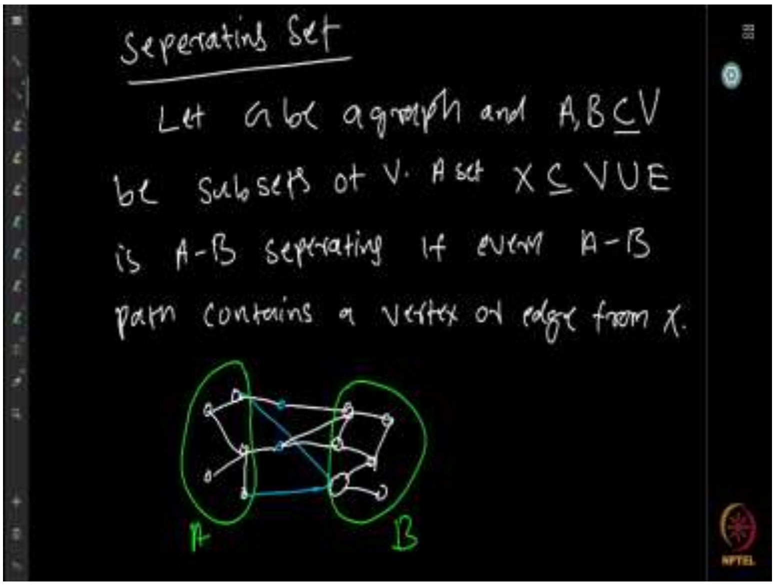

Now, we go to a few more definitions. So, given a graph G and two subsets of the vertex of let us say A and B are given. Now, we talk about the paths that go from the vertices in A to vertices in B. So, the path from A to B are basically the path which have the starting point is in the A and ending point is in the set B. Now, once you consider this AB paths, suppose you can find a subset of vertices or edges actually. So, subset of vertices and edges such that every AB path must pass through one of these vertices and edges.

You can always find this by looking at all the paths and then finding a set, suitable set so that every path from A to B must pass through one of these vertices or edges. Then such a set is called a separating set. So, if X is a subset of V ∪ E, such that every AB-path contains some vertex or edge from X. We call X as a separating set for AB, A-B separating. So, why is it AB – separating? because if you remove X, then there is no path from A to B. That is the idea.

So, the idea of separating set is important because it basically talks about, for example, you are talking about let us say the graph is going to represent connection between very important centres of let us say military intelligence and then or military stations and the edges basically represent the connections or communication channels between this centres and then a separating set is basically the nodes or the paths or bridges or whatever connection roads that can be critical in the sense that once you are able to if an enemy is able to destroy this, then the communication or contact between these centres will be lost.

So, therefore, and this can come in many, many situations not just in military planning, but it can be in communication networks, it can be in in real life networks or many other situations, but each of them basically has this abstraction that you can basically find a subset whose removal disconnects the communication between these two sets. Now, when you want to make your network very robust, you want to make sure that there is no very small subsets. So, you want to ask about such questions.

So, the notion of separating set is very important. So, here is an example in the graph given here that you have the set A. Set A is the set of vertices under this circle and here is another circle B of vertices, then the blue vertices and blue edges form separate set. So, if you just look at the blue edges and blue vertices, if you just remove these edges and these two vertices, then there is no connection between A and B, that is immediately clear. So, therefore, this is the separating set for A and B.

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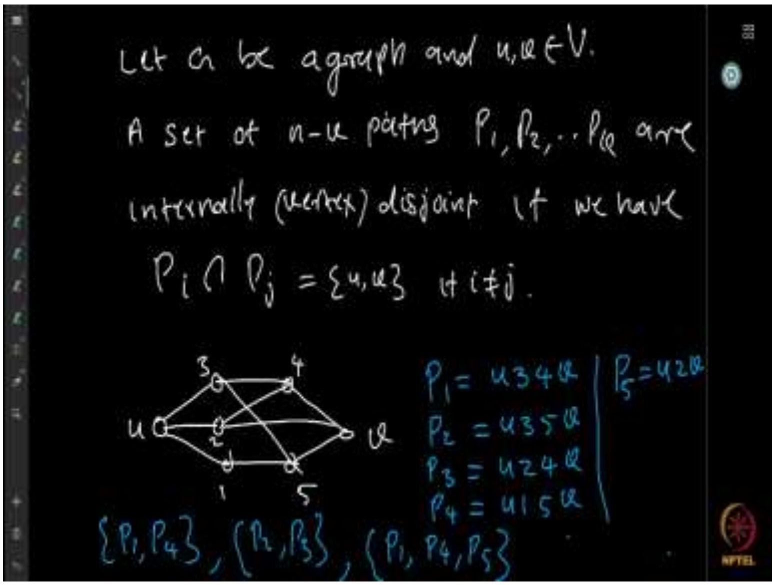

Now, a related notion is of internally disjoint paths. We will see why. So, let G be a graph and let us say U and V are vertices of the graph G. Now, a set of uv-paths okay so set of paths starting from u and ending in v that is a P_1 to P_k are said to be internally vertex disjoint or I will usually say internally disjoint or 'ivd' some times. If we have the property that for any two paths P_i and P_j where i ≠ j, the intersection does not contain any vertex other than u and v. So, only the starting vertex and the ending vertex can be common, everything else is different for any two paths.

Then they are called internally vertex disjoint. For any pair of paths, you should have the property that $P_i \cap P_j$ contains only u and v. Then we say this set of path are internally vertex disjoint. For example, in this graph here, look at the path u to 3, 3 to 4 and 4 to v. So, this is one path, then you cannot take for example, these vertices to be path of your another path if it is going to be disjoint. For example, I can take u to 1, 1 to 5 and 5 to v. These two paths are internally vertex disjoint because u and v are the only vertices.

On the other hand if you take for example, let us say u to 2, 2 to 4 and 4 to v, this path is different paths from u to 3, 3 to 4 and 4 to v but they are not internally vertex disjoint because the vertex 4 is common to both. If you look at this graph, you can find several other pairs right P_1, P_4. So, P_1 is this path, u34v, P_2 is u35v, P_3 is u24v, P_4 is u15v and P_5 is u2v.

If you look at these paths, then P_1, P_4 is internal vertex disjoint. P_2, P3 is internally disjoint. And P_1, P_4, P_5 is also internally disjoint because if I take $P_1 - P_4, P_4 - P_5, P_1 - P_5$ they all have disjoint vertex set except for u and v. So, they are all internally disjoint paths.