Combinatorics

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Mycielski Graphs

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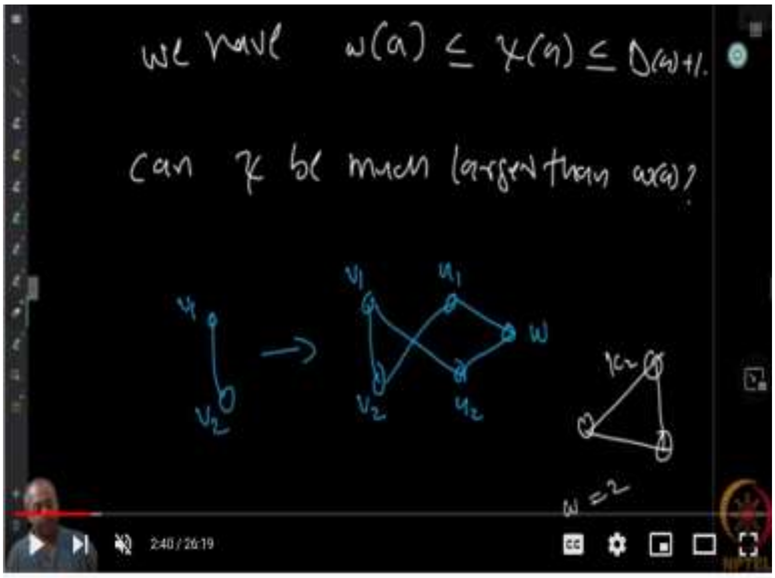

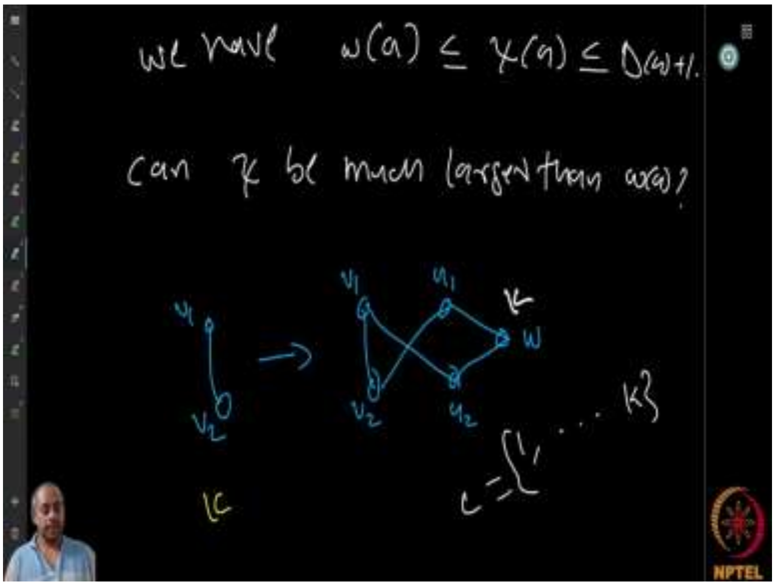

So, what we have observed this much so far is that the there is a lower bound that we found which is $\omega(G)$, the clique number and an upper bound which we found which is $\Delta(G) + 1$. the upper bound with the chromatic number. So, the chromatic number is between these two. Now, we can ask many questions like when is this lower bound attained? It is an interesting question there are several classes that we know attain this, but we still do not know universally when. Then can χ be much larger than $\omega(G)$.

We know that, the cliques will make the chromatic number large. If you have a large clique with let us say with, t vertices then we need at least t colours. Now, can you have t to be small, the complete graph omega to be small and then make the chromatic number much larger than ω, the clique size. So, because in the case of odd cycles, the Δ is just 2 but that does not say that enough for very large numbers, chromatic numbers we can have property, is it possible. So, here is an interesting result which says that we can have graph without triangles, where the chromatic number is as large as we want.

What do you mean by Triangle? Triangle is three vertices which are adjacent to each other, which is a complete graph on three vertices. Now, if I have several vertices and if there is no triangle, then I cannot have any complete graph more than complete graph on two vertices and edge because if you have any larger complete graph every vertices are adjacent so, therefore, you will automatically get a triangle as a subgraph. So, when there is no triangle, we know that the clique number omega is going to be 2. So, what we are saying is that we have graph with omega is equal to 2 and chromatic number as large as we want and how do you do that?

So, of course there are many ways to do this but one famous result was, the construction that is given by Mycielski. So, Mycielski constructions shows that if you are given a graph which does not contain triangles and have some chromatic number let us say k, then I can make a graph from this graph a new graph whose chromatic number is strictly larger than k, in fact, it is equal to k + 1 and there is still not triangle. So, I start with a triangle free graph, I make a new graph the chromatic number increases by one without getting triangle. Therefore, I can keep on doing this so I can make the chromatic number larger and larger.

So, what is the procedure? The procedure is the following. Here is a small example. Given this graph, I take the graph and keep it as it is, one copy. So, I get a copy of the graph. Now I make a duplicate of every vertex. So, duplicate of vertex means that, there is a copy of v_1 which I call u_1 and there is a copy of v_2 which I call u_2 etcetera. So, I make this the copies of this graph. And then I make the neighbours of u_1 are precisely the neighbours of v_1. So, v_1 has the neighbour v_2, so therefore u_1 has a neighbour v_2, v_2 had neighbour v_1, therefore u_2 has neighbour v_1.

Whatever is the neighbour of v_1 in the graph G, that same vertices are going to be the neighbours of the corresponding vertex here also. So now whatever graph I get from here, finally I am going to add a new vertex called w, and w is adjacent to all the new vertices that I introduce, whatever is the number of vertices here. I am going to make w adjacent to each of them. Now, u_i's form an independent set, that is immediately clear because you can see that u_i's there is no edge between them because they are not neighbours of any of these vertices in this graph. So, therefore, they form an independent set.

So, when I add w to these guys this is not going to create any triangle because if it creates a triangle that must be some edge here. Now, similarly, there was no triangle in this graph. So, now there is no triangle here now, can this create a triangle? Now u_1 is adjacent to to all the neighbours which were copies of the neighbours of v_1. So, if you look at the graph with this u_1, this u_1 cannot be part of a triangle because u1 is precisely like v_1 in the graph. If you look at and check the graph with addition of u_1.

If there is a triangle involving u_1, there is a triangle also involving v_1 because that is a triangle in this graph only. Any triangle here is also a triangle here. So, this is true for every single vertex and because there is no edge here, we do not have to look at two of them together and therefore, I show that this graph is triangle free. So, I get a triangle free graph by doing this and then now, I want to show that the chromatic number also has increased by 1.

Now, how do you show this? To show that the chromatic number is increased by 1, we have to first show that there is a colouring with one more colour and we have to show that there is no colouring with the chromatic number of G many colours. So, how do I do that? Here is an idea. Let us first show that with the k + 1 colours, the colouring is possible. That is easy because I start with a colouring of the graph G, the first graph that I started with.

Now, whatever is the colouring here, each vertex let us say v_i has some colour, I use the same colour to colour u_i also. Now because there is no edge between u_i and v_i, I can give the same colour here no problem and this colour of course is okay to give because this vertex is adjacent to only the neighbours of v_1 and v_1 I have given this colour C and because all the neighbours of v_1 gets different colours other than C, all the neighbours of u_1 also get the colour differently. Therefore, u_1 will not have any problem with the colouring.

So, this I can do for any u_i. So, therefore, I can use the colour of v_i to be equal to the colour of u_i. So, I get all these vertices, the same colour that is happening here and now for w, I give a new colour whatever its the chromatic number plus one. So, one more colour I give and that new colour. Because it is a new colour, it will not create any problem with any of the the existing colours and therefore I get a proper colouring with 1 extra color.

Now we want to show that, there is no colouring with chromatic number of G many colours. So, let us say that chromatic number of the graph is k, then I want to show that there is no k colouring of the new graph. Now why there is no k-colouring of the new graph. Suppose there is k-colour, so we will start by assuming that there is a k-colouring of this new graph.

Now, suppose there is like a k-colouring of this new graph. Then it uses this k colours atmost this k colours 1 to k on the vertices. Now given any colouring, so this is one property of colouring, given any colouring let us say using let us say colour red, blue, green etcetera. I can always change the names of the colours. I will say that all the red vertices now I am going to call new green and all the green vertices in the original graph, I am going to call it as red.

So, I just changed the names of red and blue to each other. So, red is now the new blue I mean the new green and green is the new red. So, this is just renaming the colours, it does not affect anything about the properness of the colour because earlier it was called red, now it is called blue or green whatever.

So, therefore, I can always change the names of the colours as far as I change uniformly everywhere. All the red I changed to green and all the green I changed to red. Then it is okay. So, therefore, we can assume without loss of generality that the vertex w is coloured with the colour k, just to make the argument easier. So, we will assume that the vertex w is coloured with the colour k.

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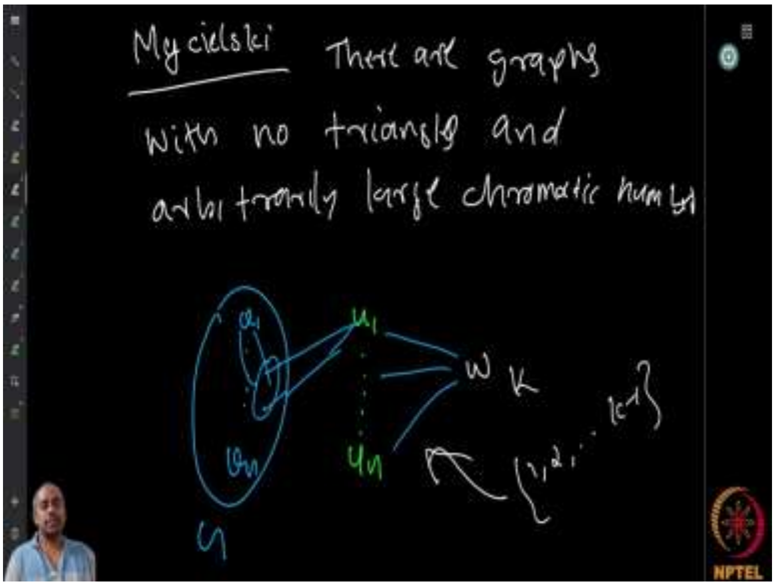

So, here is a better picture I think. So, we have this graph G and u_1 to un and u_i is adjacent to all the neighbours of v_i. So, u_i and v_i are basically clones of each other, exactly the same vertex. Just one copy of v_i. And u_i's are independent and then w is adjacent to all of the u_i. So, now, to show that there is there is no k-colouring I will assume without loss of generality that w is coloured with colour k. Now, because w is adjacent to all the u_i's, u_i's cannot be coloured with colour k. So, therefore, u_1 to u_n must be coloured with colours - 1 to k - 1, this must be the colours that is used on u_i because k cannot be used here.

Now, if 1 to k - 1 is used on u_i I claim that I can use the same 1 to k - 1 to give a colouring of G also. Why is that because, see u_i, whatever colour is for u_i that is adjacent to the vertices here. So, the colour of ui is not going to be used on the neighbours of v_i because u_i and v_i share the same set of neighbours. So therefore, since u_i is given some colour whatever the colour it is, that colour can be given to v_i also because the neighbours will use different colours.

So, this I can do for every u_i which means that the colours used here can be exactly used on the colours v_1 to v_n but this says that I am using just 1 to k- 1 to colour all the vertices of the graph G. But G was a graph with chromatic number k, it means that there is no k - 1 proper colouring. So, this contradiction proves that, there is no colouring where, only we are using k-colours. So therefore, the chromatic number is at least k + 1. We already proved that it is actually at most k + 1. So therefore, the chromatic number increased by exactly one.

So, we get a graph from triangle free graph, we get a new triangle free graph whose chromatic number increases by 1. Now I can start from this new triangle free graph and increase the chromatic number again by one. I can keep on doing this. So, I get larger and larger chromatic number. So, this is called Mycielskian construction, and such graphs are called Mycielskian.

So, given a graph G, the Mycielskian of G is the graph we construct this way. Take the graph, make copies of each of the vertices, and which means that exactly the same numbers of vertices in the graph G and because it is the new independent set, I make this new vertex adjacent to w and then I get this new graph G. So, this is the construction.

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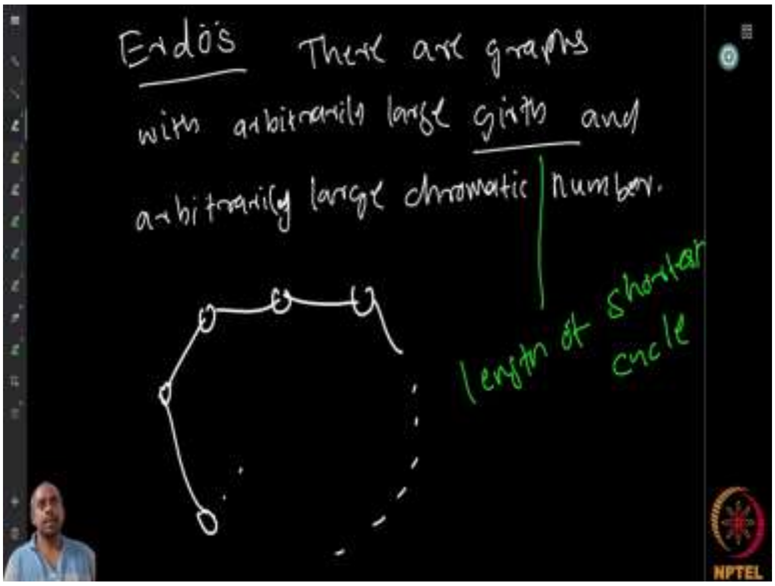

Now, there is another famous result of Paul Erdos. It says that we can have graphs with arbitrary large girth and arbitrarily large chromatic numbers. So, what is the girth? So, the girth of a graph is the length of the shortest cycle. So, we say that it is not just triangle, we can make your cycle as large as we want. The smallest cycle is as large as we want. So, there is not going to be any other edges connecting this. This cannot be an edge connecting because cycle will become smaller. So, you will have all the cycles be large enough in the graph.

So, as large as you want you can make the smaller cycle and then you can make the girth I mean the chromatic numbers also as large as we want. So, this is using a probabilistic method. We will probably not discuss the proof but we will learn the some of the tools required to work out this details. So, this is a very old result of Erdos and very influential one and you will see that this result precisely the result that Erdos proved, was used to prove the Hedetniemi's conjecture. May be I will mention it soon.

So, this is a very influential result and very famous result. And it has so many applications like on existential graph theory, where you show that you do not tell you how to come up with a graph like this but it will tell you there must be some graphs.

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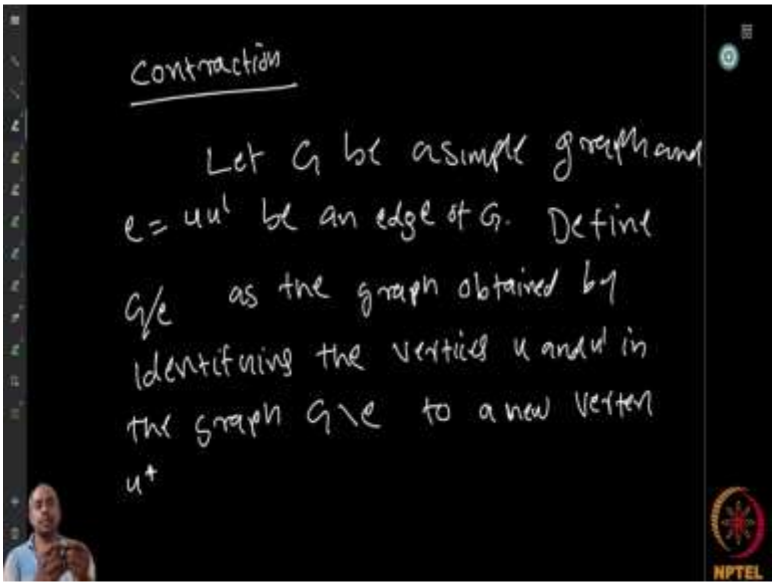

Now, a very important operation in graphs is called contraction of edges. So, the definition is as follows. So, given a graph G, it is a simple graph and some edge let us say uu′. Now what I do is that I take the graph G and delete the edge e. So, just remove the edge e, then identify the vertices u and u′ to make a new vertex, let us say u*.

I am not doing anything to any other vertex the adjacency remains the same for u, all the neighbours are going to be there except the one that we just removed and for u′ all the remaining neighbours are as it is. Now I basically identify this u and u′ which becomes a single vertex. So, all the neighbours of u other than u′ will be there in the new graph. All the neighbours of u′ other than u will also be neighbours of this new vertex. So, it is new graph. This is called the contraction of the edge.

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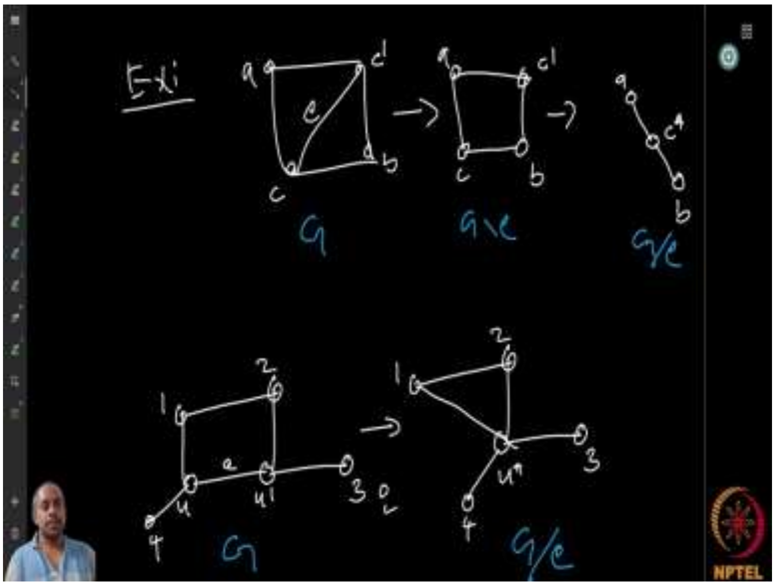

Here are some examples. I start with this graph, then I find the graph G\e by removing this edge that we fixed. So, e is the edge which is cc′. So, I remove e, I get this graph. Now, I identify c and c′ which means that c and c′ becomes vertex c*. The neighbours of c and c′, c had the neighbour b, c′ had neighbour b and they are still neighbours.

Then c also had neighbour a, c′ has neighbour a, so c* will have neighbour a. After contraction of this edge I will get this graph G. I denoted it by G/e. So, this is one example. So here is another example. So, I start from this graph, I take the edge u u′ as the edge to contract. So, it means that I delete this edge. Now I identify u and u′ to u*. So, of course the neighbours of u which will be 4 and 1, neighbours of u′ which is 2 and 3 will be neighbours of u*.

Remaining neighbourhood remains as it is and that is it. So, you get G/e. So, this is the contraction of process. Now contraction is very important. We will not go into the details of it, because, you start with any graph G and you can apply as many as deletions, contractions or vertex deletions also possible if you want. Then you will get some kind of graph from the started graph. So, these graphs that you can obtain by any sequences of edge contractions, deletions and vertex deletion are called minors of the graph.

So, there is a huge area called minor theory which studies like you know to classify graph based on its miners, what are the properties many important results have improved using this. So, there is a whole set of courses one can offer based on this minor theory, not one course, several courses. So, it is like the huge area and I really have not looked much into minor theory but it is a very interesting area with many applications.

There are some few things we will see in a graph theory course in minor theory like, things like classification of planar graphs using its minors and things like that. So, these things we will not discuss in this course, we can do this in an elective course on graph theory. So, I will not discuss minors for the time being. I just defined it because, we need this operation to look at something else. So, what is that?

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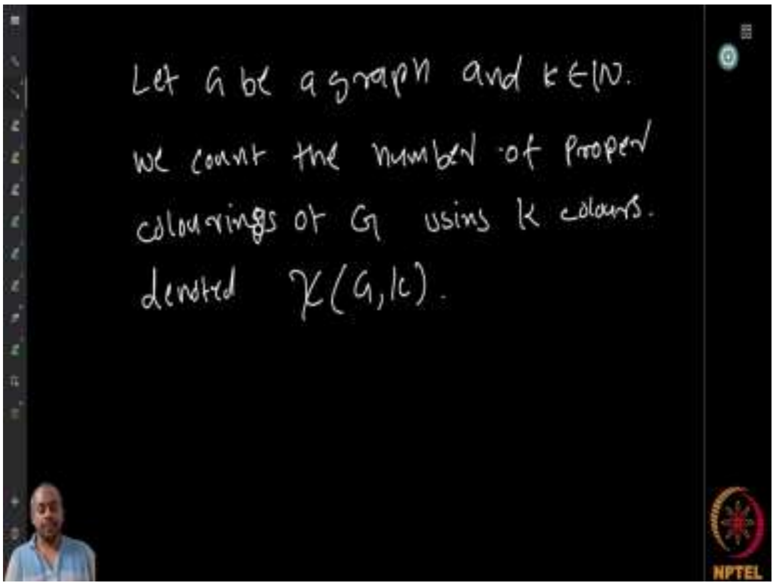

So, given a graph G. So, first thing that we said that what is the chromatic number which is the minimum number of colours which suffices to colour the vertices of the graph such that adjacent vertices get different colours. Now, another combinatorial question that one can ask is given a graph and let us say a set of colours, let us say I tell you I am giving k colours because all the sets of same cardinality are same for us.

We can just assume that, for the set once you know the cardinality, you will just assume the numbers from 1 to k are the colours. So, given a graph G and a number k, I can ask how many k-colourings how many colours, how many different colourings of the graph are possible with using exactly k-colours? So, the number of k-colouring of a graph G. So, this number is a $\chi(G)$, k. So, the graph G is given and then a number k is given and the question is that what is the number of colourings of G using k-colours.

For each k you can find this and for general k you can write it as a function. We can say this is a colouring function. If you assume k to be a variable or replace it by x. So, then I will get the chromatic function of the graph. Eventually one can show that this function is always a polynomial and therefore, its chromatic polynomial. It is a nice exercise using some of the techniques that we learned but let us not go into that. So, given a graph, we want to find out the number of proper colourings using exactly k-colours.

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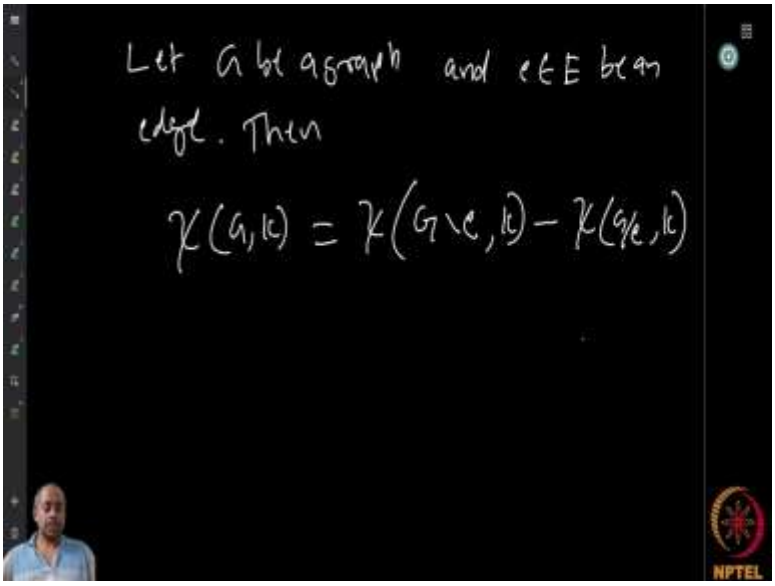

Now, my claim is that this parameter $\chi(G, k)$ satisfies the following identity. So, given a graph G and any edge e in the graph, we will assume that all the graphs are simple for the time being, even otherwise many of these hold but we will not work with that assumption. So, given a graph G and an arbitrary edge e, then $\chi(G, k) = \chi(G\backslash e\ , k) - \chi(G\ /\ e, k)$

Similarly, for any graph. So, I want you to think about this and try to prove his identity as well.

If you think about what is the proper colouring and what exactly happens when you take the contraction or edge deletion, you can immediately come up with this identity, it is not a difficult thing, just think about what is meant by proper colouring and what happens in G contraction e and what happens in G\e.

What is the difference between these two graphs and what is the similarity between these two. This will allow you to give a proof for this. So, I think that is all we have for today. And we will look at a few more things in the next class and then we will go to some other topics for this course.