Combinatorics

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Graph Colouring

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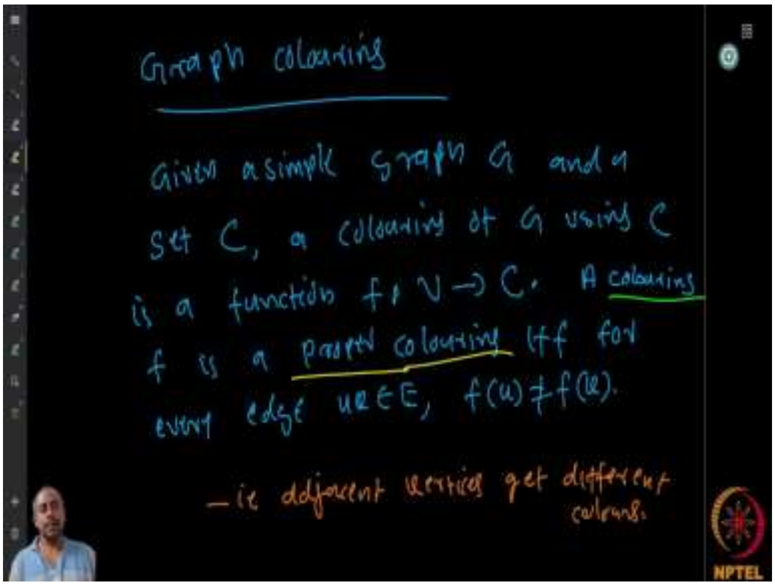

Now, let us look at, so we will look at the remaining question maybe later or you can work out this detail if you learn more sophisticated techniques. So, here we look at an introduction to graph colouring. So, graph colouring is a very useful subject. It is one of the major areas in graph theory. Because there are so many different versions of colouring and these colouring can represent many different problems from real life. So, there are so, many kinds of questions that arises which can all be converted to graph colouring.

So, this has become a huge area in fact major part of graph theory is now graph colouring. So, I did my PhD in graph colouring for example during my PhD time and there are many other related colouring notions. We will look at one for the time being and maybe later we will look at another related notion but there are dozens or even hundreds of colouring notions etcetera and then you can we can add many many constraints to the question that we ask. So, here is the general idea of colouring.

So, colouring basically a fancy name for partitioning of the vertex set. So, given a simple graph G and let us say a set C, a colouring of G using the set C is a function f from V to C. So, that is all colouring. Now, a colouring is so, most of the time when we use, we are going to use this.

A colouring f is called a proper colouring, if it gives different colours to adjacent vertices. So, if u and v are adjacent in the graph, then f(u) must be different from f(v). In that case, the colouring is proper otherwise the colouring is not proper. So, even if it is not proper there are application for that. But most questions will look at proper colouring. We will see many versions of proper colouring if you will look at.

So, basically a vertex colouring is what we immediately when we say graph colouring, it is the first thing is the vertex colouring. There are other notions of colouring, edge colouring and face colouring, there are many things we can do. And these all depends on the cases but when we say colouring, it is usually vertex colouring. And most of the time we will just write colouring to say that proper colouring of the vertex where the adjacent vertices get different colours.

Now, if we are using a colouring which is not proper, we will mention it exactly. So, if I just say ‘colouring’, without mentioning anywhere that we are not looking at proper colouring, it is assumed to be proper colouring. This is the convention that we make.

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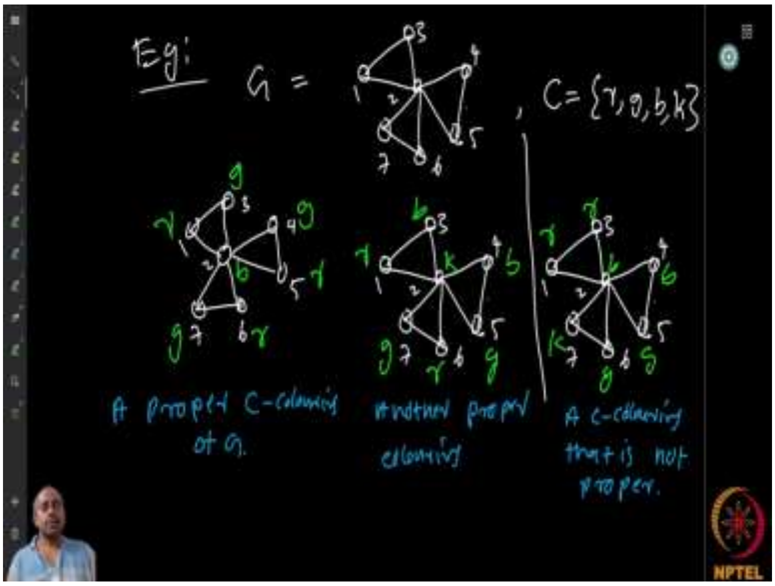

So, here are some examples. So, we have a graph G with some colourings here. So, let us say set C = {r, g, b, k}. I am going to colour using these four possible colours, Red, Green, Blue and grey. So, what are the colourings possible? Like let us say, here is the proper colouring of G using the colours in C where I colour the central vertex in b, then these vertices cannot be coloured b, because I am looking at proper colouring.

So, this must be different from b, it must be let us say g. This must be different from b and g because they are adjacent to each other. So therefore, this is all different three colour. But on the other hand, I can use g again here because these are not adjacent. So, I use the same three colours to colour all the vertices I do not even use the black. So, with just three colours I coloured the entire vertices, it is that proper colouring, it uses only colours from set C.

So therefore, it is a proper C-colouring of G. Now here is another proper C-colouring of G so I use the colour black also. So, k is used in the middle in this case and then I used r and b here. I can either use r and b or b and r here b and g and whatever I place. So, I get another proper colouring. Here is another C-colouring but which is not proper.

So, I use colour b here I use b and g here and I have used g and k here, but now I use r and r here but r and r are adjacent therefor it is not a proper colouring but it is a C-colouring. So, we have this kind of all kind of colourings we can look and see which are the ones which are proper. Then we can ask, can we count the number of proper colouring? What is the minimum number of colours that will give you a proper coloring? So, these are interesting questions.

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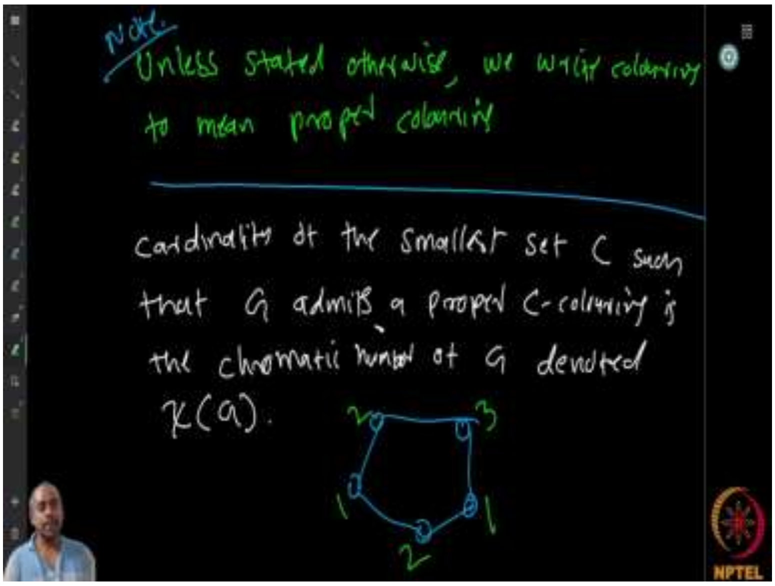

We will look at the graph colouring. So, again as I mentioned before unless stated otherwise when I write colouring, it is proper colouring. Now, the smallest integer that is a cardinality of a set C, such that the graph G admits a proper C-colouring is called a chromatic number of the graph G it is denoted by $\chi(G)$. So, chromatic number is the smallest number of colours that will allow a proper colouring.

For example, let us say that I take this graph or maybe this graph. Now, what is the minimum number of colours that will allow to colour this graph? How do I argue this? Suppose I start with like colour 1, 2, 3, 4 etcetera so that I can use the minimum number. So, I start with the colour 1, so I colour this vertex with 1. Now, I know that because I coloured this with 1, I cannot use colour 1 here. So, it must be different from 1 so, therefore, I can use 2.

Now, there is a question whether I want to use colour 2 here or I want to use a new colour here. I do not know which is the best but for the time being, I will say that okay I can reuse the colour because I want to minimize. So, let me say that I am only using two colours. So then, in that case I have to use 2 here because I cannot use 1. So, I can safely assume that I am using colour 2 here.

Now since I have used colour 1 here and colour 2 here, I cannot use colour 2 here. Using only two colours, I have to use one more I mean I have to use the colour 1 here because I cannot use colour 2. Now because this is 1 and this is 2. So, because it is 2 well here it is 1 and here it is 2, I cannot use 2 here. So, I have to use maybe 1 but I cannot use 1 here because I have already used 1 here. So, therefore we know that we need to use atleast one more colour, so we need three colours to give a proper colouring.

So, the number 3 one can convince that is minimum required and because there is a colouring with three colours, we see that it is a chromatic number of this colouring. So, we showed that with less than 3 is not possible and 3 is possible. So, that shows that there is colouring 3 and that is the smallest. So we get three to be the chromatic number of this graph. So, this question is like you know, can we find the minimum number that is required, that is for an arbitrary graph I take. While we are not going to really discuss this, one can show that this question is not an easy question.

And in fact, if you want to find the value algorithmically, then it is a question that is not known to have any polynomial, it is not going to have any polynomial time algorithm. So, it is an NP hard question and we still don't know whether we can solve it in polynomial time. So, it is going to be difficult question to actually find the smallest number, to be a precise number for general graphs. Now there could be several graphs, classes of graphs where we can easily find it that we will discuss in a graph theory course.

But in this course, we will not look much into it except we will look at some bounds. So, what we are going to look at is, can we say that what is the smallest number which is necessary. The largest number that you can say is necessary and the smallest number that will suffice. So, we will get a lower bound and upper bound and then we will get to reduce the gap between these and see how much close we can go without spending too much. So, this is another interesting study. So, there are many questions one can come up with and many other parameters. But, but I just wanted to point out that finding a minimum makes sense.

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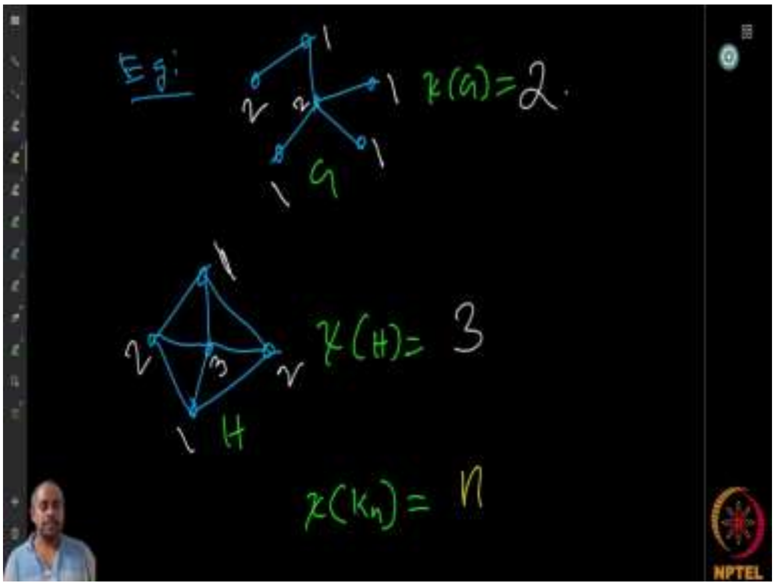

We will discuss some examples of applications of this. Now, here is a graph what is the chromatic number. Let us say, what is a smallest. So, can you find out what is the minimum number of colours required for this? So, one thing that we can immediately say is that if the graph has at least one edge, you need at least two colours. If there is an edge, the endpoint must get different colours. This is clear. So, therefore, if the graph is a nonempty graph, then we will need at least two these two colours. So, now, the question is what is the minimum colours that will suffice?

So, in this case one can show that two is sufficient. So, what I am going to do is that let me start with the vertex here with the colour 1 maybe I will use some other colour, colour 1. Now, let me take another vertex. So, let us say that I take the vertex which is neighbouring this. Neighbouring this vertex this is going to get colour 2 because I cannot give the same colour but now I say that these neighbours of this 2 can all be given colour 1 because there is no edge connecting.

So, for this I can give colour 1, now I can give colour 2 to here and therefore, I get that two colour. So, because two is necessary, we see that this is equal to 2. Chromatic number of graph is equal to 2 and we should include. Now, what about this graph? So, the graph H, what is the chromatic number?

Well, we can try using some method. Now, let us say that I start with the colour 1 here. Now, I can colour this with colour 2. Then I know that I will need colour 3 here because I cannot use colour 1 or 2 here. And now, here right here, I can use either I can use only 2 because 1 and 3 are not possible. So, I use colour 2 and here I use colour 1 again. So, I can do with three colours. And since I can see that at least 3 is required from this part because all these three are adjacent to each other. So, therefore, we have to give different colours to each of them.

Therefore, I will say that this required atleast three. So, it will be equal to 3. Now, one of the observations that we made during this is that like if two vertices are adjacent, then they must get different colours. So, if all the vertices are adjacent, all of them must be of different colours. So, chromatic number of the complete graph on n vertices must use at least n different colours. Every vertex must use a different colour. So, this is actually equal to n. So, the chromatic number of a complete graph is the number of vertice ($\chi(K_n) = n$)

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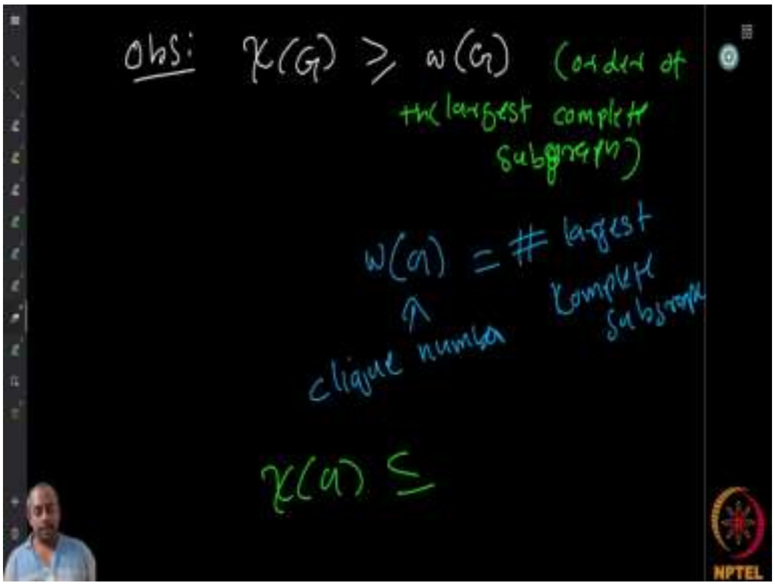

Now, here is an observation that we actually observed in the previous examples that the chromatic number of a graph is at least the smallest complete subgraph which is sitting inside, if there is a complete graph sitting inside, all of the vertices must get different colours and ω is the cardinality of the smallest, the largest complete graph not smallest, the largest complete graph sitting inside. So, ω is the largest complete graph sitting inside.

So, $\omega(G)$ is cardinality or order of the largest complete graph, complete sub graph. So, chromatic number is at least the $\omega(G)$. This is the clique number. So, this kind of complete graph which are sub graphs are called cliques in a graph. So, the clique number is small omega

G. So, the chromatic number is at least the clique number, that is immediate because you know that all the vertices of the clique must get different colours.

So, immediately we got a lower bound, we know that it must be at least $\omega(G)$ for lower bound of chromatic number. Now, we can think about upper bounds. Can we say that okay, $\chi(G)$ is always less than or equal something and if we say $\chi(G)$ is less than or equal to something of course, the immediate thing that you can say that it is less than or equal to total number of vertices, but it is a useless bound, because we know that we can not even give more colours anyway.

This is a trivial count, which is not very useful for us. So now, the question is that what can we say about a number which is different from n? Can we say still that it is less than or equal to something so, maybe you think about this for a few minutes right and then come up with a number. Then we can we can continue.

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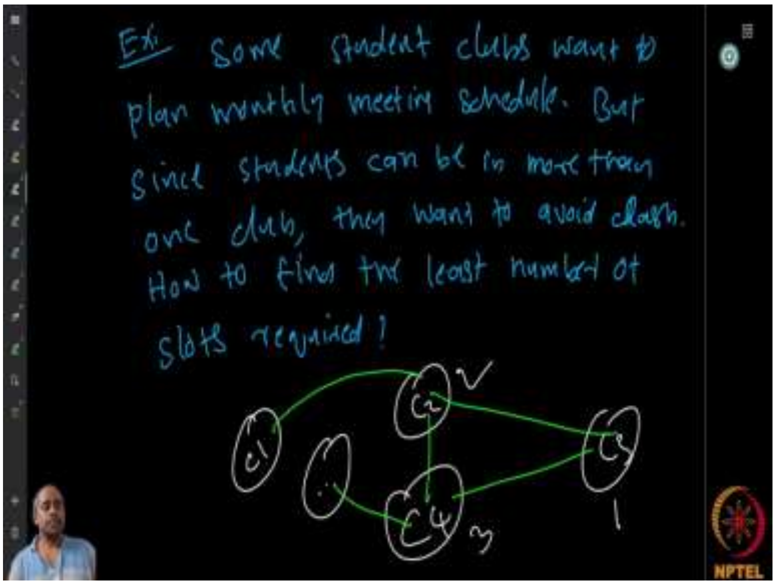

So, let us look at something. Now, before going into an upper bound, let us look at one application of colouring. So here are some student clubs and there are a number of them and they want to plan monthly meetings schedule. So, every month they have allotted a weekend, on Sunday, let us say you can arrange your meetings on this day. Now, the problem is that, when a student belongs to two clubs, and if both clubs are meeting at the same time, then this person cannot attend both the meetings, but of course, they do not want to miss the club meeting.

And therefore, what you can do, right so you want to find out or we have to make sure that when we assign or decide the time schedules, we have to make sure that they do not intersect. So how many different slots are required? Now, of course, if every club has a different slot that will be required but if the number of clubs is large maybe we do not want to spend too many hours for it, maybe we have more than 10 clubs and then maybe we will find out that we don't want to spend 10 hours in a Sunday.

For this so let us try to see what is the smallest number possible. So, can you do with less than the number of clubs? So, of course this is the question. We can design as a graph colourings. So, how do you design it as a graph colouring? So, each club, let us say becomes a vertex of the graph. So, this is club 1 and club 2, club 3, club 4 etcetera.

So, so we write this clubs as vertices. So, we get several of these clubs and the several of vertices. Now, I will define the graph, which means that I have to define the edges. So, what are the definition of edges? So, the edges are there between two clubs, if the clubs share some member. So, if the club 1 and club 2 has some common member then I put an edge between. When there is at least one member, I put an edge between them. Then I defined the graph like this. So, whatever is there I will say that okay these are the relations.

So, once I have this graph I say that if I find a chromatic number of a graph that tells me the least number of slots that will work for doing this club meeting schedules. So, how do you do that? Well, let us say that each slot is given some numbers, one to 10. This slot is now a colour, numbers can be colours because our element of sets are colours. So, now what I am going to do is that I am going to colour the vertices in a proper manner with the least number of colours possible.

So therefore, if I start colouring let us say one of these clubs with colour 1. I mean not like this actually, I want to find the minimum and this result will not be the minimum, but now just to give you a flavour, so let us say that this vertex is given colour 1, then I know immediately that because the two is adjacent to C3, C2 cannot get colour 1. So, it has to use a different slot let us say 2.

Now, C3 and C2 are adjacent to C4 so therefore, C4 must not because they have common members, so therefore, I need to find a different slot for C4 also. So, this way, the colouring will give you a schedule and if you find the minimum number of colours that will give you the minimum number of slots required for the meetings schedule. Something if you think about few minutes, it will be much more clear to you.

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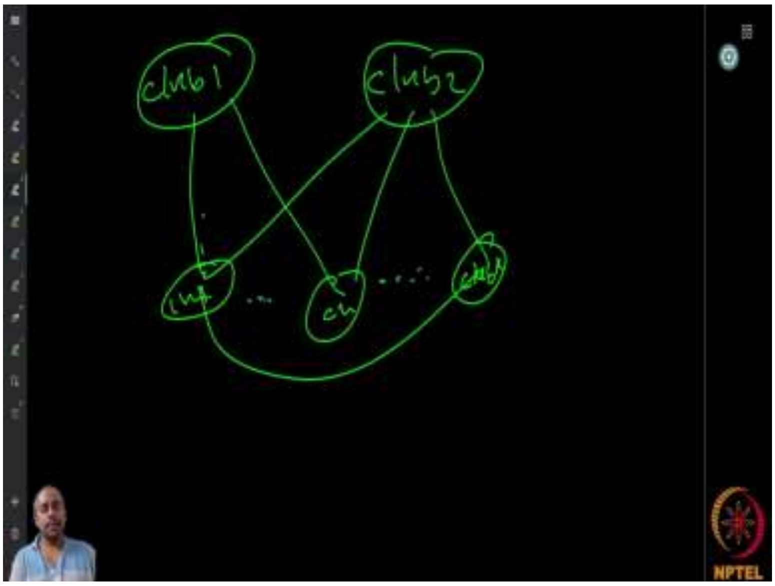

So, I think this is the same thing that I was mentioning. So, finding the chromatic number of the graph will tell you I will help you to resolve this.

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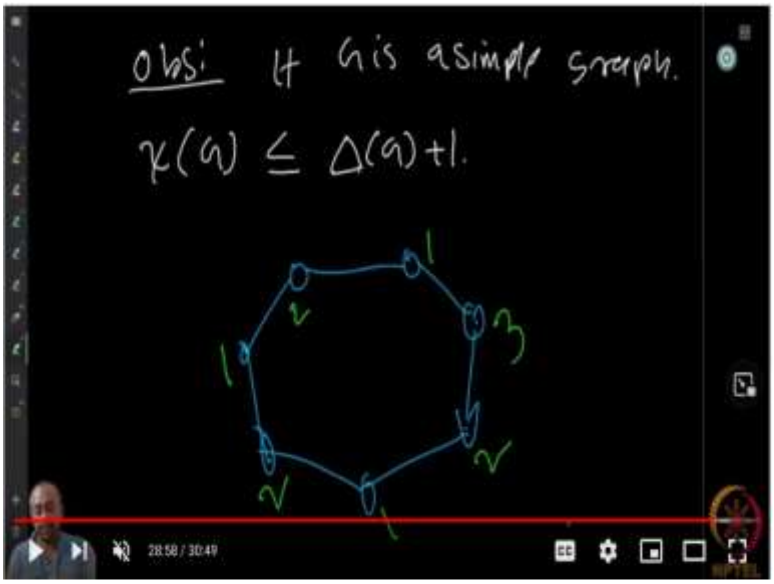

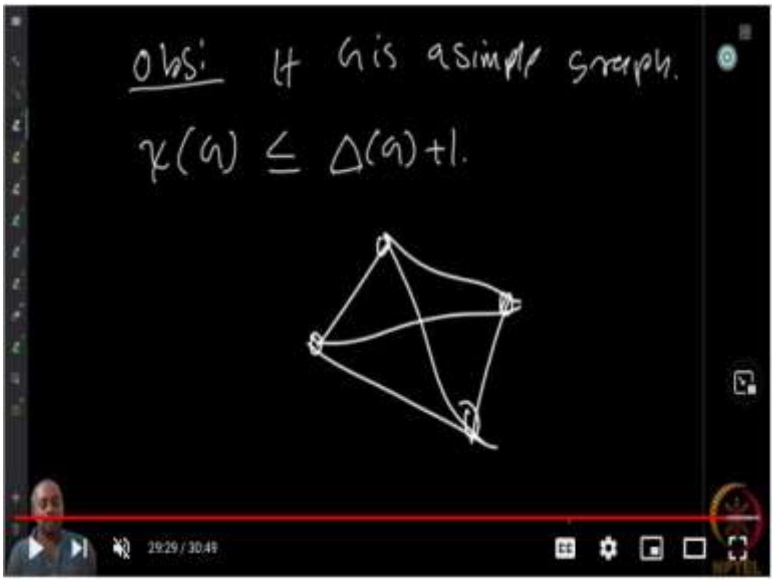

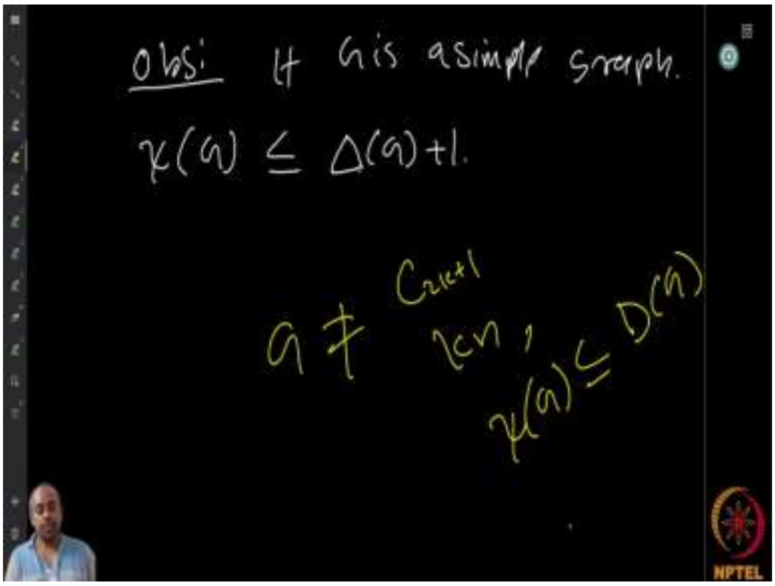

Now, here is an observation about the upper bound. So, if G is simple graph, then the chromatic number of G is at most the maximum degree of the G plus 1 (That is, $\chi(G) \leq \Delta(G) + 1$). So, given any graph G, so the $\Delta(G)$ is the maximum degree, the degree of vertices are there, d1 to dn and then the maximum amount of the degrees is called the maximum degree Δ, this is something we defined earlier. So, whatever is the maximum largest degree in the graph, that plus 1 is an upper bound for the chromatic number.

Can you see why? You need to think about this for a few minutes figure out why this is an upper bound. So maybe, here is the reason. So, I just take the graph, and then start colouring. So, what I am going to do is that I pick a vertex, whatever vertex I want, which is not coloured, and try to give it a colour. Now we are now going to take this graph pick a vertex and then I am going to colour this vertex with some colour.

So, let us say that the degree of the vertex is d, the vertex is u and the degree is d. Now, with the degree of the vertex is d, I am going to pick I have already picked this vertex and now, we want to colour it. So, I want to associate some colour to this vertex. So, what are the colours that I cannot give to the vertex u? I cannot give a colour to u, only if that colour is used in one of its neighbours either here or here. So, we look at all possible d neighbours. Each of them can use at most one colour.

And in total, it can use that most d colours. So, if I have more than d colours available in the set, in C if I have more than d colours, then I can use one of the colours other than this d colours. This would be less than d but at most d. So, one of the colours other than this d will be available and that colour I can use here. So, if I have at least d +1 colours in the set, I can always colour the vertex with degree d.

Now this is true for any vertex. No matter what the previous vertices are coloured with. This vertex can always be coloured with if I have at least d + 1 colours available. Now, Δ is the largest such d, the maximum so, we are looking at maximum overall d degree.

So, maximum overall u of all the vertices d of u, that is delta and therefore, d is going to be less than or equal to delta for every vertex. d is going to be less than or equal to Δ. So, when I am taking $\Delta + 1$ colours right I have at least one colour available even if colour all the delta neighbours with different colours. So, therefore, I see that I can always colour with the delta plus 1. So, the chromatic number is less than or equal to $\Delta(G) + 1$. This is immediately clear.

Now once you have something like this, so, in graph theory like the question that we come up with are like this. Suppose you have inequality then the first question you can ask is that when is the inequality actual equality? Can you use it or ever attained? Is it possible to get a graph where you actually need $\Delta(G) + 1$? Or is it possible to do with less than $\Delta(G) + 1$ if the graph is not one of some specific graphs or in all cases.

So, these kinds of questions are interesting. So, what is this, the question that we asked immediately is that, precisely which cases can attain this bound as equality and then if the equality is not attained, then what can you say about the other graphs and what is the largest bond and things like that. So, this is solved by a very famous theorem called Brooke's theorem.

So, Brooke's theorem says that there are only two classes of graphs which require $\Delta(G) + 1$. So, the first one is what is called the odd length cycles. So, odd length cycles are the cycle graph where the number of vertices is odd 1, 2, 3, 4, 5, 6, 7. So, let us take an odd length cycle. Now, the odd length cycle has all the vertices have degree exactly 2.

So, the maximum degree is also equal to 2 but it requires three colours because, as we observed in the case of the C5 earlier, if I start with one colour, then I am forced to give colours 2 to these guys if I am using at most two colours. Therefore, consequently we have to use colour 1 here and here and then I started with the yeah, colour 1 here and here and then I have to use the colour 2 here it, but here, but I cannot use colour 2 here and here because they are adjacent. So I need to use a third colour.

When this holds, then we can say that this is the minimum. So, for all cycles we can immediately see that we need three colours. So, what are the other graphs which requires delta plus 1? So, it turns out that the other graph which require $\Delta + 1$ are the complete graph K_n. So, if the graph is complete, we said that all the vertices must get different colours. So, you take any complete graph, we know that every vertex must get a different colour because there are all adjacent but what is the degree?

The degree is the number of vertices minus 1. So, the Δ is actually n - 1 but we need n colours so, that is what exactly equal to delta n minus 1. So, if the graph is the complete graph K_n or it is an odd cycle C_{2n+1} , we need a $\Delta + 1$colours. So, G is not C_{2n+1} and K_n, then the chromatic number of G is less than or equal to $\Delta(G)$. We don't want a plus one. So, this is something which we need to prove, this is just a claim. This is called Brooke's theorem and I give this to you as an exercise to work. It takes a little bit of work, but it will be very interesting.

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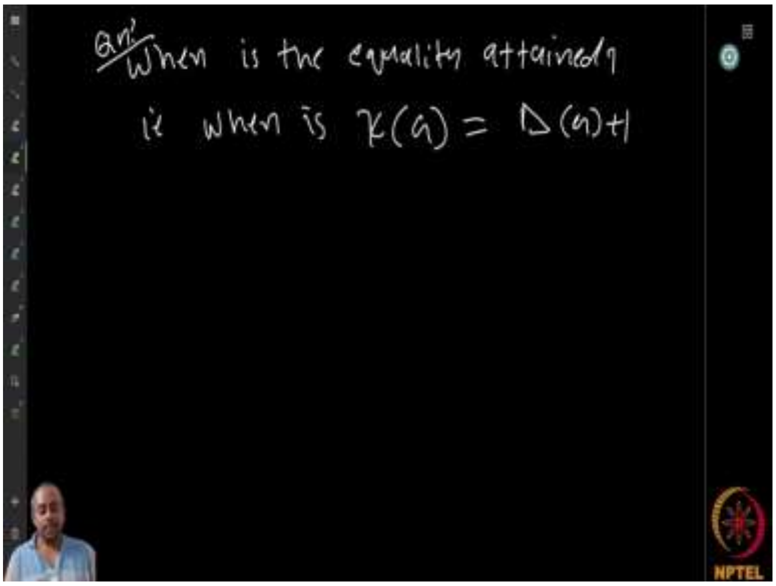

So yeah, so the question is that when is equality attained? When is $\chi(G) = \Delta(G) + 1$? The equality is mentioned, it is when the graph is a complete graph or odd cycle.

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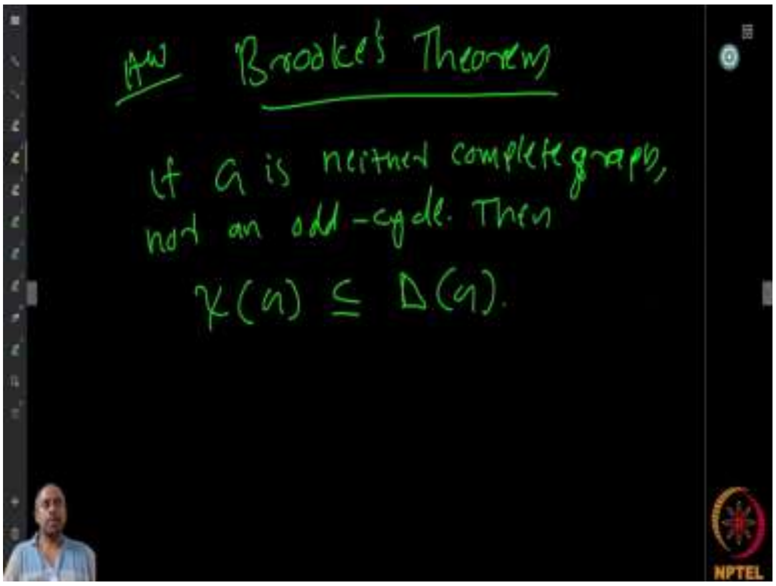

So, what I want you to show is the prove the following Brooke's theorem. If G is neither complete graph nor an odd cycle then the chromatic number of G at most $\Delta(G)$, which is the maximum degree.