Combinatorics
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Lecture 32
Paths, Walks and Cycles

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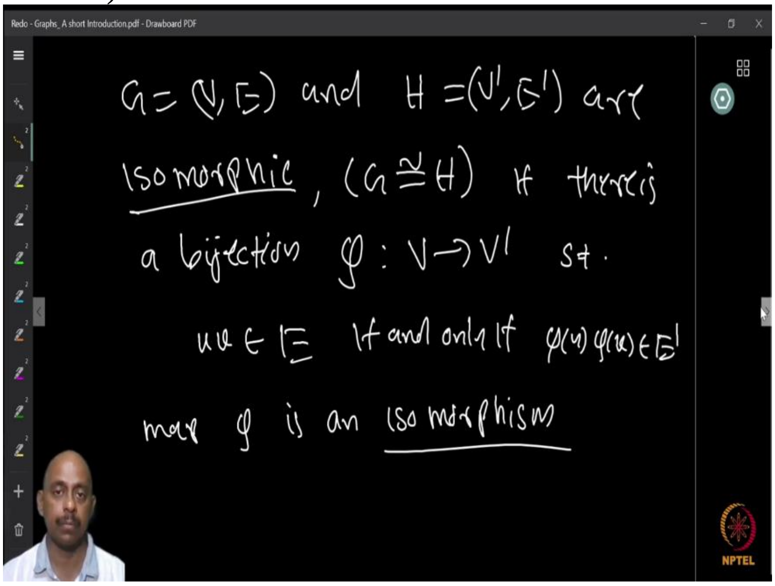

In the previous lecture we defined what are graphs, then subgraphs, and some related notions like edges, adjacency, and things like that. Now, let us look at a few more definitions before we go into applying these things. We have, let us say a graph G, we denote by (V, E), and we have a graph H with the vertex at V′ and edge at E′.

We say that the graphs G and H are Isomorphic, they are said to be isomorphic, and we write offcourse G ≅ H in this way. If one can find a bijection between the vertices, V and V′ such that uv is an edge if and only if $\phi(u)\phi(v)$ is an edge. So, what we are saying is that there is an edge preserving bijection, that is the edges are preserved in both ways.

So, you have the graph G, then you have bijection from the vertex set of graph G to the vertex set of the graph H, and we are saying that whenever uv is an egde in G, then $\phi(u)\phi(v)$ is an edge in the graph H.

And if uv is not an edge, then $\phi(u)\phi(v)$ is not an egde and vice versa. If $\phi(u)\phi(v)$ is an edge then we want uv to be an edge, and if there is no edge between $\phi(u)\phi(v)$ then there is no edge between u and v in the graph. So, such a bijection is there then we say that the graphs are isomorphic. And the map, which does this job; shows the isomorphism, that they are isomorphic is called as isomorphism. There could be several isomorphisms, but one of such map is an isomorphism. If you have an isomorphism then the graphs are isomorphic.

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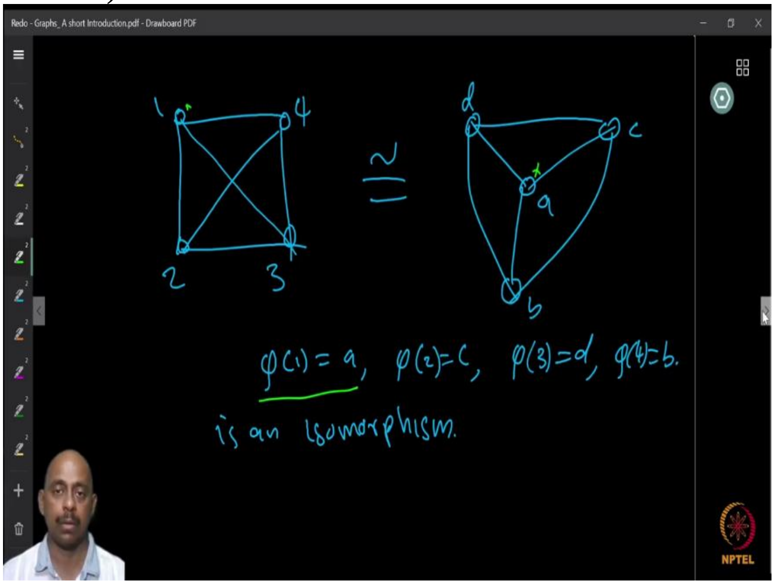

Here is an example; you have the graph, complete graph. I have represented it in this way 1, 2, 3, 4. And I have a very different representation of the same graph, but the vertex set is different. But it is again a complte graph on 4 vertices, where I labelled it with a, b, c and d. Now, I have represented it in very different manner, but these two graphs even though they look different, are exactly the same in the sense they are isomorphic. So, what is the isomorphism between these two graphs? Well, here is one, I can say that 1 goes to a, 1 is going with a by isomorphism. $\phi(1) = a$, then 2 goes to c, 1 goes to a, then 2 goes to c, 3 goes to d and 4 goes to b, this is an isomorphism.

And one can verify it is an isomorphism by loking at whenever there is an edge, say $\{1, 2\}$ is an edge, so therefore, correspondingly what I want, a and c must be an edge. But now the graph is complete so all possible edges are there. It is a trivial isomorphism, so we do not have to discuss it further.

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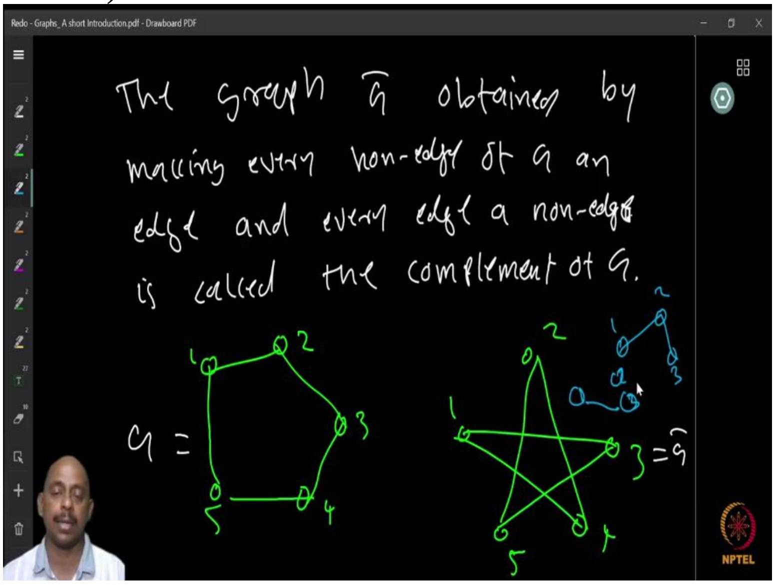

Here is another example of an isomorphism. So, I have the graph G here, 1, 2, 3, 4, and 5, and then I have this graph $\bar{G}$, let us say 1, 3, 5, 2, 4, and 1. So, these two graphs are isomorphic. Now, these two graphs are isomorphic because one can see that there is an isomorphism. What is isomorphism? You can just see like if you take 1 let us say, then let us say that 1 is mapped to 1, then 1 2 and 1 5 are edges so therefore, I want to make sure that wherever these guys are going that should be also edges. So therefore, I want 2 to be going to let us say 3 or 4, one of these I have to go.

So, then once I map then I can say that 2 is adjacent to 3, therefore if 2 is going to 3, then whatever that is, it should be mapped to, 3 is adjacent to 5 here, therefore whatever is that adjacent to here. 4 must be mapped to this, and similarly you can find out the rest. So, this will give you an isomorphism and one can verify that it is indeed isomorphism.

Let us define something else, let us take a graph G, and then whenever there is an edge, I will make a new graph on the same vertices. Whenever there is an edge in the graph G I will say that, that edge is not present in the graph $\bar{G}$. If there is no edge between two vertices in the graph G, I will say there is an edge in the graph $\bar{G}$.

So, I take the graph G, every non-edge of G becomes an edge and every edge becomes a non edge. Such a graph is called complement of graph. So here I have the graph G and its complement. In this particular case it happens that graph G is isomorphic to its complement, it is not always the case you can just see if you take some other example, like; let us say you take this graph 1, 2 and 3. Then what is the compliment of this? I have 1, 2 and 3, now {1, 2} is an edge therefore there is no edge between 1 and 2 in the complement, {1, 3} is not and edge, therefore, {1, 3} will be an edge, {2, 3} is an edge, therefore, {2, 3} will not be an edge so I get this graph. These graphs are not isomorphic.

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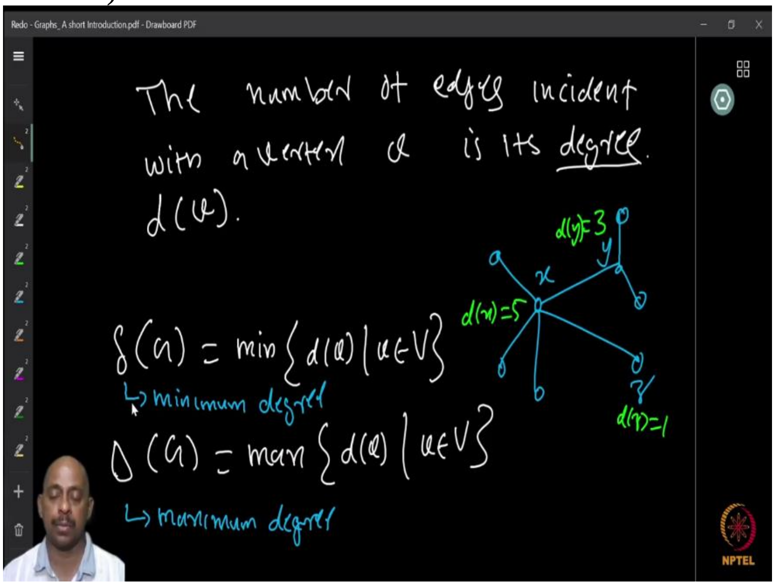

Now, the number of edges incident with a vertex of a graph is called the degree of the vertex. Now, this is a very impotant notion, we will use it quite frequently, and especially in combinatorics we use it for counting things and you know studying several properties of objects.

So now, let us look at some examples, you have this graph drawn here, and in this graph I have the vertex y let us say, if I take y. Now, the vertex y has three edges which is incident with it. You can see it from the picture, so that is the advantage of using the picture. If I have the set theory description, you have to count how many occurance of y as one of the end points.

So, here it is very clear, I have three edges. So, these three edges are incident with y therefore, the degree of y is three. Similarly, if you take the vertex x, it has 5 edges incident with it, therefore, its degree is 5. Similarly, vertex z has only 1 edge, therefore its degree is 1. Now, if you look at all the vertices, look at their degrees, you can collect them together. If you look at the smallest value among these numbers, so degrees of each vertex; look at the smallest value. The smallest value is called the minimum degree of the graph, because, the vertices with the smallest degree its degree is the minimum degree in the graph G. So, I denote this by $\delta(G) = \min\{d(v) : v \in V\}$, the minimum degree of the graph G.

Now if you look at the largest degree, maximum of all the degrees, then it is called the maximum degree of the graph and it is denoted by $\Delta(G) = max\{d(v) : v \in V\}$. Again, these notaions we will use very often so you should try to remember.

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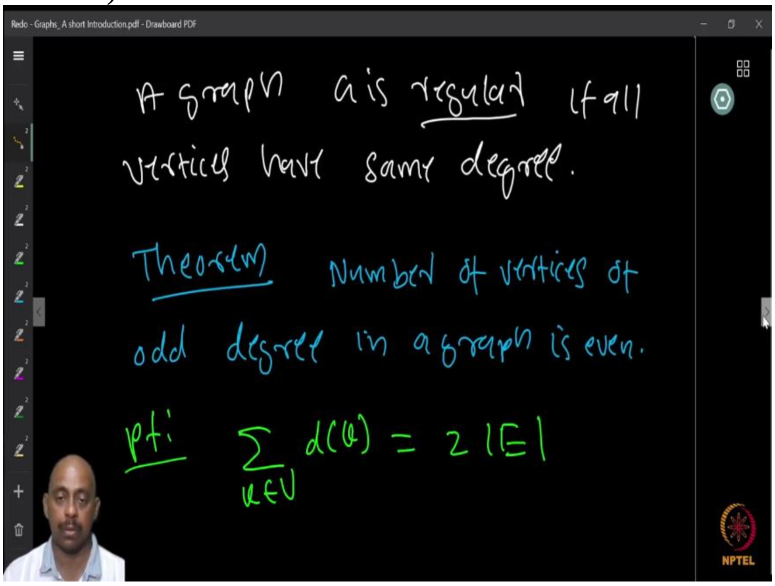

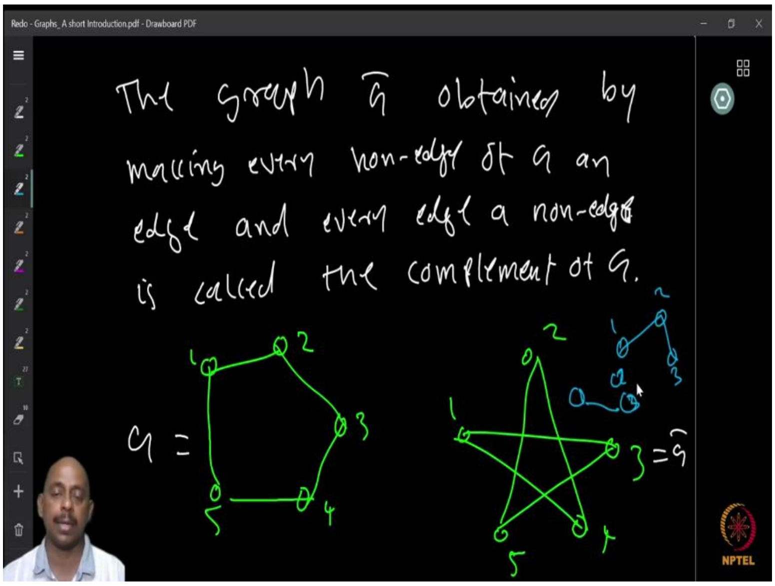

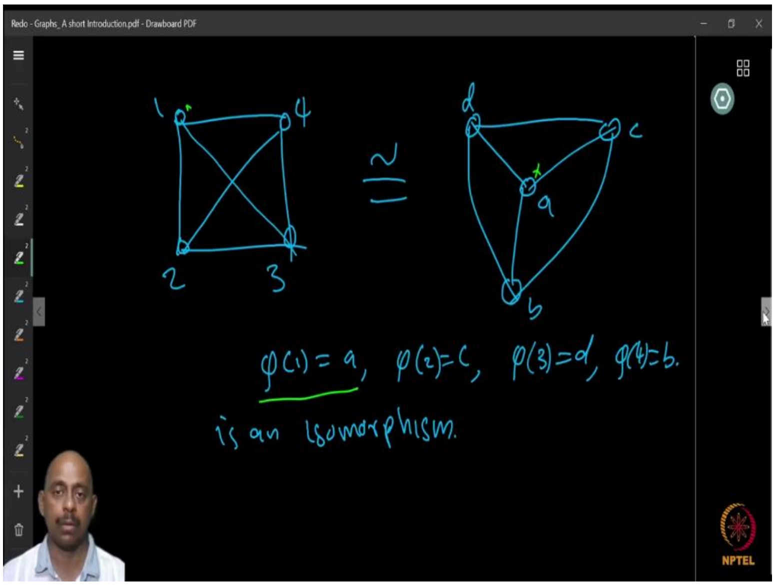

Now, let us take a graph and suppose all its vertices have the same degree then we say the graph to be a regular graph. You saw some examples of regular graphs before, for example this graph was a regular graph. You had, every vertex had degree exactly 2 edges. Similarly, its complement it is same.

And we had the complete graph, on 4 vertices. Every vertex of this graph has degree exactly 3; therefore, it is also a regular graph. So regular graphs are important because it has more nice properties. So, we can see those properties, and therefore that comes quite frequently. We now study the first theorem of graph theory. What is the first theorem of graph theory? The number of vertices of odd degree in a graph is always even. So, you cannot have a graph where you have odd number of odd degree vertices. And why is this?

The proof is immediate, because the sum over all the degrees is always twice the number of edges. Suppose you have sum over all the degree is twice the number of edges, then the theorem is immediately clear, why? Because if you have a graph where there are odd numbers of odd degree vertices, then remaining vertices, if there are, will all have even degrees.

Let us look at the sum of the degrees, and then you have even times something plus odd times odd. Therefore, I have an even number plus an odd number, which is an odd number. But odd number cannot be twice the number of edges, which is an integer. So therefore, if I have this theorem then I know that the number of vertices of odd degree must be always even.

Now, how do you prove this theorem, that, the summation of degree is twice the number of edges? Well, it should be very clear if you think about for a few seconds, but let me explain. If you take the graph and sum over all the degrees, so what happens? So, every vertex you are going to count the number of edges incident with it.

Now, if you look at in terms of the vertices, this is what we are doing. But now let us look at what happens with the edges? Now, an edge contributes to the degree of the vertices which are incident with it. So, if I take an edge it has exactly two vertices incident with the end points of the edge e.

Look at the end points of the egde e, e contributes to the sum of the degree, we are looking at the degree sum, and this degree sum the contribution of an edge is basically 2, because it contributes one to one of the end point and another one to the other end point. So, both end points contribute one or this edge is counted by both of the end points exactly once.

So, two times we are counting the edge, therefore, we are counting all the edges exactly, two times each, so therefore, I have two times the number of edges. The total degree is twice the number of edges which is an even number. Therefore, you cannot have odd number of odd degree vertices.

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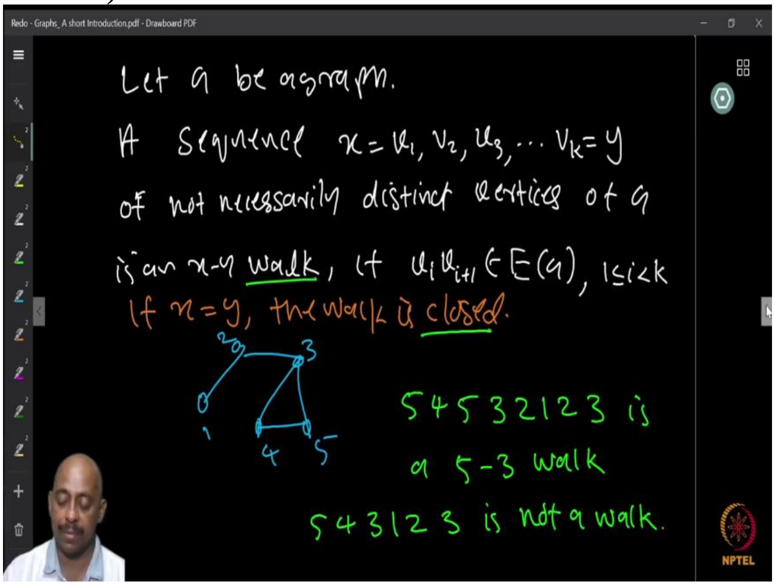

Now, let us take a graph G, and then what we do is to, let us look at this example first. Let us say that I start from a vertex we will say 1. So, I start from 1, then I go to 2 by an edge after that I proceed to the vertex 3, then I decide I will go back to 2. Now, I go again to 3, then I go to 5 and then I decide to stop. So, basically, I have taken a walk, supposing that this is a road network, I can think of this as a walk from the point 1 I go to 2 by the road, then 2 to 3, then for whatever reason I decide to walk back to 3 and to 2 again. Then I go back to 3 once again and then go to 5 and then I decide enough of walking and I stop. Therefore, this is called a walk.

So, a walk is basically a sequence of vertices or sometimes it is defined as sequence of vertices and edges, such that, consecutive vertices is always an edge. So, if you have $v_1, v_2, v_3, \dots, v_k$. some labels of the vertices, which form a walk in the graph if $v_1\ v_2$ is an edge, $v_2\ v_3$ is an edge; etc. $v_{k-1}v_k$ is an edge.

These vertices need not be distinct, you can repeat them. For example, as I said I can go from $v_1\ \ to\ v_2$, then I can go to v_1 again then go to v_2 then go to v_3, again come back to v_2 and v_1, so all these things are possible. So, a sequence starting with let us say x, then which is $v_1, v_2, \dots, v_k$, is said to be x-y walk where v_k is y, if $v_i\ v_{i+1}$ is an edge for every $1 \leq i \leq k$.

Now, if it happens that the starting point is same as the ending point then the walk is said to be closed walk. For example, if I start with 1 then I go to 3…2, 3 then go back to 2 and then back to 1. Then that is a closed walk in the graph. So here are some examples, 5 4 5 3 2 1 2 3, in this graph. 5 then 4 then 5, 3 then 2 then 1 and 2 and 3, it is a 5 to 3 walk.

Similarly, but on the other hand if you look at 5 4 3 1 2 3, this is not a walk, because 5 4 3 is okay, but then 3 to 1 is not an edge, 3 1 is not an edge, so I cannot say that this is a walk, because 3 1 is not edge. Now suppose you have a walk in which I do not allow the repetition of edges then we say it is a trail.

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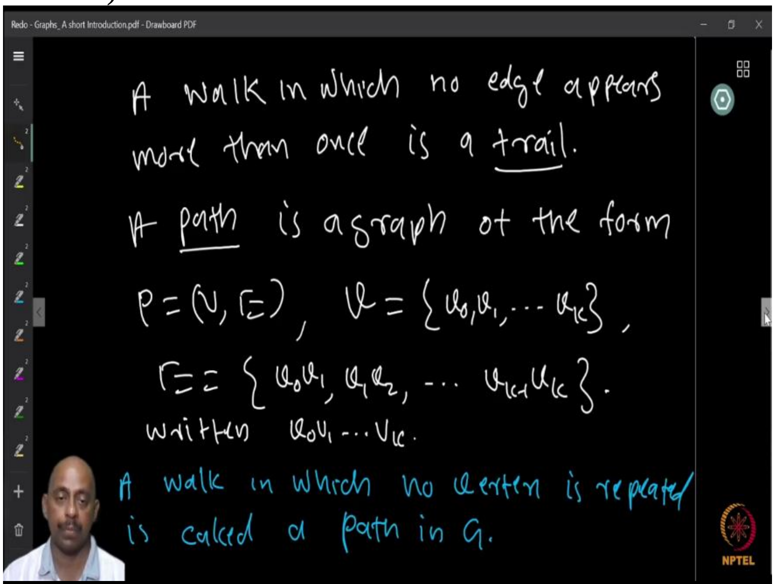

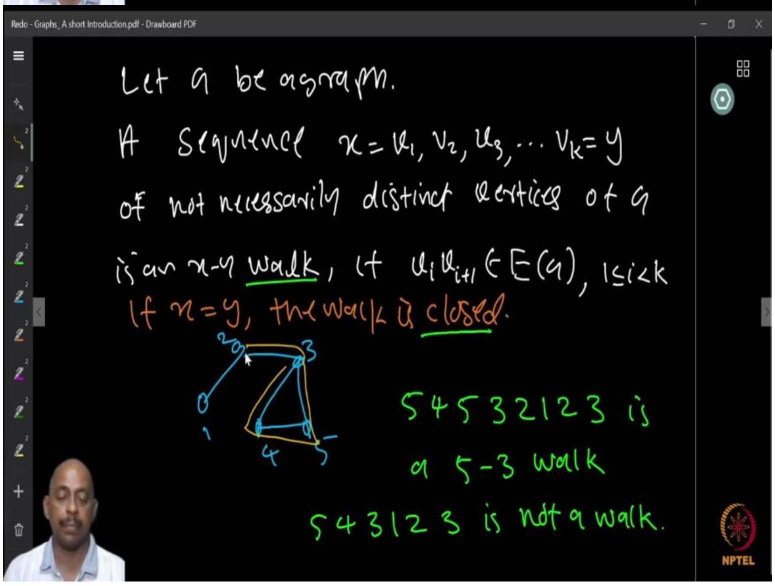

So, a walk in which no edge appears more than once is called a trail. So, trails are paths where you are not allowed to repeat the edges, but you are allowed to repeat the vertices. For example, you can see the same graph, I can define the following trial, I can go from 1 to…or maybe I say that I start from 3 4, 4 5, 5 3, 3 2. That is a trail, because 3 to 4 is an edge, 4 to 5 is an edge, 5 to 3 is an edge, 3 to 2 is also an egde.

And I do not repeat any edge, what I repeat is the vertices. Therefore, we have a trail, so the trails can repeat vertices, but not the edges. If I do not allow repetion of any vertex also then a walk is called a path. The path is a graph of the form let us say P equal to V comma E where $V = \{v_0, v_1, \dots, v_k\}$, and E is the set of edges $\{v_0\ v_1, v_1 v_2 \dots\ v_{k-1}\ v_k\}$.

There are no other edges; the precise set of edges is given here. Then the graph is called a path graph. On the other hand, if I am talking about a graph and then looking at a subgraph which is isomorphic to a path then that graph can have other edges, but the path that we are looking at it is a subgraph of the graph.

So, if a subgraph of a graph is isomorphic to a path graph, then we say there is a path in the graph G. Another way to define a path in the graph G is to say that it is basically a walk where I do not allow the repetition of any vertex. Then you can clearly see that the consecutive vertices form edges. Since there is no repetion of vertices, there cannot be repetiion of edges. Then we will see that you have precisely the set of edges as we want in the path, so I get a path as a subgraph.

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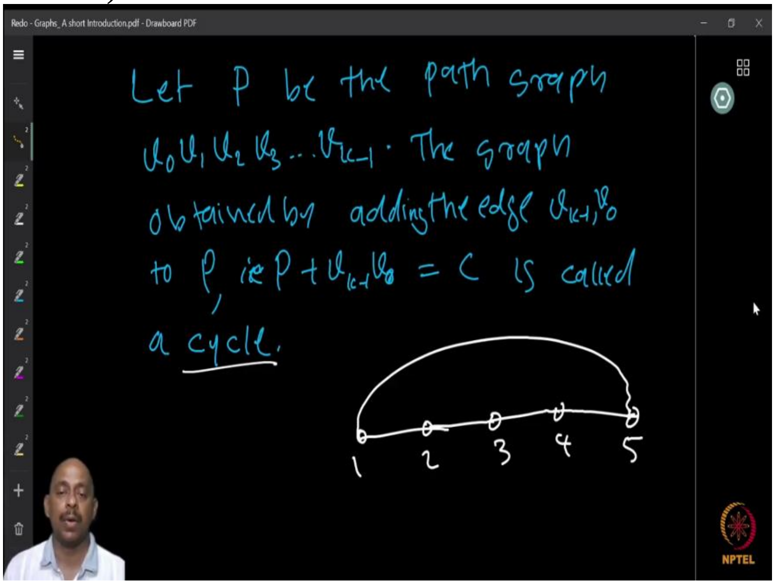

Now, suppose it happens that you have a path like let us say 1 to 5 in this example, and then suppose I also add one more edge starting from end point of the path to the beginning of the path. If we add an additional edge, then we say it is a cycle. So if we have a path $\{v_0\ v_1 v_2\ ...\ v_{k-1}\ v_k\},$ then if I add the edge $v_{k-1}\ to\ v_0$ then I will get a cycle. So, a path together with one more edge connecting the end point of the cycle. So, here is an example of a cycle, we have 1, 2, 3, 4, 5, and then 5 to 1. Again, cycles also can be subgraph, if there is a subgraph isomorphic to a cycle graph then it is basically a cycle in the graph.

Now, if you want to define in terms of walks, you can see that basically cycle is like a walk where we do not allow repetition of vertices except for the starting vertex can be repeated exactly once, so we can come back to the vertex. It is basically a walk where we do not repeat any vertex except for the starting vertex and ending vertex can be the same.

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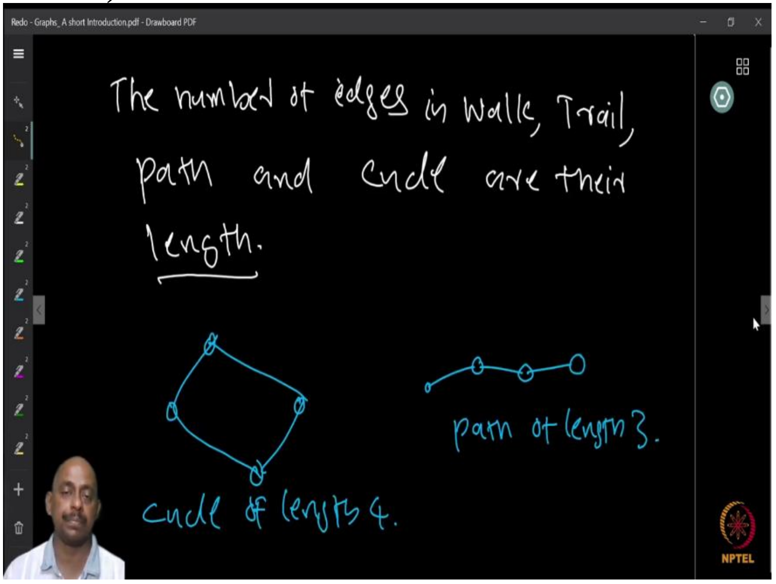

Now, for all these walks, paths, trails, etc. you can define the measure I call the length of the walk. So, the lengths of the walk, trail or path are basically the number of edges in this. We usually denote the number of edges to talk about the distance or length. Here we have a path of length 3, you have 4 vertices and then you have 3 edges. So, you have a path of length 3. Then you have an example of a cycle of of length 4, you have 4 vertices which forms a cycle. So, the length is again the number therefore, the length of cycle is 4 and length of the path here is 3.

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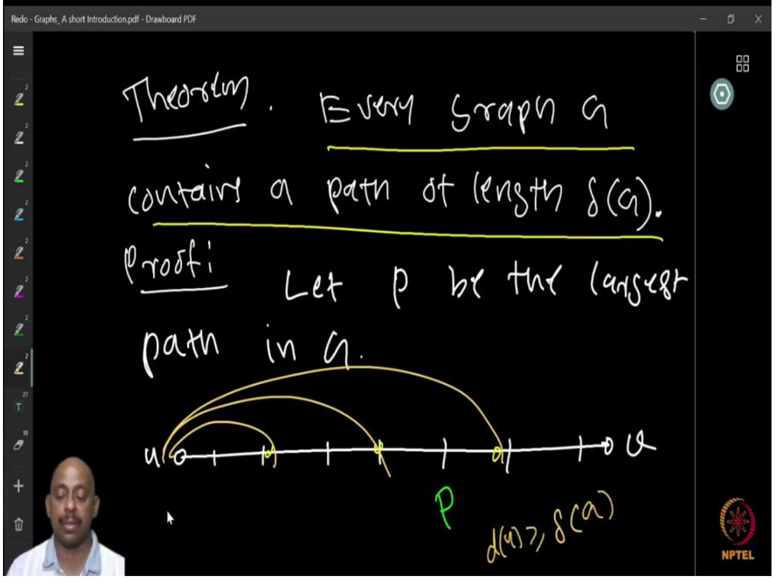

Now, here is the very interesting theorem, what I want you to do is to think of a proof, I have given a hint here. What you want to show is that, every graph G contains a path of length $\delta(G)$. Now, why is this true? So here is a hint; let us start with a path of the largest length; maximum length in the graph G. So, P be the largest path in the graph, so we can denote P by some u-v path where u is one of the end point and v is the another end point of the path.

Since it is a path, I know that all these edges are present in the graph. You have u-v path which is the largest path in the graph. Can we use this information to say that every graph G contains a path of length equal to minimum degree? So, here is the proof, if you think about it sometime, you will get it, so you stop and think about it for few minutes. Then I can continue, so what I do is that I look at the u-v path which is the longest path in the graph. Since it is the maximum length path, let us look at the vertex u.

If the minimum degree of the graph G is 1, since there is at least 1 edge in this graph, you know that there is a path of length 1. If minimum degree is 1 then I am done, so I have an edge and therefore, I have a path of length 1. Suppose it is not the case, if the minimum degree is more than 1 then the vertex u which is the starting point of the path will have at least some additional neighbours, δ neighbours are there. Only one of them is present here, all $\delta - 1$ neighbours must be present in the graph.

So, let us look at the remaining neighbours of u, where can the neighbours of u go? I claim that there cannot be a neighbour of u outside this path P. Then, all its neighbours must be in the path P itself. Why is this? If I look at the vertex u, suppose it has a neighbour outside this path P. If there is a neighbour outside the path P then, let me start from this neighbour wherever it is.

So, let us say that it is x, x to u I go, then u to v I go, because the vertex x is not part of the path P. This is a path of longer length, I can go from x to u first and then I can go from u to v taking the path P, via the longer path, but we started with the assumption that P is the maximum length path.

This tells us that, all the neighbours of u must be in the path P itself. All the delta neighbours of u are in the path P including the neighbour in the path itself. Therefore, every neighbour of u must be in the path P. Now, the degree of the vertex is…the minimum degree of vertex is delta, so u has at least small delta of G many neighbours.

So, what does it says? It says that the u-v path contains at least δ vertices, which are the neighbours of u. Apart from u you have δ more vertices in this path, if the path have at least delta verticespresent, then the length of the path must be at least δ, because apart from u you have δ more vertices, $\delta + 1$ vertices must be there in the path. Tha is there could be more, there is at least small $\delta + 1$ in the path, therefore, its length must be at least δ. This is why every graph must contain a path of length of minimum degree of G.

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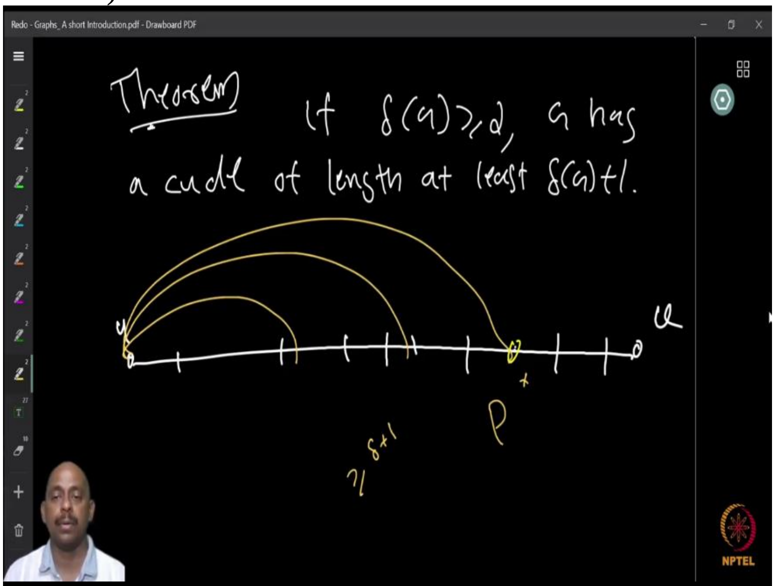

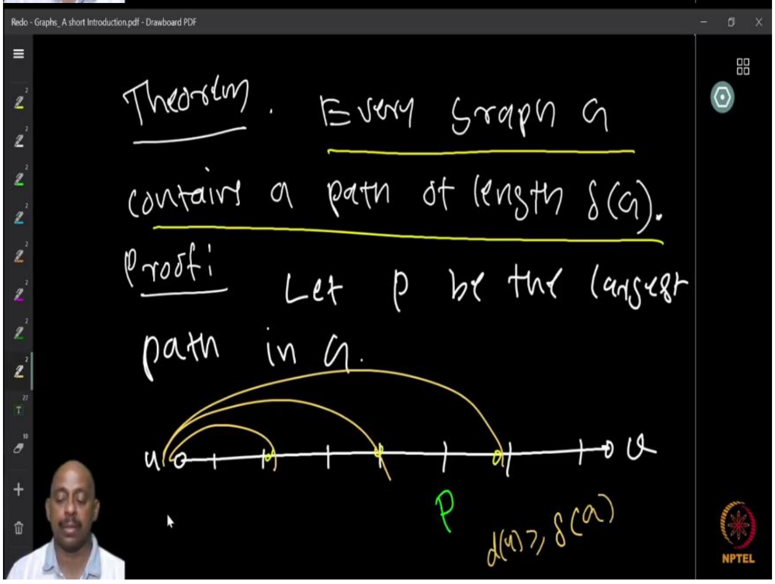

Using the same idea, we can prove that if minimum degree is at least 2 then the graph has a cycle of length at least $\delta(G) + 1$. If a graph has minimum degree at least 2 then it definitely has a cycle. Then it says that the cycle has length at least small delta plus 1, I mean there is such a cycle, there could be smaller cycle, but you have at least a cycle whose length is minimum degree plus 1.

The proof is absolutely the same, if I look at the maximum length path in the graph, say P. As we mentionied, all the neighbours of u must be in P itself. Since there is at least δ neighbours for P, all the neighbours of u are in the path, look at the farthest neighbour.

This neighbour, since all the neighbours are happening between u and this vertex, you know that there is at least $\delta + 1$. So now, I have this path from u to P, I mean u to this vertex, but then ofcourse, because this vertex is a neighbour of u, I can go back to u from this vertex.

If I look at this path of whatever length greater than or equal to $\delta + 1$, that plus this edge, defines a cycle whose length of course is at least $\delta + 1$, because $\delta + 1$ vertices are here, we need at least δ edges and then together with this edge it will be small $\delta(G) + 1$. With the same idea we have proved two theorems. First theorem was, if you…have a graph G then it has at least a path of length minimum degree of G and if the minimum degree is at least 2, there is a cycles and the cycle has length greater than or equal to $\delta(G) + 1$.

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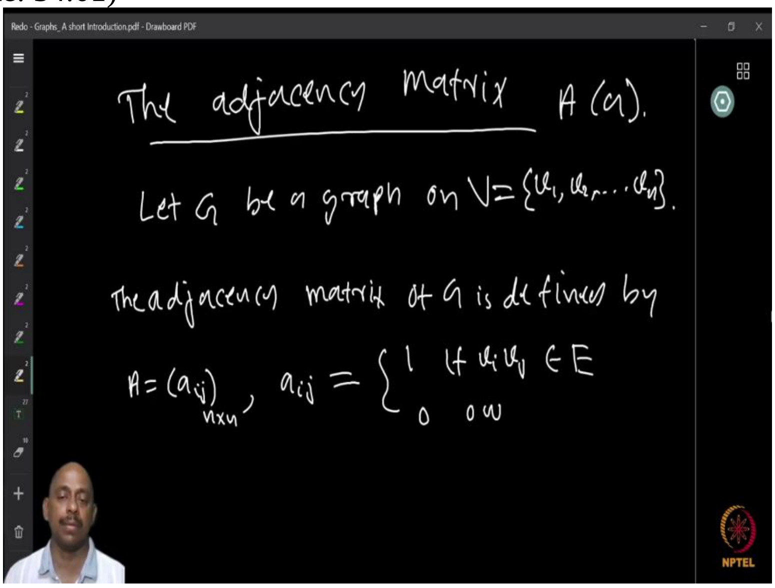

Let me define what is called the adjacency matrix of the graph G. So, this is a way to represent a graph, because whenever we use computers and all, you need to represent the graph in some other way. But ofcourse there are some other uses of the adjacency matrix, we will see some of the uses soon.

So, what is the adjacency matrix? We have the graph G with vertex at say $v_1, \ldots, v_n$. The adjacency of the matrix G is defined as, the matrix is defined as, $A = \left(a_{ij}\right)_{n \times n}$, $a_{ij} = \begin{cases} 1, if\ v_i v_j \in E \\ 0, otherwise \end{cases}$. It is very simple definition. Let us look at an example.

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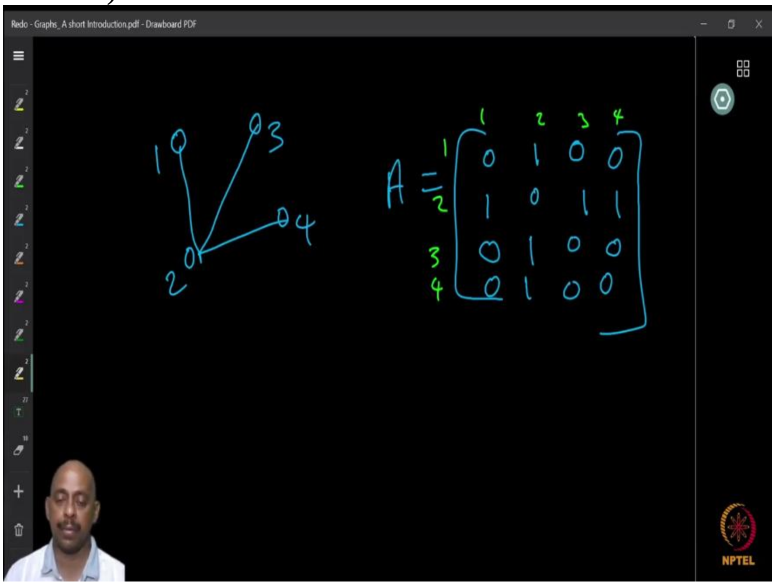

We have a graph here, {1, 2}, {2, 3}, and {2, 4} are the edges. Then I define the adjacency matrix as this, I write it like this. I have this matrix with the following entries 1, 2, 3, 4 are the corresponding vertices, the labels of the rows and the columns, so (i, j) is an entry if and only if $v_i v_j$ is an edge. Since 1 to 2 is an edge I have this entry, similarly, this entry because 2 to 1 and 1 to 2. Similarly, {2, 3} is an edge so therefor this and {2, 4} is an edge therefore this, similarly {4, 2}, and {3, 2}. So, these are the non-zero entries then remaining entries are all 0. Therefore, I have the adjacency matrix.

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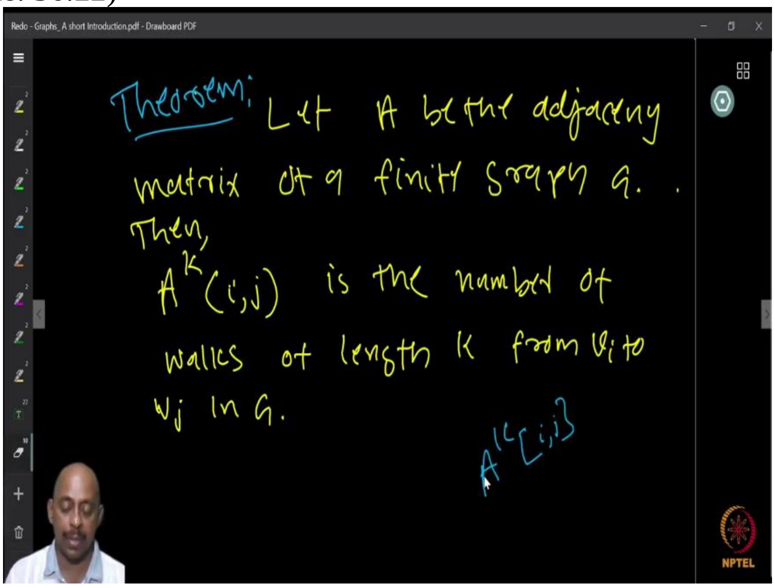

Now, an immediate theorem is that if you have the adjacency matrix of a graph is given, let us say it is A, then if you look at the k th power of this matrix (Power means that you are taking the product of the matrix with itself; because it is a square matrix I can multiply it many times) so I take A^k which is the k th power of the square matrix A.

Now, I look at the (i, j)th entry of this matrix A^k. The claim is that the value in this position in the matrix A^k is the number of walks of length k from v_i to v_j in the graph G. So (i,j)th entry is the number of walks of length k from v_i to v_j. This is fairly easy to prove, I want you to think about it and try to prove it by yourself, but we can discuss it. If you want you can stop and try to find the proof yourself.

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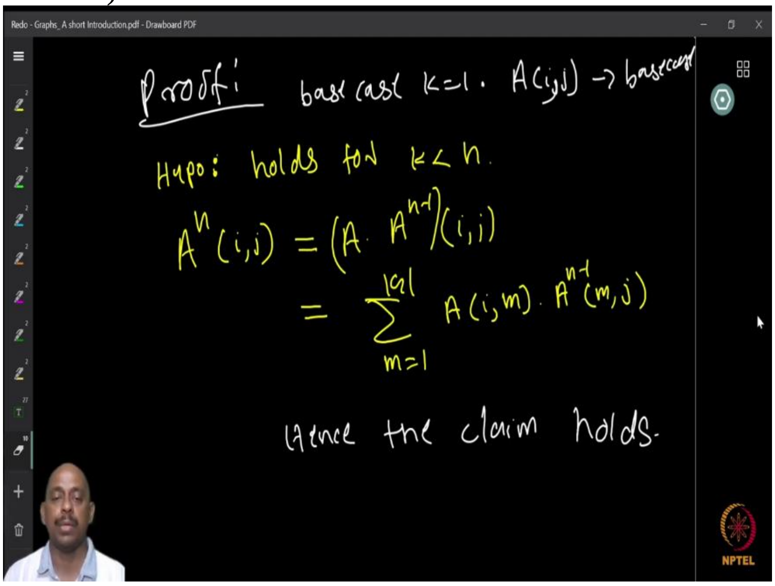

Now, here is the proof. We use induction on the value of k. So, the base case is k is equal to 1, because I start with the adjacency matrix and the base case is clear because (i, j)th entry is defined to be 1 in the adjacency matrix if v_i v_j is an edge in the graph, so there is an edge then it is equal to 1 otherwise it is 0.

Therefore, the base case is clear. The non-zero entries are precisely when there is an edge, therefore, there is a path and that is precisely 1 and there is an edge only one edge. Now, because of this we can now use induction. The induction hypothesis is that the result holds for any value of k which is strictly less than n then we do it for n.

Look at the nth power of A, what is the nth power of A? I look at A^n and just look at the (i, j)th entry of A^n . A^n is $A \times A^{n-1}$, so look at that $A \times A^{n-1}(i,j)$. Now, by the definition of matrix products, we have $A \times A^{n-1}(i,j) = \sum_{m=1}^{|G|} A(i,m) \times A^{n-1}(m,j)$

Ofcourse A(i, m) says that there is an edge between i to m if the value is non-zero, and then $A^{n-1}(m,j)$ by induction is the number of walks of length n - 1, from m to j. If there is an n - 1 length path from m to j and there is a path of length 1 from i to m then there is a path of n from i to j.

So, since $A^{n-1}(m, j)$is precisely the number of walks of length n - 1 from m to j, I can create a path of length n precisely when going through this vertex m precisely if {i, m} is an edge. So, I am basically counting with this; I look at every vertex which has the path of length n - 1 to the vertex j and then I see if there is an edge from vertex i to m, direct m, then I can make an n length path and I can add this number.

I just add them together and the non-zero entries will sum to number of paths of length n and that is the proof.

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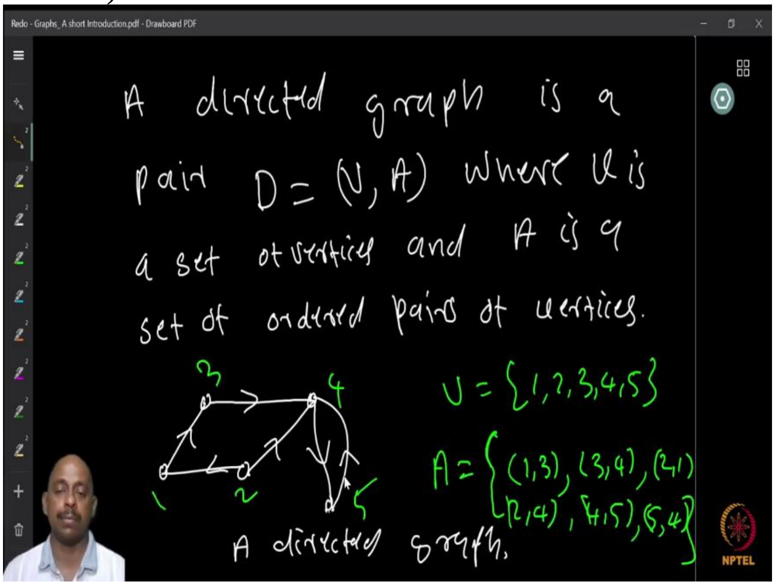

Now, I will stop this with a definition of what we call a directed graph. When we define the graph earlier, we have said that we are looking at just 2 elements of it. In directed graph we want to say that the order is important. So, I will say that I have an edge from u to v, but there is no edge from v to u, basically it is a one way traffic let us say.

From point A to point B this road is going from A to B but you cannot take your vehicle from B to A because it is one direction. So, this way we want to generalize this idea then we can say it as directed graph. A directed graph is a pair let us say (V, A) where V is a set whose elements are called vertices as usual, and A is a set of ordered pairs of vertices which we call arcs.

Sometimes it is also called edges but we can call them as arcs. So, what is an example, let us look at the given graph below with the vertex set V + {1, 2, 3, 4, 5} and the arc set {{1, 3}, {3 ,4}, {2, 1}, {2, 4}, {4, 5}, {5, 4}}. It says that {1, 3} is an arc it means that I can go from 1 to 3, it is directed. In the graph represented in the picture we will also put an arrow mark to say that the edge starts from 1 and ends at 3.

So, it is 1 to 3 with the arc direction, similarly, I can have an edge from 3 to 4, 4 to 5 and also I can have 5 to 4 because I allow this multiple going from 4 to 5 and also from 5 to 4. So, therefore, I have this graph, which we call as directed graph.

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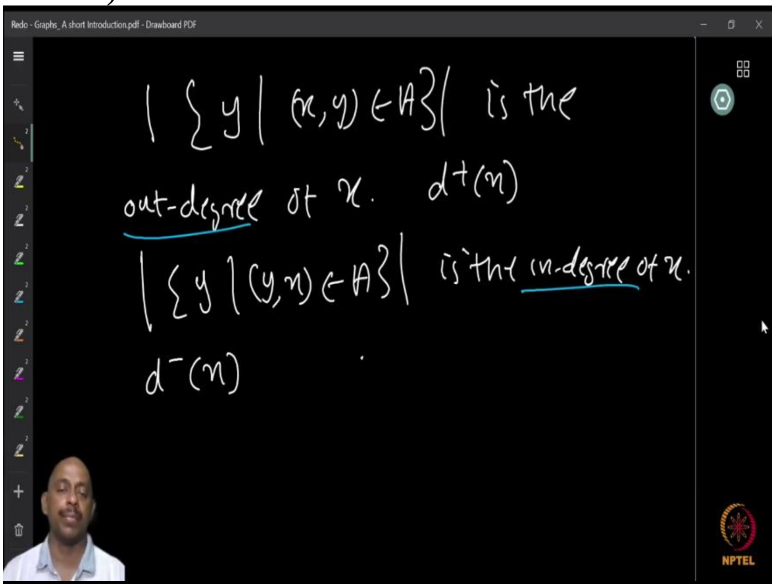

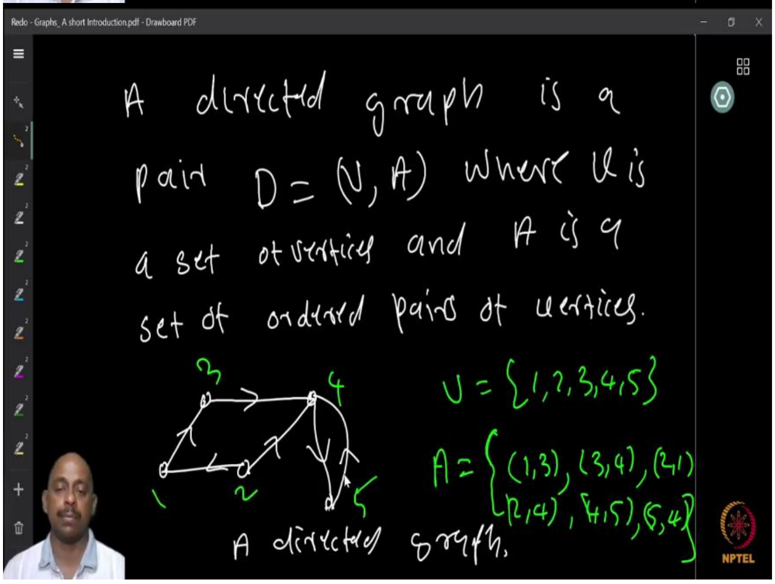

If you look at the directed graph, the degree can be defined as, we can count separately the edges which are going out and which are coming in. The number of edges which are coming in is the in degree and the number edges which are going out are the out degrees. Out degree is the set of all y such that xy is an edge, ($|\{y: (x, y) \in A\}|$)so that is the out degree of x. Similarly, set of all y such that yx is an arc is the in degree of x (($|\{y: (y, x) \in A\}|$)). So, usually denote by d^- and d^+ for the out degree. So, I think with this we can stop and we will continue with further topic in the next lecture.