Combinatorics
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Lecture 33
Digraphs and Functional Digraphs

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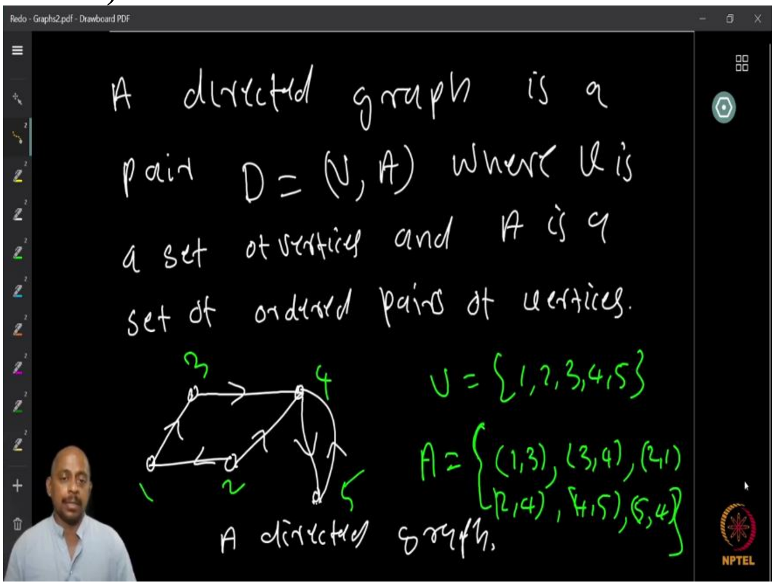

In the last lecture we defined what is a directed graph and what was it? It is basically a pair of sets, the set V of the vertices and then set A of arcs which are ordered pairs of elements of V. The difference from that of the ordinary graph is that instead of two element subset we have ordered pairs of vertices. Now, this enables us to visualize the graph as having some kind of direction for the edges like if (1, 3) is an ordered pair, I can represent by drawing a curve connecting 1 to 3, but with an arrow which says that it is actually going from 1 and going to 3.

Therefore, the direction is represented by the arrow and that tells you also what is the arc what is the ordered tuple. When I want to ask the elements of the two-element set then the direction comes by saying that there is an arc from 2 to 1 therefore, (2, 1) must be the corresponding arc. Here is an example of a digraph where V = {1, 2, 3, 4, 5} then we have A = {(1, 3), (3, 4), (2, 1), (2, 4), (4, 5), (5, 4)}. Now, there is (4, 5) as well as (5, 4) which enables us to go from 4 to 5 as well as from 5 to 4 by taking an arc.

On the other hand, if you look at 2 and 4, I can go from 2 to 4 but there is no way to go from 4 to 2 in this case. It is important how this direction and that allows us to represent more information about the network. For example, when you have a traffic network where some of the roads are one-ways then we know that we cannot have bi-directional commutation and hence, this will allow us to represent traffic networks better.

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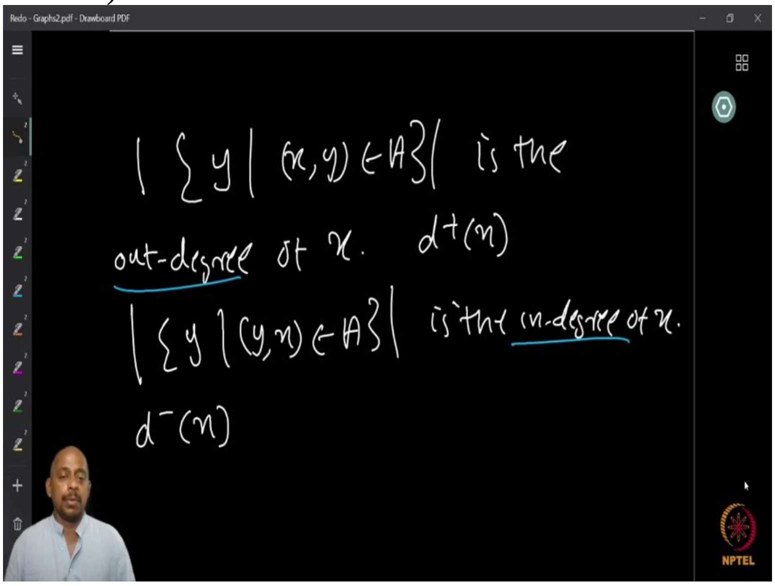

Now, if you look at the multi-set of all vertices such that (x, y) is an arc it is called the out-degree of x. These are the number of edges which goes out of x, start from x and go to some other vertex. Then it is usually denoted by $d^+(x)$. That is $d^+(x) = |\{y: (x, y) \in A\}|$.

Similarly, the multiset set, set of all y such that (y, x) is an arc is the in-degree which means that all the arcs which are coming into y denoted by $d^-(x)$. That is $d^-(x) = |\{y: (y, x) \in A\}|$. We have the out degree as well as the in degree. If you think about this one can see that if I have an arc from any vertex let us say, x to y then of course it contributes 1 to the in-degree of x and 1 to the out-degree of y.

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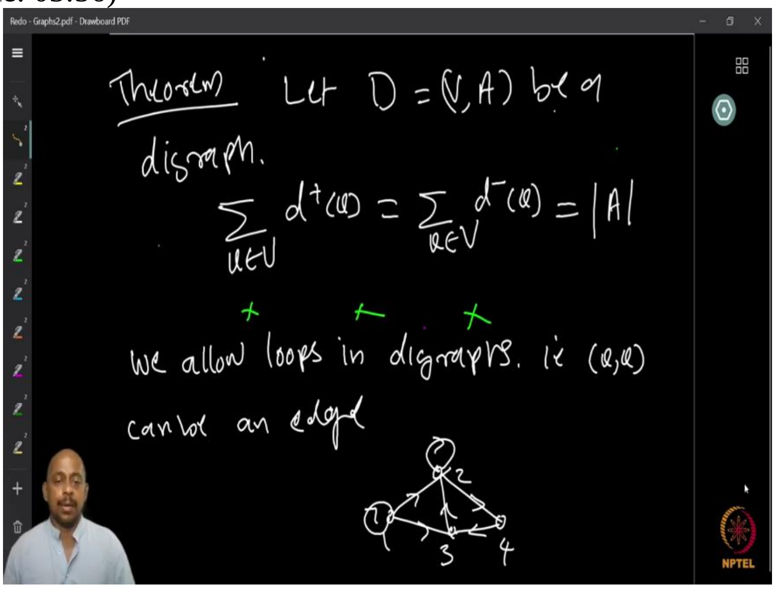

Therefore, the following theorem is immediate that if you look at D = (V, A) as a diagraph then $\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |A|$.

So, the out-degree sum must be equal to the in-degree sum, which is equal to the cardinality of the arcs. This is immediately clear and proof also we have just mentioned.

Now, in digraphs we usually also allow loops, that is you can have v to v as an edge where I go from v and then comes back to it and these are called loops, and often we consider this kind of loops in our digraphs, but of course we also discussed loop less diagraphs most of the time and then only when we require, we will mention it and then we use it.

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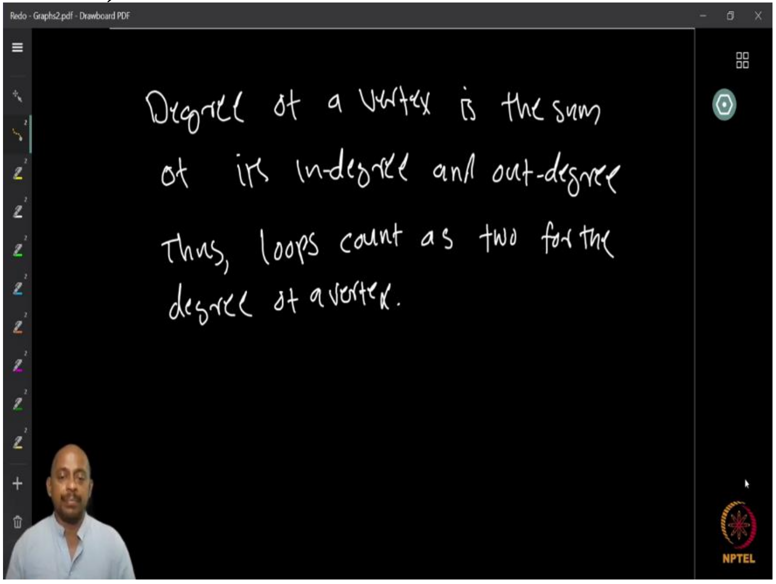

The degree of a vertex is the sum of the in-degree and out-degree. That is the number of edges which is actually going. You can also see this as the degree of the underlying graph where we discard the directions. So, if you see the arcs as edges rather than directed edges then you will get a multigraph, you can get a multi graph where you allow multiple edges between a pair of vertices, and then you can see that it is basically the degree of the multigraph.

Therefore, one can talk about the underlying graph of a digraph. Then the loops usually count for 2 for the degree of a vertex, because the in-degree is there and out-degree is there and for the undirected case also most of the time we will do that.

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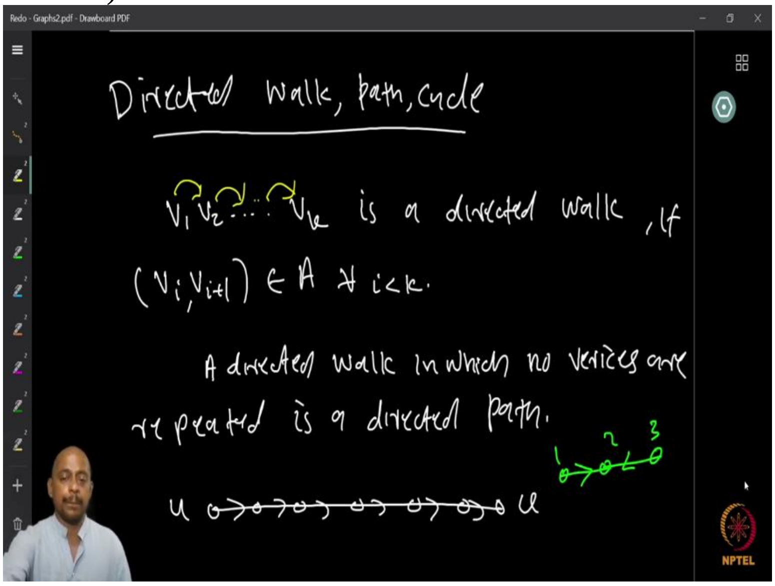

Then very similar to what we did in the undirected graphs, we can define directed walk, path and cycle. A directed walk is basically a sequence of vertices v_1 v_2 ... v_k, where v_iv_{i+1} for every i < k is an arc, we need to have arcs in that particular direction. This is a directed walk. If the vertices are not repeated and vertex only appears once then a directed walk as in the case of the undirected graph also, is called a directed path. Here I have a directed graph, u and v are there and if you have arcs going from u to v, then uv is an example of a directed path.

On the other hand, 1 2 3 here I have written in yellow or greenish yellow is basically not a walk or a path because 1 to 2 is an arc but 2 to 3 is not an arc. So 1 2 3 cannot be a path or a walk. On the other hand, 1 2 is a walk and a path, 3 2 is a walk and a path.

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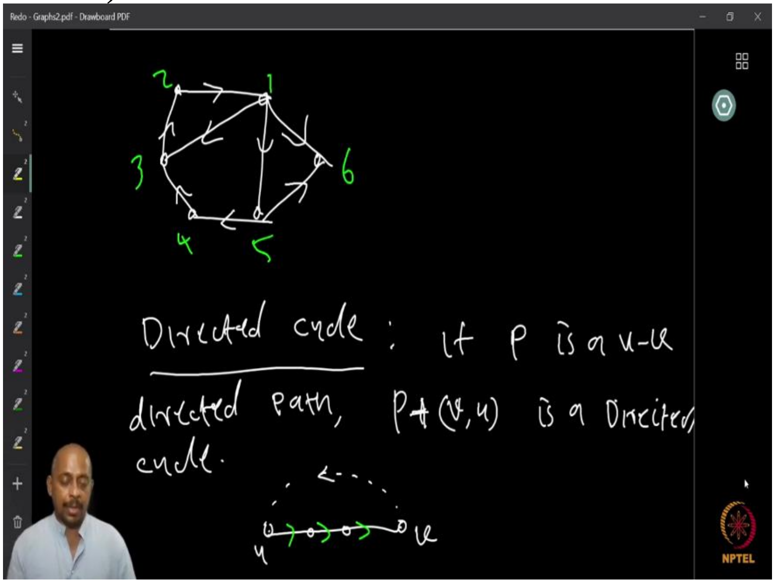

A directed cycle just like in the case of undirected case, if P is a u-v directed path then you add the arc v to u also then that is a directed cycle. So, u to v and then coming back to u from v directly. Examples are in this graph above, you can see that 1 to 3 then 3 to 2 and 2 to 1 is a directed cycle. Then you have 1 to 5, 5 to 4, 4 to 3, 3 to 2, 2 to 1 this is another directed cycle, but on the other hand 1 to 6 or like 5 to 6, 6 to 1 and 1 to 5, even though it is a cycle, it is not a directed cycle. It is a cycle in the undirected graph, but on a directed graph it is not a cycle.

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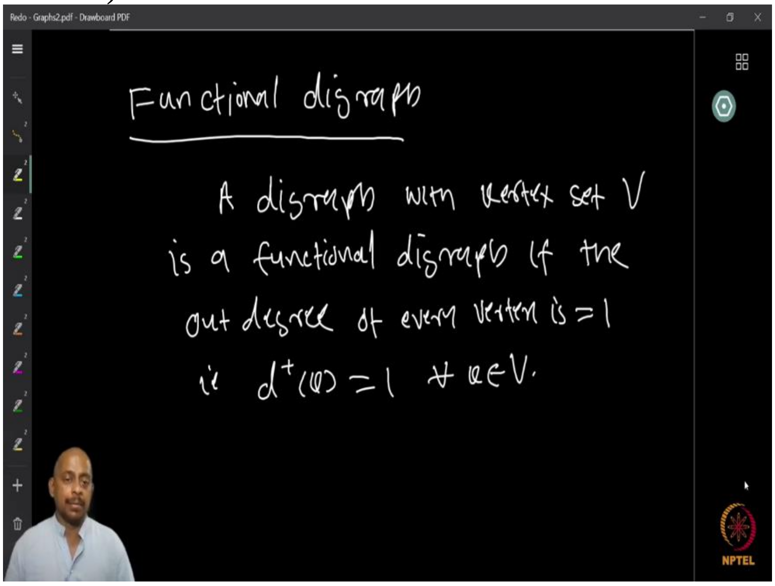

We define a special type of digraph, it is called functional diagraphs. Functional diagraphs are basically digraphs which can represent endofunction, functions from a set to itself. Let us take a diagraph with a vertex set V and then we say it is a functional diagraph if the out degree of every vertex is actually equal to 1.

Every vertex has exactly 1 edge going out of it. It could have many coming inside but going outside is exactly 1, and this you can see immediately why it should be the case because if you are defining a function it can only have a unique image, it cannot have multiple images for an element.

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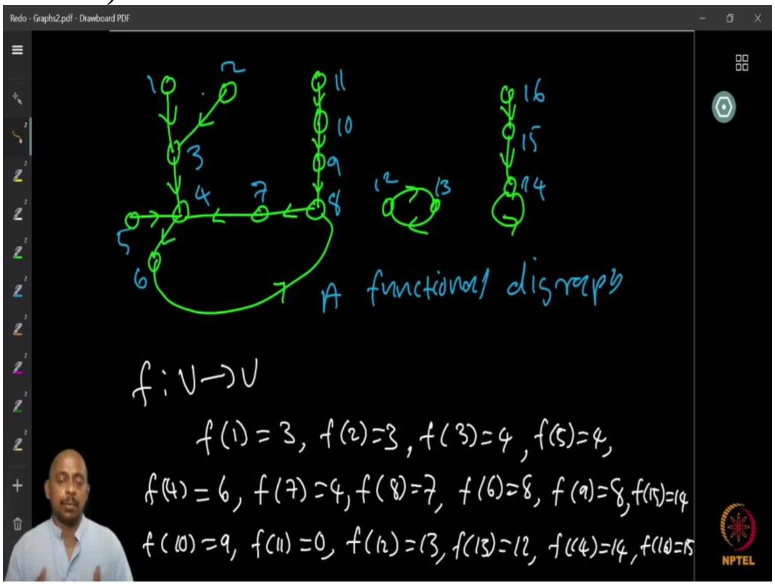

So, here is an example of a functional digraph where if you look at any vertex, you will see it has exactly 1 outgoing edge. Let us see that, vertex 2 or 1 has exactly 1 then vertex 3 has 2 incoming but 1 outgoing, 5 has 1 outgoing, 4 has 1 outgoing, 5 6 then 8 7 and I know like then 11, 10, 9, all these things has exactly one outgoing arc, and you can verify for other vertices.

So, what we have is a functional diagraph. Why is it called functional digraph? Because it actually represents a endofunction from V to V. You can see the function by just looking at what each element is mapped to as the out neighbor of the vertex. Out neighbor of a vertex is the vertex which is obtained, it is the neighbor through the out going arc. In this particular example we have f(1) = 3, because 1 is going to 3.

Similarly, 2 is also going to 3, f(2) = 3; f(3) = 4; f(5) = 4 and similarly you have all these values f(14) = 14, because there is a loop. This is how it is. What I want you to do is to draw a few functional digraph by yourself and then try to see whether you can observe any nice properties about this, you think about this and then come up with my observation that would be very nice.

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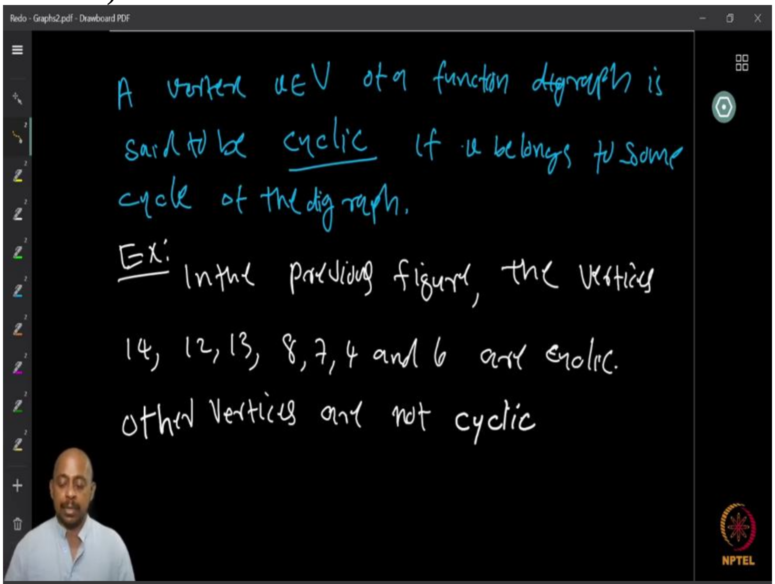

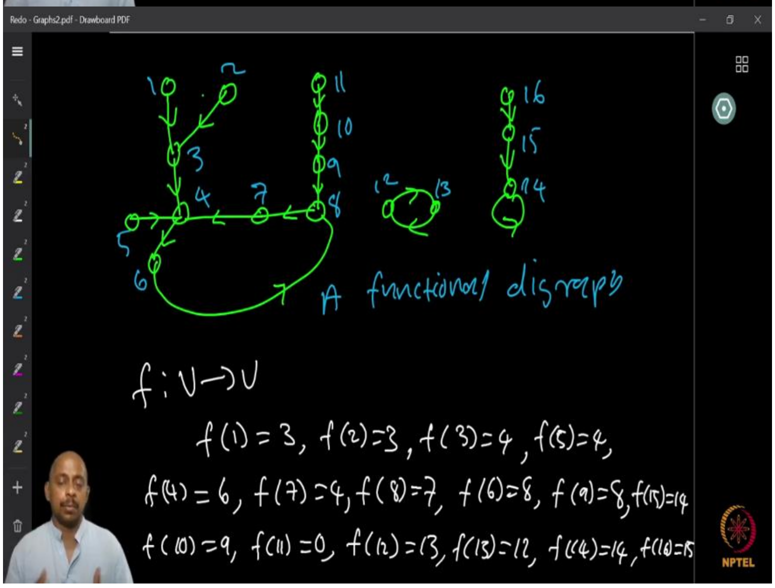

We say a vertex in a functional digraph to be cyclic, if it belongs to some cycle of the digraph. Any vertex that is part of a cycle is called a cyclic vertex. So, in the previous example we saw that like the vertices 14 is cyclic because 14 is in a loop then 12 and 13 are cyclic because 12 to 13 and 13 to 12 are also arcs, which forms a directed cycle, two cycle and then we have this other part where we have 4 going to 6, 6 going to 8, 8 going to 7 and 7 going to 4 again. That is another cycle. 4 6 7 8 are again part of a cycle, therefore they are cyclic vertices. We have this 1 2 3 4 plus 3 7 cyclic vertices. These are the cyclic vertices and other vertices are not cyclic because they are not part of a cycle.

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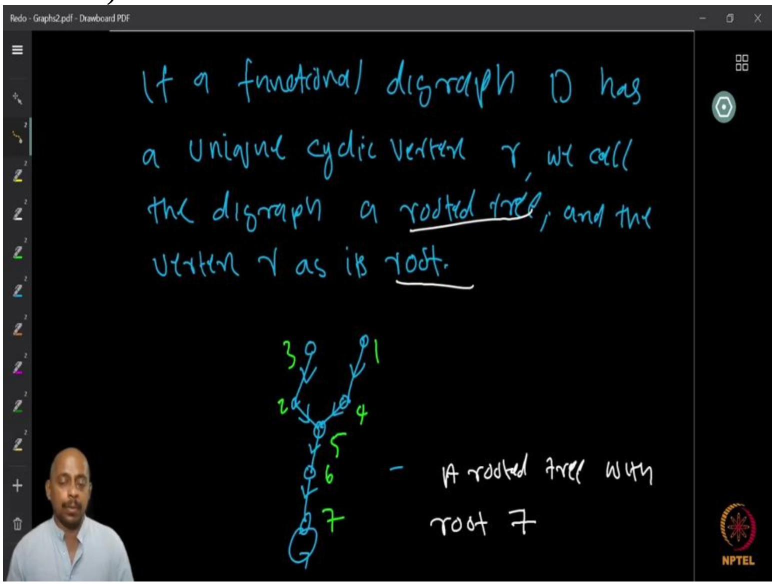

Suppose you have a functional digraph and if this graph has exactly one cyclic vertex, in the entire graph there is only one cyclic vertex then we call this digraph as a rooted tree. The rooted tree is basically a functional diagraph where we have exactly one cyclic vertex and the root of the vertex is the vertex having a cycle. In this example you have 7 as a root of the functional digraph, because 7 is a single cyclic vertex which is in a loop and every other vertex you can see has exactly one outgoing edge. Therefore, it is actually a functional digraph. So, you have a rooted tree graph. Rooted tree is basically a functional digraph with exactly one cyclic vertex.

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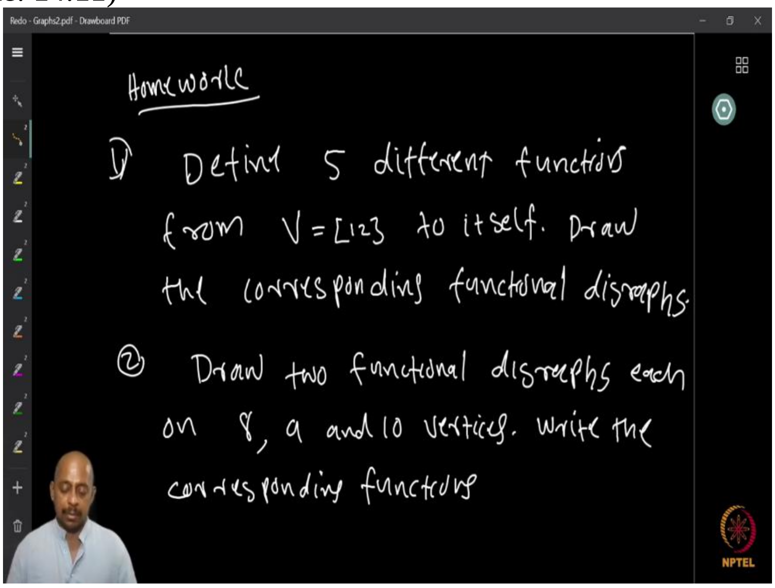

I give you some homework questions, first define 5 different functions from a vertex set 1 to 12 to itself and draw corresponding functional diagraph and once you draw this look at the properties. Draw the functional digraphs, two functional digraphs each on 8 9 and 10 vertices then write the corresponding functions.

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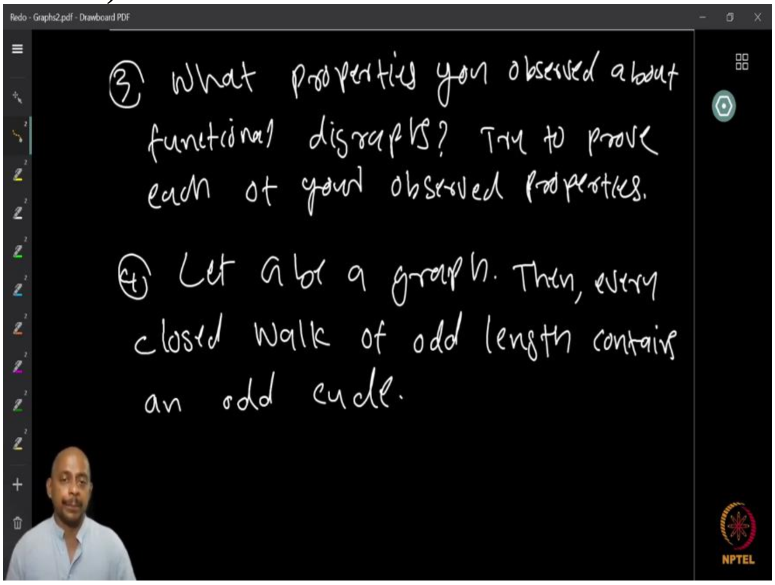

Now, if you have observed some properties while drawing several of these, you must have seen some few properties then what are these properties that you have seen, make a note of that and try to write it. Try to prove each of your observations and see whether this observation is actually a property for the entire functional diagraphs or not?

Now, if you have a graph let us say G then every closed walk of odd length contains an odd cycle. This is not for directed graph; I am asking you to draw for a normal graph. Given a graph G then every closed walk of odd length contains an odd cycle. This homework you must do because we are going to use it to prove something else.

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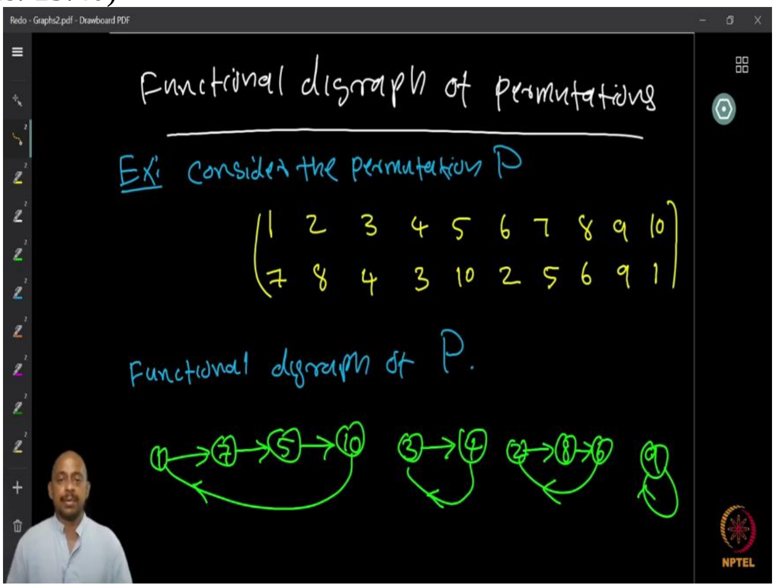

Some properties of functional digraphs, we are looking at functional graph of special type of functions now which are the permutations. Permutation is a bijection from a set to itself. So we have the following permutation let us say P which takes 1 to 7, 2 to 8, 3 to 4, 4 to 3, 5 to 10 etc.

What is the functional digraph of this permutation? If you look at the functional digraph of the permutation, you will see something interesting. If you look at this functional digraph you will see that well you have, 1 is going to 7, because 1-7 is the map then 7 what happens to 7? 7 goes to 5, so to draw the digraph you basically do this define the function and then you look at what happens.

So, 7 going to 5 then 5 goes to 10 then 10 goes to 1. Now, that is one cycle then we have to start with the next available one. I start from 3, 3 to 4 then 4 to 3 again comes back another cycle then you start from 2, 2 to 8, 8 to 6 and 6 to 2 again and similarly, 9 is mapped to itself. Therefore, it is a cycle there, if you look at this you will see that, it is basically a collection of cycles.

Every vertex is part of a cycle. This one can see why it would be true in the case of permutation. This also gives us a linking that why basically a permutation can be represented as cycles. We have a cycle representation of permutations which comes from the observation that the digraphs of the permutations are basically a collection of cycles. You can try with several examples, try to define certain permutations, draw the functional digraphs and see whether you have the same property.

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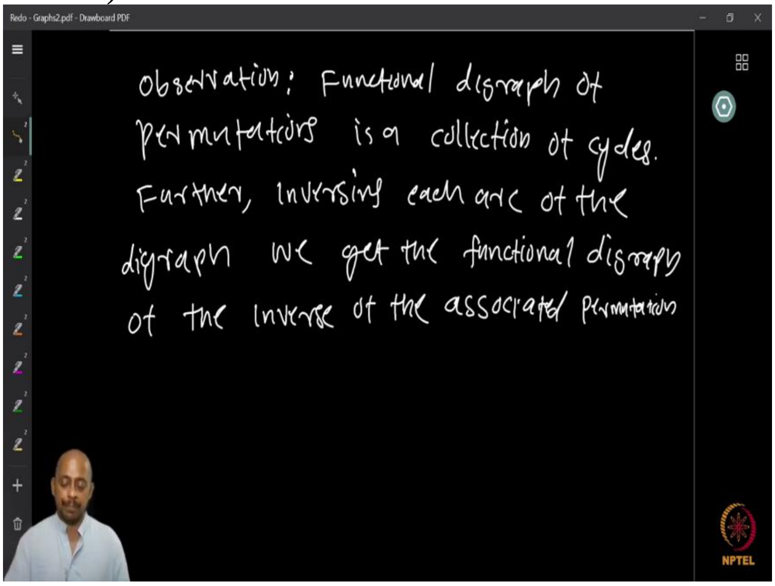

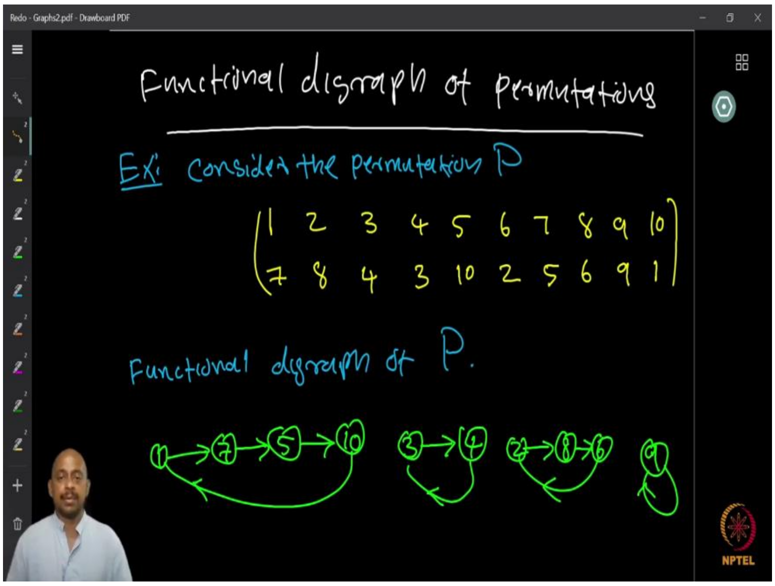

Now, the functional graph of permutations is a collection of cycles, this is an observation that you can immediately make, and then if you inverse each arc of the digraph then we get the functional digraph of the inverse of the associated permutation. Again, if you map from 1 to 10, 10 to 5 then 5 to 7 and 7 to 1 extra then you will get, what you get is basically the inverse of the associated permutation and the corresponding functional digraph.

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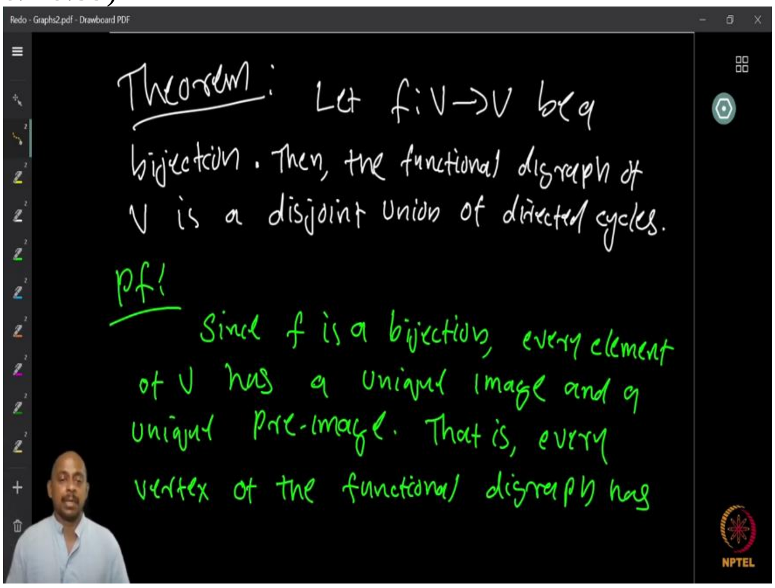

Now, here is the theorem we want to prove. If f is a bijection from V to V then the functional digraph of V is a disjoint union of directed cycles. We can now prove it formally, I want you to think about the proof before you go ahead with this proof. It would be nice to do that.

Now, how do you prove this? We know that f is a bijection; it is a bijection which means that every element of V has a unique image and a unique pre-image. By the definition of bijection there is a one-to-one correspondence. Every element has an image and it also has a unique pre-image, which means that every vertex of the functional digraph has exactly one outgoing edge because it is a functional graph anyway but it also has exactly one incoming edge.

So, the outgoing edge is one and incoming edge is one, so which means that if you look at the the underlying graph without the directions then what is it? It is basically, every vertex has degree exactly 2 because one is actually the outgoing vertex, outgoing arc and one is the incoming arc, these two contribute degree two. Therefore, we see that in the underlying graph every vertex has degree. It is a simple exercise to show that if a graph has the property that every vertex has degree exactly two then it is basically a collection of cycles.

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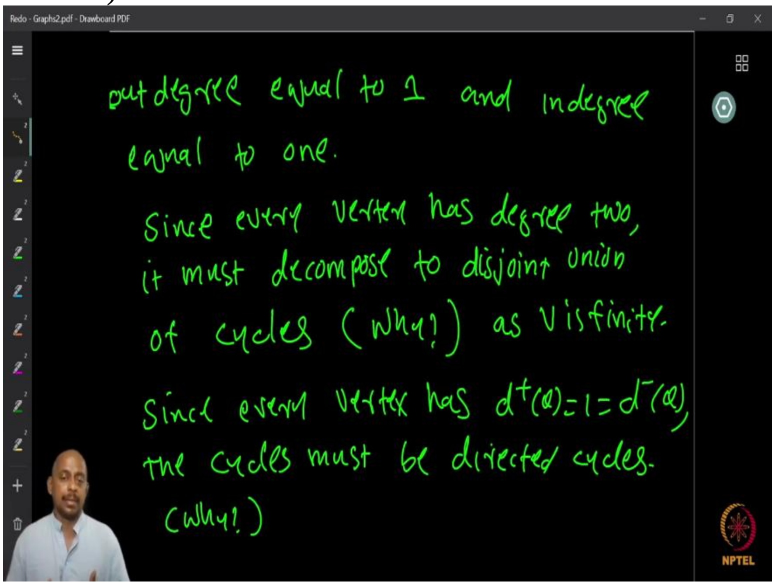

This will allow us to see why it should be in the case the permutation must have a collection of cycles. Out-degree is equal to 1 and in-degree is equal to 1 since every vertex is degree 2, it must decompose into disjoint union of cycle. I want you to prove this formally write a proof, but it is kind of obvious but then try to write it formally.

And this is true only because we assume that V is finite, if for infinite vertices you can have just a path where I mean we can have a tree for example where every vertex has degree exactly 2 and then it need not hold. Since every vertex has $d^+(v) = 1$ and $d^-(v) = 1$, cycle must be directed cycles. Can you see why? Every vertex has the out-degree and in-degree equal to 1 then the cycles in the graph must all be directed cycles, because if it is not directed cycles you will see that some vertices cannot have this property.

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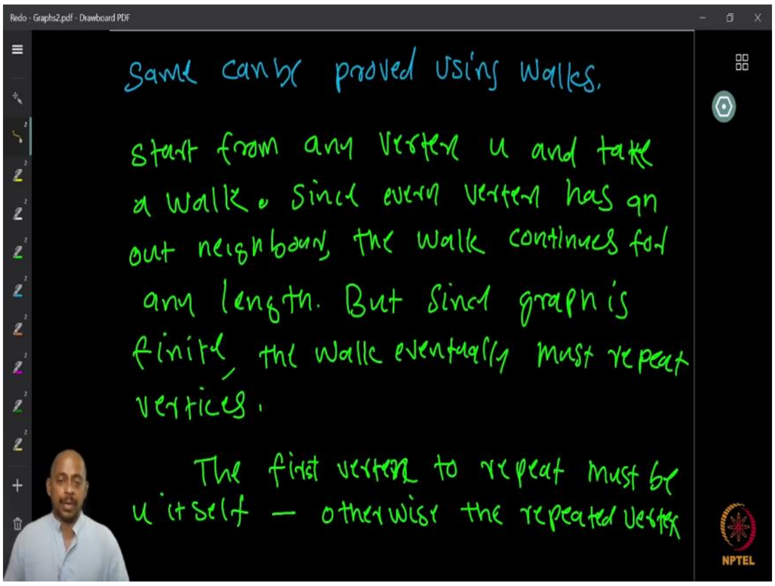

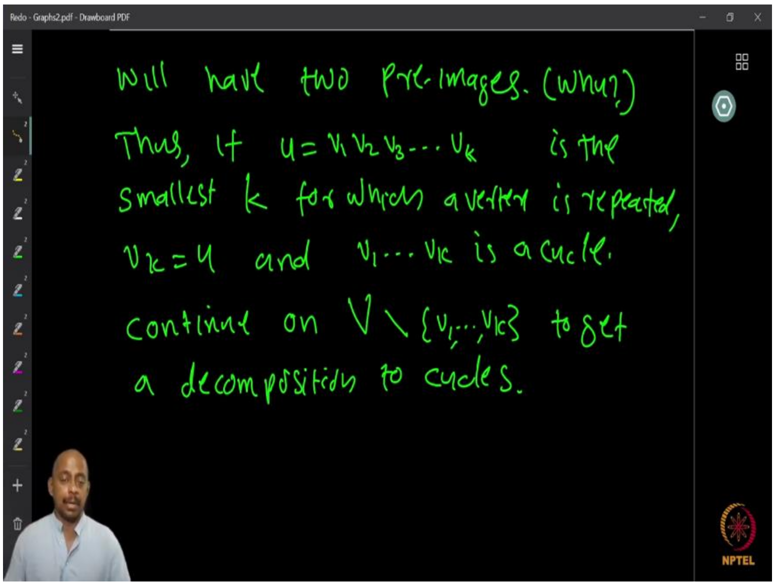

Now, we have another proof using walks. Start from any vertex u and take a walk. Since every vertex has an out neighbor, the walk continues for any length, because I start from a vertex then I go to the next neighbor taking the out degree. I have taken an incoming edge to go here therefore, now I can also go out of this, so I go out of it to another vertex and then I continue. I keep on continuing this and I can do it as much as I want because every time I reach a vertex, I can go out of that vertex.

Now, since the graph is finite, this one cannot be infinite, walk can be infinite but the walk must repeat vertices because we have only finite vertices. I start from a vertex then eventually it must come back to some vertex. The first vertex that repeats must be the starting vertex u itself, why is that? can you think of this? Try to see if I have a functional digraph then we start from a vertex then you go, because every time you can go out of it and then if eventually a vertex is repeating in the case of permutations, eventually if a vertex is repeating then it should be the starting vertex.

It need not be the case for functional digraph, arbitrary functional digraphs, but for the permutation case it must be the vertex u itself. Why is that? Because if that is not the case, the repeated vertex, what happens to it I start from u I go to let us say v and then go further and then suppose it comes back to v not to u then if you look at the vertex v, v already had an incoming edge from u and then outgoing edge that we took as part of the walk and then coming back without visiting u, it means that I am basically having another incoming edge to the vertex v.

So, I have two incoming vertices, so in-degree is at least 2, but we said that the in-degree is exactly 1, because it has only unique pre-image. So, therefore, we see that like if I start from a vertex after sometime it must finally come back to the starting vertex. So, that forms a cycle by the definition of cycle, I have a sequence of vertices without repeating anything except the first one is repeated. Then you get a directed cycle. Now, I can throw away these vertices, because all the incoming and outgoing edges are counted now for this vertices I throw away, and remaining I can start again. So, therefore, I get a decomposition into cycles, so here is another difference.

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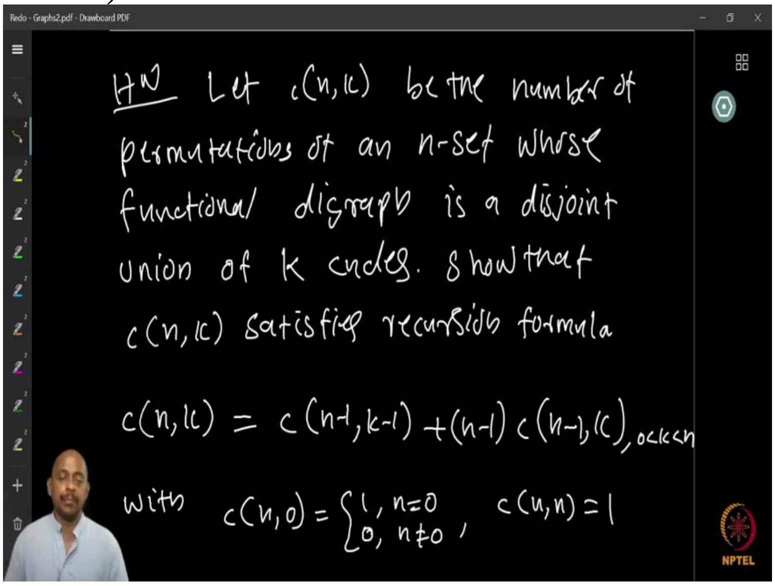

Now, as homework I define the following. Let c(n, k) be the number of permutations of an n-element set whose functional diagraph is a disjoint union of k cycles. So, that c(n, k) satisfies the following recursion formula

c(n,k) = c(n-1, k-1) + (n-1) c(n-1,k) , 0< k <n

with c(n, 0) = $\begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$ and c(n,n) = 1

These conditions lets you calculate c(n, k) uniquely and find out this so you can think about this.

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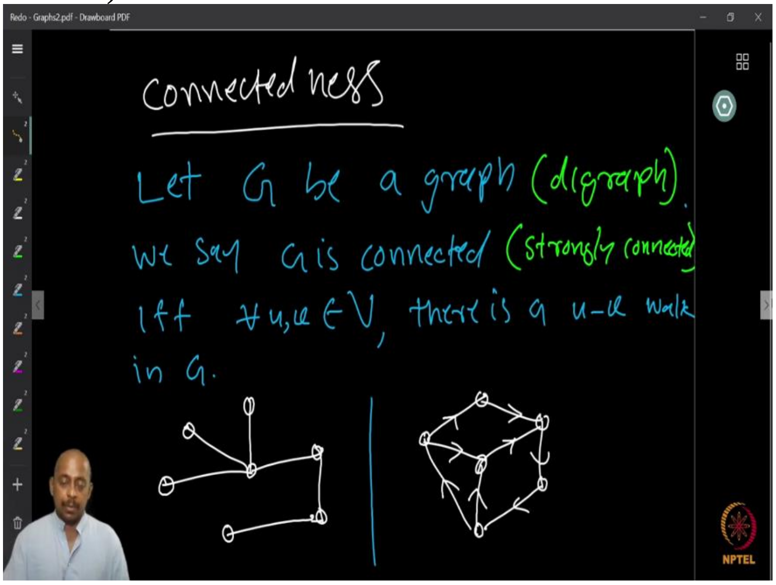

The next notion I want to look at in this case is that of connectedness. Given a graph G, we say the graph G is connected if and only for every pair of vertices let us say u, v in the graph G, if there is a u-v walk for every pair of vertices then the graph is said to be connected. If you are looking at digraphs, we say the digraph is strongly connected if for every pair of vertices there is a u-v directed walk in the digraph.

In the case of graphs we say the graph is connected if any two vertices has a walk between them or a path between them. Similarly, we say a graph a digraph is strongly connected if for every pair of vertices I can reach any vertex from any other vertex.

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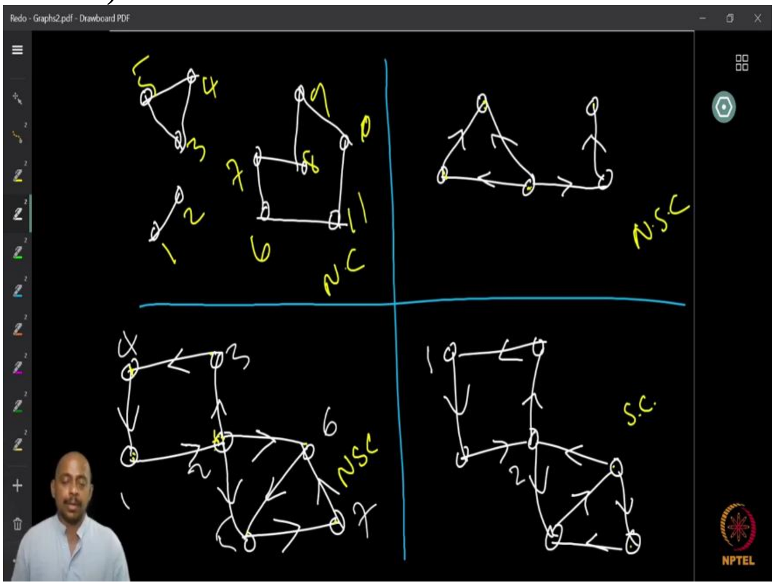

So, here are some examples, the first one is basically an undirected graph where which is not connected because it has several parts where I can go from 1 to 2 and 2 to 1 but I cannot go from 1 or 2 to let us say 3, there is no walk or path from 2 to 3. Therefore, this graph is not connected.

You will see that there are several such parts and each part is basically connected. Then you have the directed graph which is not strongly connected even though the underlying graph is connected. You have this second case where you cannot go from one of the vertices to the other I know for any pair of edges.

For example, I can go from this vertex to here, but I do not have any way to come back from that vertex to this vertex. Therefore, it is not strongly connected. Then again another example of a not strongly connected graph because in this I can go for example, from let us say this vertex and let me give me names this is a 1 4 5 6 and 7. I can go from 1 to 2, 2 to 3, 3 to 4, 4 to 1 etc. But if I want to go for example from 6 to 2, I cannot go from 6 to 2, I can go from let us say 2 to 5, 5 to 6, I can go from 5 to 7 and 6.

Similarly, I can go from 6 to 5 and 6 to 7, but I cannot go from 6 to 3 or 6 to 2. Therefore, this is also not strongly connected. Then here is a strongly connected example where I can go from any vertex to any because I can go from for example vertex 1 to vertex 2 by taking any path 2 to 1 and similarly, from 2, I can go to here and come back and then you will see that all this can be reached from any other vertex and therefore we have it strongly connected.

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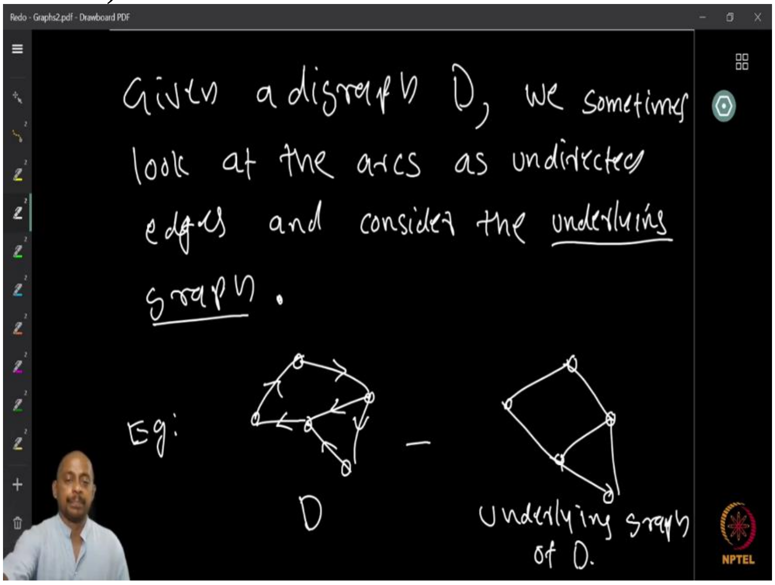

Now, given a directed graph D, we sometimes look at the arcs as the undirected edges, as I mentioned before and consider the underlying graph. An example the direct graph D is represented here and the underlying graph of D where we just discard the directions. All the ordered tuples now become just 2- elements sets. This could be useful in several occasions we will see that.

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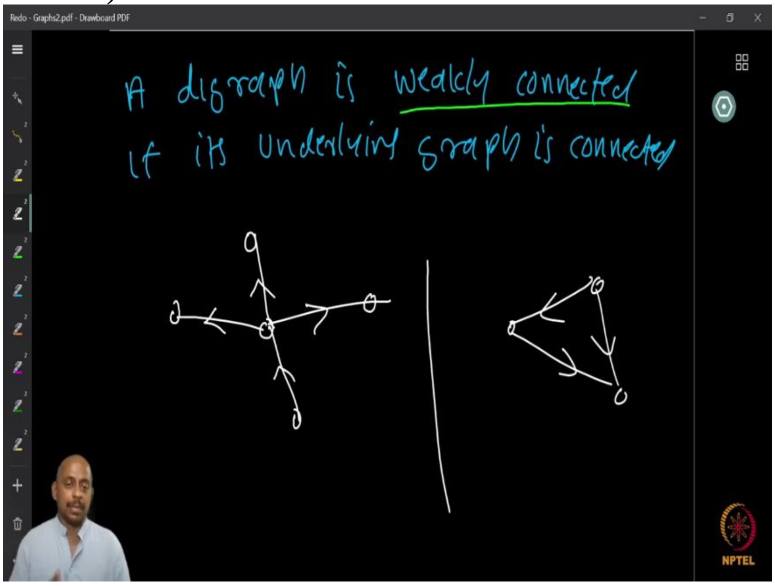

A digraph is weakly connected if its underlying graph is connected. If you just look at the underlying graph which says that it is actually connected then we say that digraph is weakly connected. It is not strongly connected because I cannot go from anywhere to anywhere but it is weakly connected in that sense that from the underlying map allows us to go.

For example, if I have a transport network where some paths are one ways then of course you cannot go from a vertex to, a point or another point may be taking an edge but on the other hand it says that in case of some special situations there is a possibility of reversing the decision to make it one way and then you can allow other direction passing.

It is not the excellent example, but to say that why the weekly connectedness can be useful in some times. So, a strongly connected graph is of course weakly connected because if you can go in the directed graph you can always go in the undirected graph.

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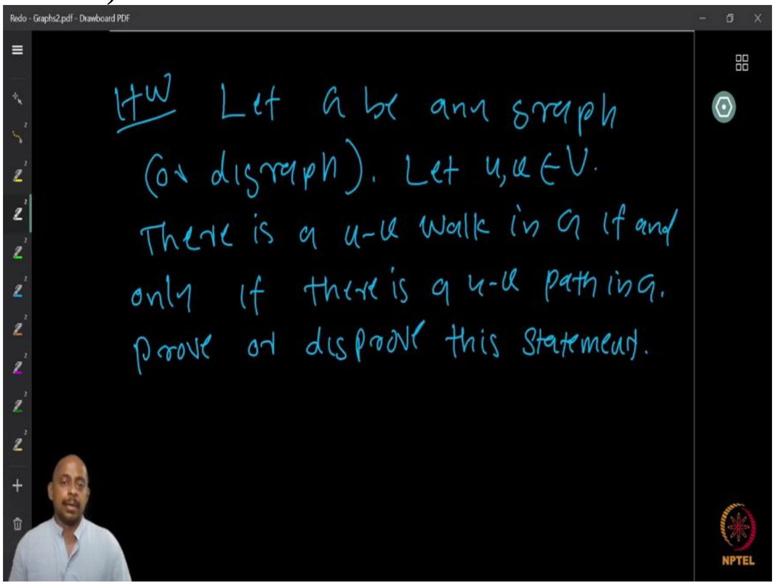

As a homework question, you can think about the following that G is any graph or a digraph and if you have a pair of vertices u and v then there is a u-v walk in G if and only there is a u-v path in G. For graphs and digraphs one can show, so u-v walk if under leave there is a u-v path. This you can try to prove or disprove if it is not true for either graph or digraph and this will be a nice homework.