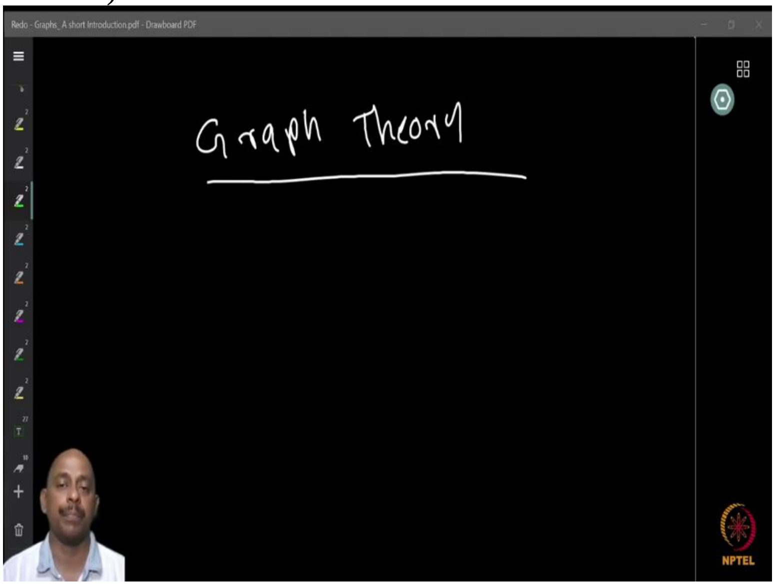
Combinatorics
Professor Doctor Narayanan N
Department of Mathematics
Indian Institute of Technology, Madras
Lecture 31
Graphs - Introduction

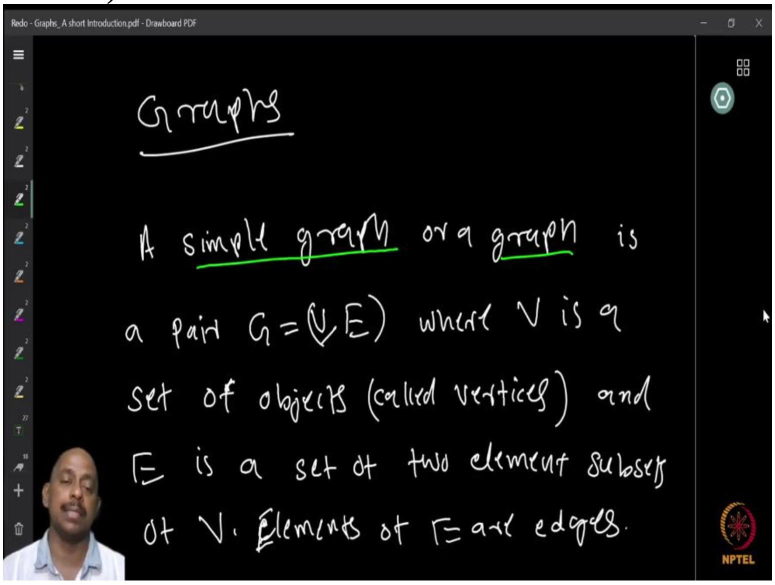
(Refer Slide Time: 00:14)



Hello and welcome. In the next few lectures, we will discuss topic from Graph Theory. Graph Theory is one of the most popular areas in Combinatorics. One of the main reasons it is so popular is that it is very easy to convey the ideas of a graph theory question, and explain it to a person, who is not an expert in that area.

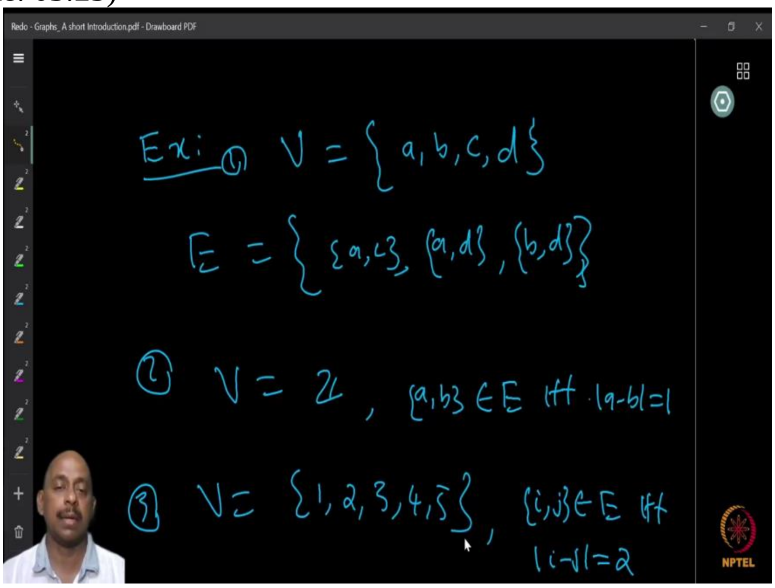
But even more importantly, it is popular because it has a wide range of applications in many areas, especially a topic from computer science, biotechnology, chemistry, engineering branches and several other topics. This is because graphs as we will define soon are representing relations between pairs of object, and it is a way to visualize the relations in a nice way. So, that one can see what is happening in a pictorial manner. This is one of the main reasons the area became popular, and we will see what this is about. We will look at some of the basic results and notions and some few important theorems from graph theory.

(Refer Slide Time: 02:12)



Let us start with looking at what are graphs. For the time being we will define what we call simple graphs, and we will just call it as a graph. For the time being when I say graph, what it means is simple graph. And what is a simple graph? A graph is a pair of sets; let us say G = (V, E), where the elements of the set V are called vertices, so this could be a set of objects. And the set E is a set of two element subsets of V, so the elements of E are called edges. For our purposes a graph is this, that you have a set V whose element we call vertices, and a set E of what we say binary relation or set of two element subsets of V which we call the edges.

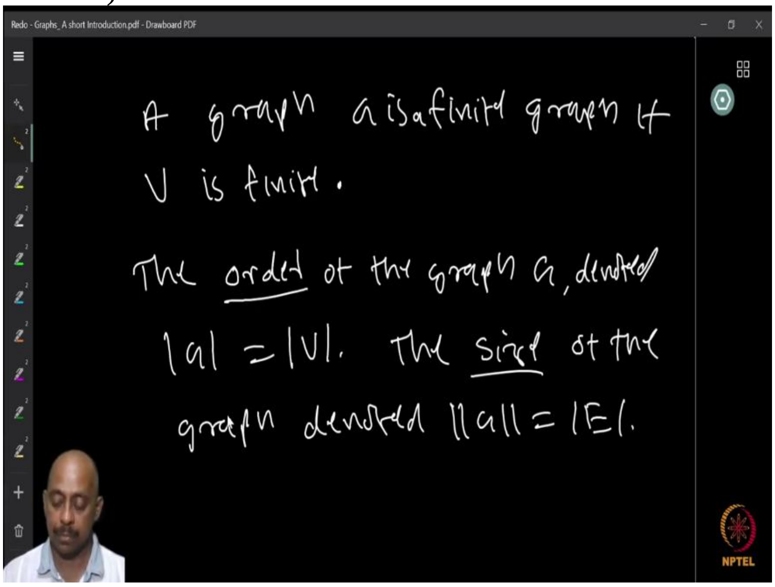
(Refer Slide Time: 03:25)



Now, let us look at some examples. We start with one simple example where you have 4 elements in the vertex set V, which we call V={a, b, c, d}. The edge set is a set of two element subsets of V, and here we define E = {{a,c}, {a,d}, {b,d}}. So, these are the three two element subsets which form the edges of the graph.

Another example I give, let us say V is ℤ, which is set of integers and two element a and b belong to edges if and only if their difference is 1. In this case it happens that the vertex set is not finite. Let us look at one more example, we look at the set V ={ 1, 2, 3, 4, 5}. Now, in this graph, let me define edges as the set {i, j} when the difference between i and j is actually equal to two. Whenever the difference of two numbers is equal to two then we say that there is an edge between them. So, these are examples of graphs or simple graphs.

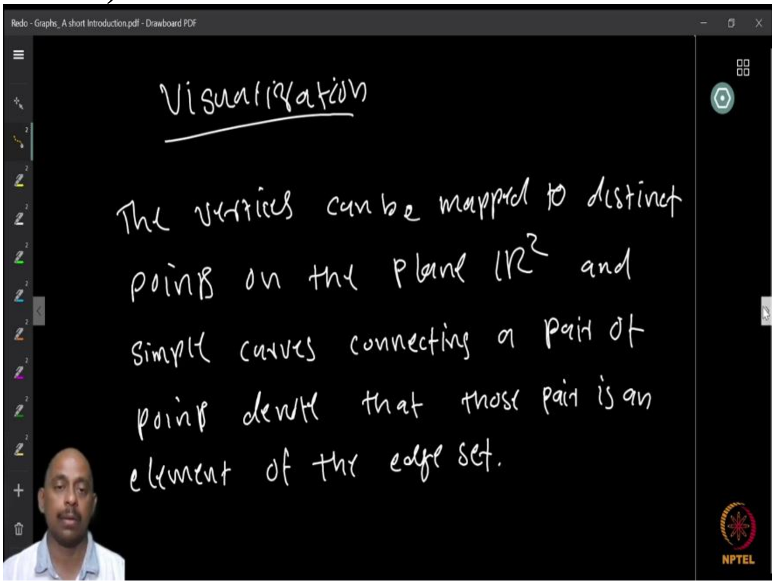
(Refer Slide Time: 05:15)



Now, a graph G is said to be finite if the vertex set and edge set are finite. Since we are looking at simple graphs, once the vertex set is finite we will also have the edge set is finite. So, for our purpose, if the graph is finite, the vertex set V is finite. The cardinality of the vertex set is called the order of the graph, which is the number of vertices in the graph. Now, we usually use to denote the order of the graph is G within the graph name within 2 horizontal bars and which is the order of the graph |G|.

The size of the graph, on the other hand is the number of edges of the graph, which is denoted by the symbol ||G||. It is not the norm, and in this context, we will use it as size. So, the size of the graph is always the number of the edges and order of the graph is its number of vertices.

(Refer Slide Time: 06:49)



As I mentioned the main advantage of using graphs is the ability to visualize relations. So, as we mentioned the edges are two element subsets, and the two element subsets can be thought of as binary relations. Whenever there is relation between two objects, we can say that there is an edge between them. So, this is one of the reasons it is applicable in many areas.

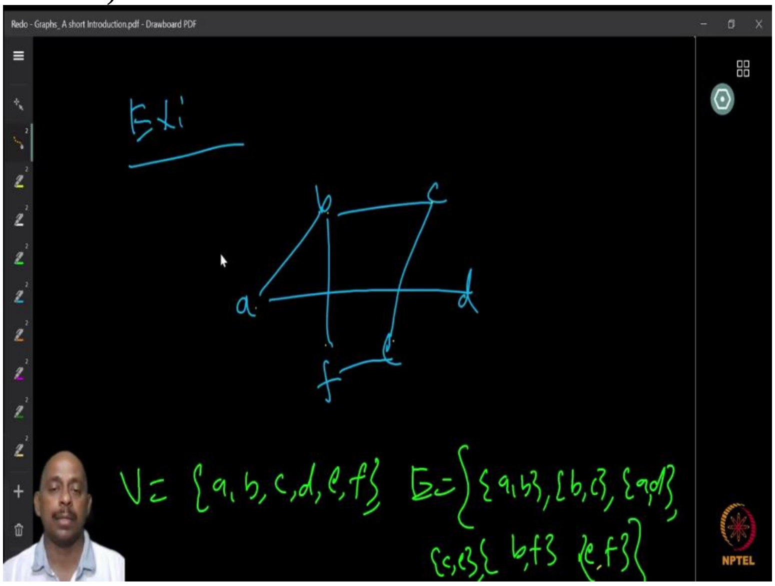
For example, you want to study the transport network of different cities. You have the cities and you have the connection between the cities with different routes. Now, these routes can be made as edges, if there is a direct path from, road from city 1 to city 2 then I can say that there is an edge relation between city 1 and city 2,{ city 1, city 2} is an edge.

This can be carried to other areas like, when we are talking about people, the relation between people, or in a computer you can talk about different components and the connection between them. Or in a chip for example, or in a World Wide Web, or social networks, neural networks, like electrical network, all these things can be represented as graphs. So, this is one of the main reasons why it has a wide range of applicability.

So, it will help if we can actually visualize these relations. So, what we do to visualize is to represent it as a picture, and how do we do that? We take your graph; look at the set of vertices. Now, the vertices we will map to distinct points on the plane, for the time being we will assume we will draw it on the plane. So, we will map the vertices to distinct points on plane.

Now, when two points or two vertices are on an edge relation, when there is a relation between them, it forms an edge. Then what we will do? We will draw a line segment or a curve connecting these two points, the corresponding points on the plane. So, this will give you a picture of the graph. Let us look at an example to see what we mean.

(Refer Slide Time: 09:49)

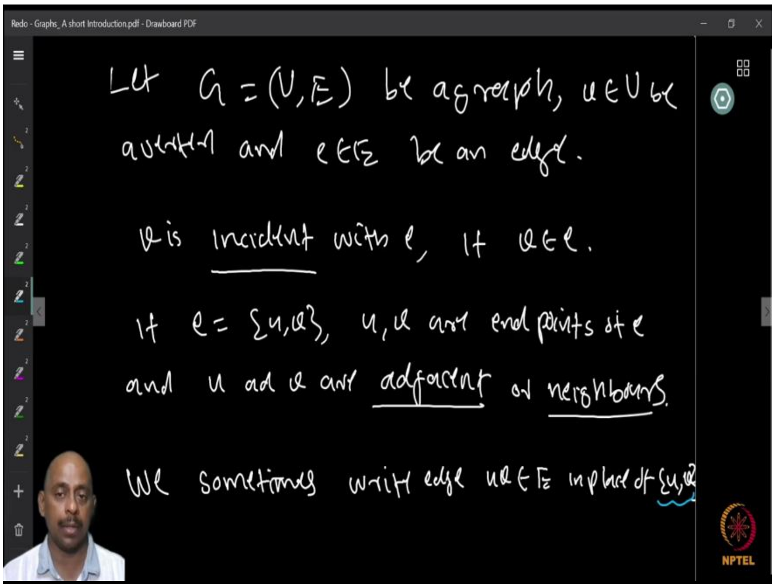


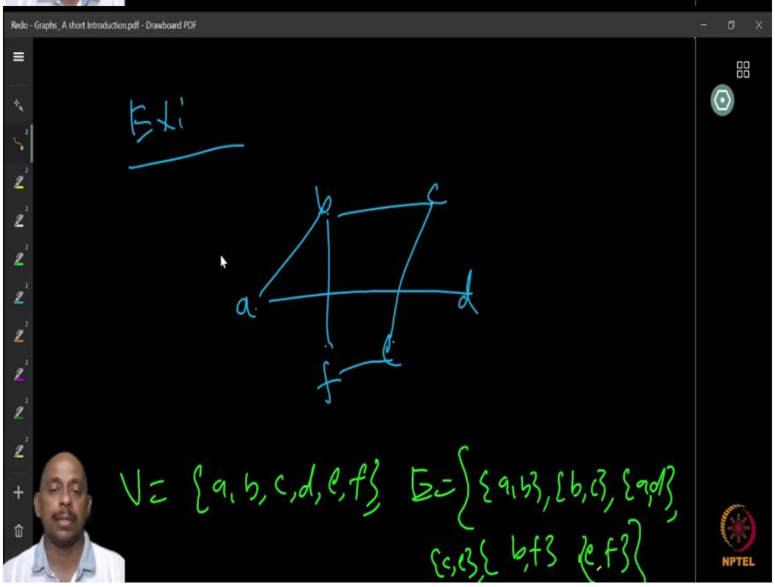
So, here I have the graph with the vertex set V = {a, b, c, d, e, f } and the edge set E = {{a, b},{b, c}, {a, d}, {c, e}, {b, f}, {e, f}}. Now, what we have done here is to map these points a, b, c, d, e, and f to different points on the plane. For example, a is mapped to some point here, b is mapped to some point here, d is mapped to another point, e and f. So, all these vertices are mapped to distinct point on the plane.

Since {a, b} is an edge, I draw a line segment connecting a and b. Similarly, since {b, c} is an edge, I have this connection. Then I continue, I have {a, d}, so I add a line joining a and d, then I have {c, e}, so c and e I will connect, b and f, I will connect and similarly e and f , I will connect. So, then I will get a picture, so this is a picture that we got. This picture tells you many things about the relation that we may not immediately see from looking into the description, the set theoretic description.

Now, if you look at this graph for example, you will see there is some kind of closed cycle here; we will define this thing formally, a closed walk here where you can find b to f, f to e, e to c and c to b. It is immediately clear from the picture. Here you need to find it out by going through each one and then see which one forms a cycle. In the picture it is immediately clear what forms a cycle and what does not form a cycle. So, this is one of the main advantages of using graphs, one can visualize many relations. We will see many other examples, and then how the visualization helps, in due course of our lectures.

(Refer Slide Time: 13:08)





Now, we will naturally have to define few terms so that we will be able to look at more advance stuff. Let us start with some definitions. Let G = (V, E) be a graph, we start with this assumption. And you have a vertex; let us say ‘v’ in the vertex set V. Let me call by ‘e’ as the name for an edge, so we know that e is basically a two-element subset, we denote it by some name e and that is it.

We say that vertex v is incident with edge e if v is an element of e. So, what I mean is that if you look at this graph, ab is an edge, a is the vertex which is part of the edge {a, b}. So, {a, b} is an edge, and because it is a two-element set, I can talk about the vertex a being an element of the 2-element set {a, b}. So, a is now an element of the edge, the vertex is the element of the edge.

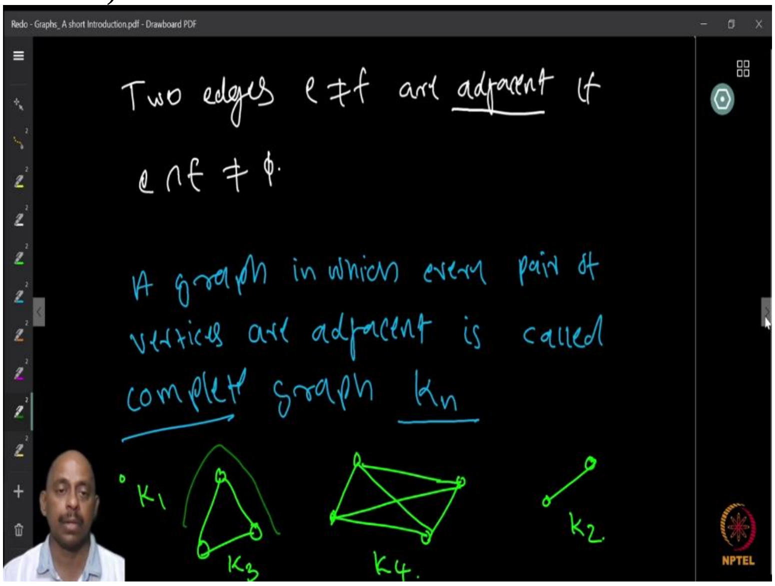
So, if that happen, I say that the vertex v is incident with the edge e. If the edge e equal to let us say {u, v}, then we call u and v to be the end points of edge e. So, end points of e are the two vertices in that edge. Now, the names like end points, incident, vertex, all these things come from the pictorial representation.

For example, if you look at the representation of the graph, you will see that ab is the line segment or the curve in the pictorial representation and a and b are basically the points, points a and b correspond to the vertices a and b. So, the points are basically the end points of the curve. The curve starts at a point let us say a and ends at a point b. So, this is the reason why we call it as end point and why we call it as incident, because one can see the incident of line segment. So, these are the geometric concepts.

Now, we say that 2 vertices u and v are adjacent in the graph G, if u and v form an edge. If there is an edge connecting u and v then we say vertices u and v adjacent or they are neighbours. I can say that u is the neighbour of v or v is the neighbour of u, etc. We often write the edge, {u, v}, in the graph, {u, v} is a 2-element subset, let us say that it is one of the edges of the graph.

Then to save the space of writing we often use the shortcut just uv belongs to E. So, uv if I write in our usual context, if I write uv, then u and v stand for vertices and uv is an edge automatically. So, unless I state otherwise, we will use this notation and of course sometimes we will have to change the notation depending the type of names that we use, or in some context we cannot use this notation. We will use standard notation u comma v or some other notations, in that time we will mention it explicitly. For the time being let us assume that if I write $uv \in E$ that means uv is an edge.

(Refer Slide Time: 17:54)



We say that two edges are adjacent, two distinct edges are said to be adjacent if they have non-empty intersection. Since the edges are distinct, we say that they are adjacent if they have at least a common vertex. Now, for our purpose since the graphs are simple, we always have an intersection; if they are not adjacent then the intersection is empty or otherwise it will have exactly one element.

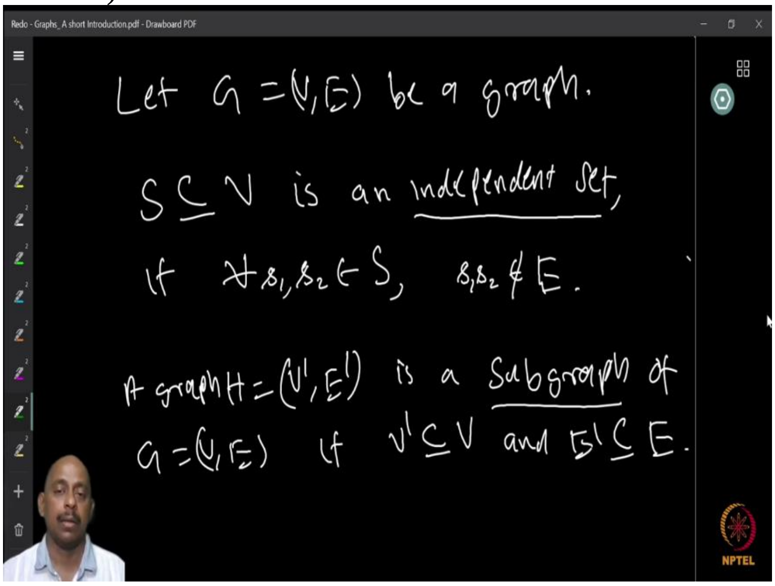
But when we talk about other types of graphs, not necessarily simple graphs, we will also come across the cases where the end points of the two edges could be the same. That we will not discuss in this course. Now, when you have a graph where you have a vertex set and any 2 vertices are adjacent in this graph. So, if any possible pair, every pair of vertices are adjacent then we say the graph to be a complete graph.

So, if there is an n vertex graph in which all the edges are present, all pairs of vertices form an edge then we say the graph to be complete, because we cannot have any more edges. We need to select any 2-element subset and then decide whether it is an edge, but if we select all 2-element subset then we cannot add anything further. Therefore, we call it as complete graph and we usually denote it by the letter K_n.

Here are some examples of complete graphs. You have the first one is K_1, which is a one vertex graph and there are no edges, because it has only one vertex, so you have this one. And you have the one vertex complete graph. Then you have the three-vertex graph that is K_3, where you have the three possible edges, and here it is.

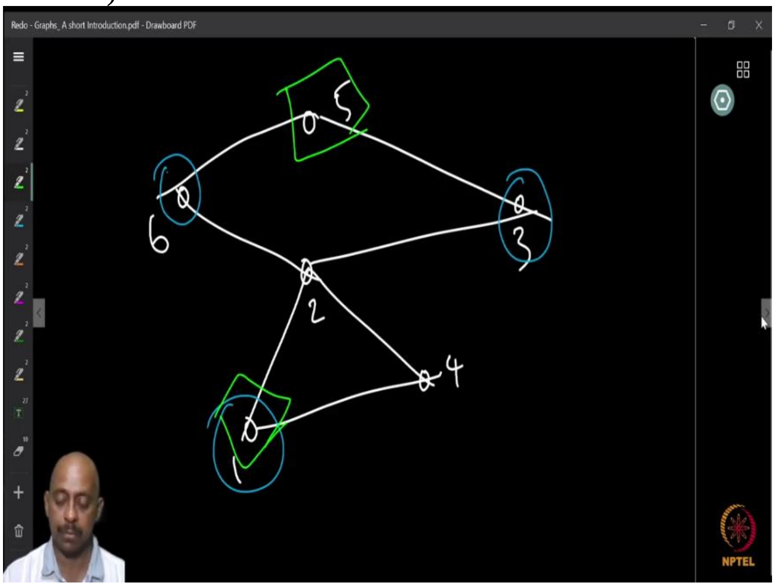
Then you have the 4-vertex complete graph where you have all the 4 vertices and then you have all possible 6 edges 4 choose 2. Two element subsets there are 4 vertices and therefore 4 choose 2 edges will be there. So, you have all the 4 choose 2 edges in graph K_4. You have a two-element vertex set and then you have a complete graph on two vertices that is K_2. So, these are examples of complete graphs.

(Refer Slide Time: 21:26)



Again, let G = (V, E) be a graph. We look at a subset of vertices let us say S, and we say that this set is an independent set in the graph, if between this subset of vertices there is no edge present. For any two elements in the subset they do not form an edge in the graph G then we say this subset is an independent set in the graph G. Now, let us look at an example for an independent set.

(Refer Slide Time: 22:17)



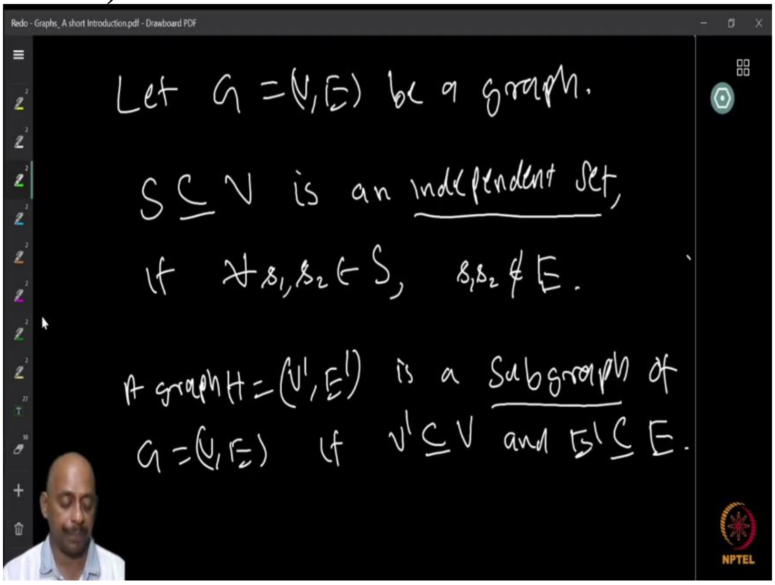
So, here is a graph, and in this graph what I am going to do is to pick some vertices, a subset of vertices. I pick let us say 1. Once I pick 1, since I want the subset to be independent, I will not be able to pick 2 or 4, because 2 is adjacent to 1, 4 is also adjacent to 1. If 1 is already there, I cannot pick 2 or 4 if I want the subset to be independent.

Then I have the choice of either 6, 3 or 5 one of these three. Suppose I choose 3, if I choose 3 then I can say that 1 3 is an independent set, because 1 itself is independent, 1 3 is independent set. Now, I can see, can I add any more vertices? So, I see that 5 is adjacent to 3, I cannot add 5. But 6 is not adjacent to 1 or 3, so I can add 6.

So, I selected these three and I got an independent set of size 3 in this. So, I have a subset that is 1, 3 and 6, which is the subset of the vertex set, and it is an independent set in the graph. I can talk about other independent sets, like maybe I started with 5. Now, once I choose 5, I would not be able to 3 or 6. I can pick maybe 2 or 4 or 1. Let me say that I pick 1.

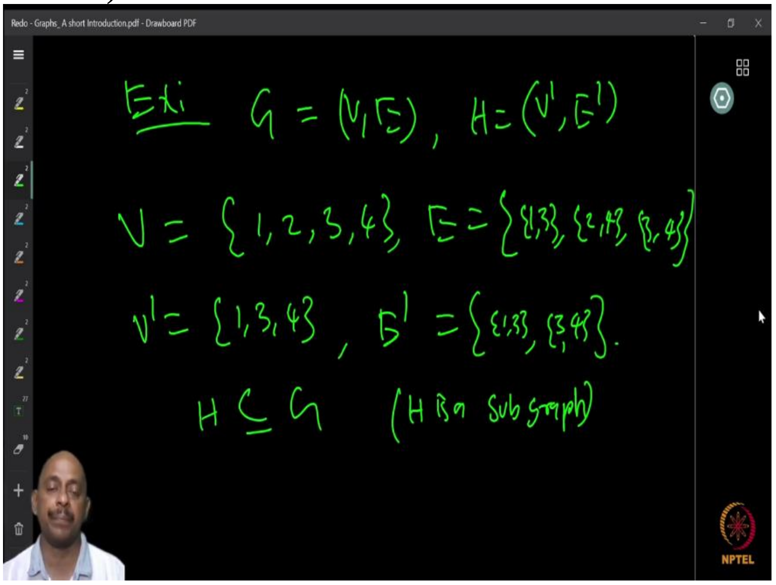
So, then 1 comma 5 is an independent set. Once I pick 1, I cannot add 4 or 2, so I cannot add any more. I have a two-element independent set 1 comma 5; I have a three-element independent set {1, 3, 6}. Similarly, I can find other independent sets.

(Refer Slide Time: 24:47)



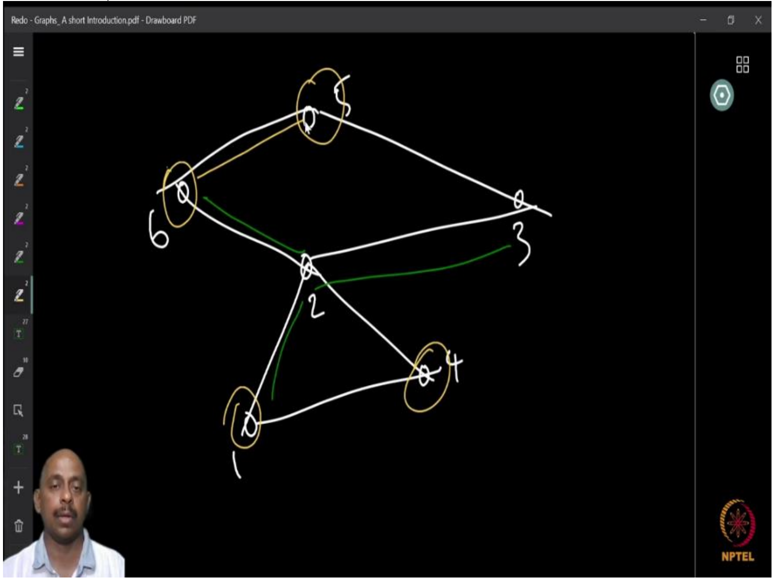
Now, let me define another graph what we call Subgraph. So, let us say that we have a graph G and you have a graph H. H is the graph on the vertex set V′ and edge set E′. So, (V′, E′) is a graph, we call the graph as H. Now, we say that graph H is a subgraph of graph G = (V, E), if V′ is subset of V and E′ is subset of E. So, a graph H is a subgraph of G if V′ is a subset of V and E′ is a subset of E, so this is very important, we will come across the independent set and subgraph quite often. So, let us look at some examples.

(Refer Slide Time: 26:00)



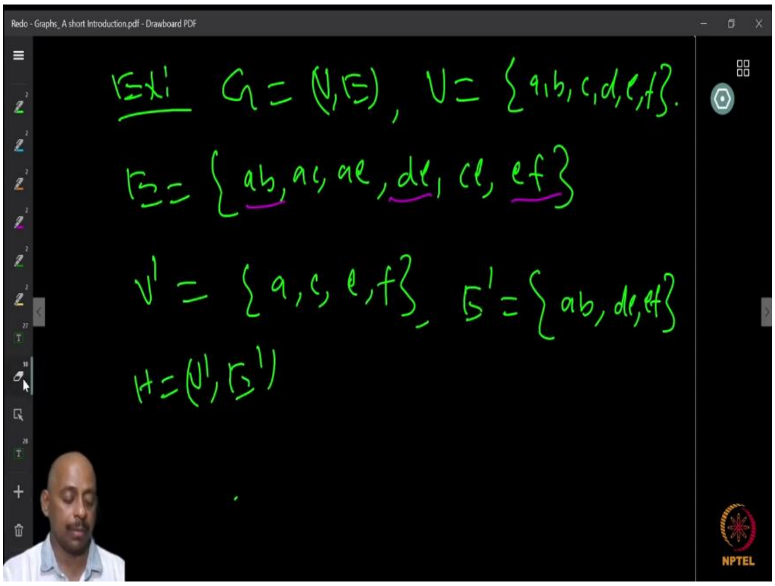
So, here is an example. Let G = (V, E), where V = {1, 2, 3, 4} and E = { {1, 3}, {2, 4}, {3, 4} }. Now, I have H = (V′, E′) ,where V′ = {1, 3, 4} and E′= { {1, 3}, {3, 4} }. Now, H is a subgraph of G, because we can say that V′, which is {1, 3, 4} is a subset of {1, 2, 3, 4}, and E′ is a subset of E. Therefore, the graph H is a subgraph of the graph G.

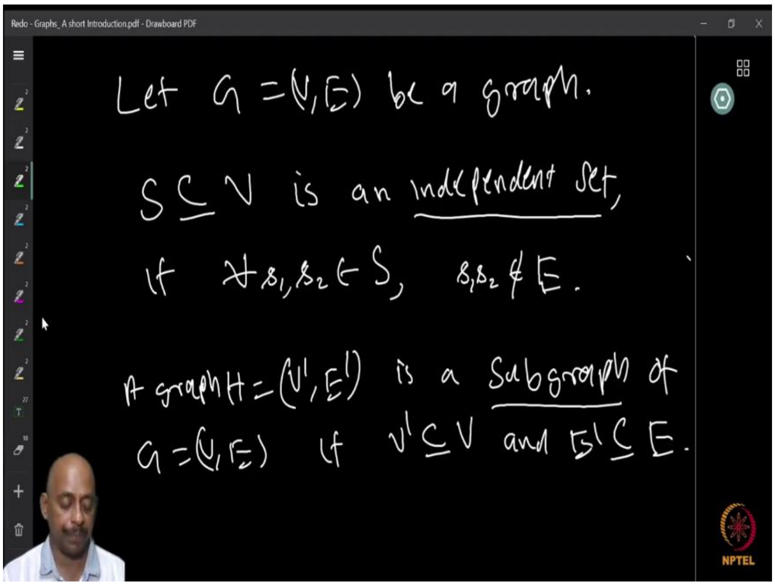
(Refer Slide Time: 27:11)



In this I can look at another subgraph. For example, our independent set was a Subgraph, there is no edge but it is a subgraph. Let us say it is the empty set, which is the subset of edge set anyway. If you look at this graph you can clearly see many of its subgraphs immediately. For example, I can just look at the graph formed by 1, 2 and 3 which is a subgraph of course. So, the vertex set is a subset, edge set is a subset and therefore, I see a subgraph clearly. Similarly, I can say that I can add more, I can add maybe 1, 2, 3 and 6, so I have a subgraph on {1, 2, 3, 6}. Similarly, I can see other subgraph. For example, if I choose 4 and I select 5 and I select 6, I select 1, then I choose the edges 5 and 6. So, now in this case, for example, we have {1, 4, 5, 6} as the vertices of the subset and then the edge 5 6 is the subset, so therefore, it is a subgraph.

(Refer Slide Time: 29:03)





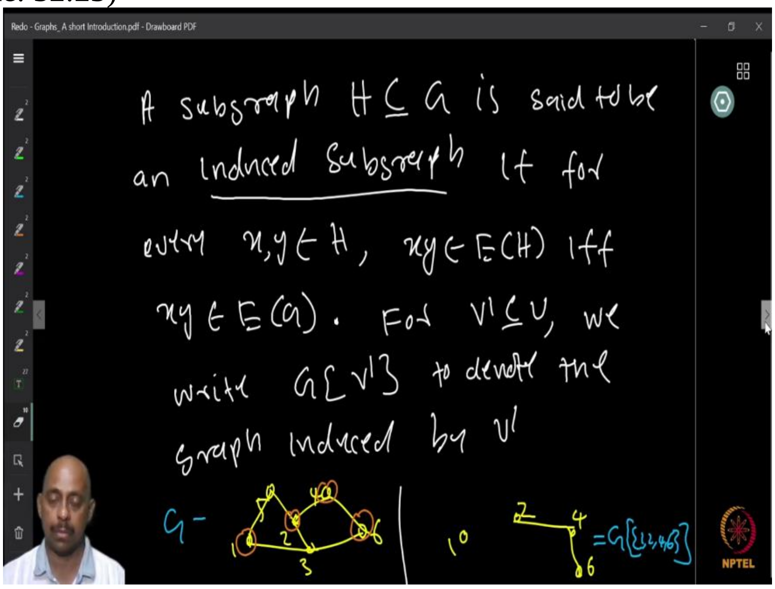
Now, let us move to another example. I have the graph G = (V, E) where V is the set {a, b, c, d, e, f} and E is the set {{a, b}, {a, c}, {a, e}, {d, e}, {c, e}, {e, f}}. So, I have this graph G then I have the graph H let us say, with vertex set V′, which is {a, c, e, f} and E′ = { {a, b}, {d, e}, {e, f}}. Now, clearly you can see that V′ is a subset of V, because {a, c, e, f} is the subset of {a, b, c, d, e, f} and E′ is a subset of E.

But now I claim that H is not a subgraph of G. Why is this? If you have been careful, you should have already noticed it. It is because our definition of subgraph says that H is a subgraph of G, we have, a graph H equal to (V′, E′) is a subgraph of G if V′ is a subset of V and E′ is a subset of E.

Now, what is the definition of a graph? We said that a graph having a set V as the set of vertices and a set E, which are two element subsets of the set V, then we say it is a graph. Now, what we have here for example, in this example, V′ is {a, c, e, f} and E′ is {{a, b}, {d, e}, {e, f}}. but d is not the vertex of V′. Since d is not the vertex of V′, d cannot be an edge in the graph with the vertex set V′.

So, therefore H equal to (V′, E′) is not a graph. Since H is not a graph, it is not a subgraph, because our definition says H should be a graph first and then only it can be a subgraph. So, it is not just sufficient to have subset of vertices and subset of edges, but we should also make sure that the edges are using only the vertices of subgraph that we are looking at.

(Refer Slide Time: 32:23)



Now, a subgraph of, subgraph let us say H of a graph G is said to be an induced subgraph if for every, let us say pair of vertices x and y in H, the edge xy belongs to the graph H, if and only if the edge xy is an edge in the graph G. So, whenever the edge is present in the graph G, I will add it to the graph H, of course only between the vertices of H.

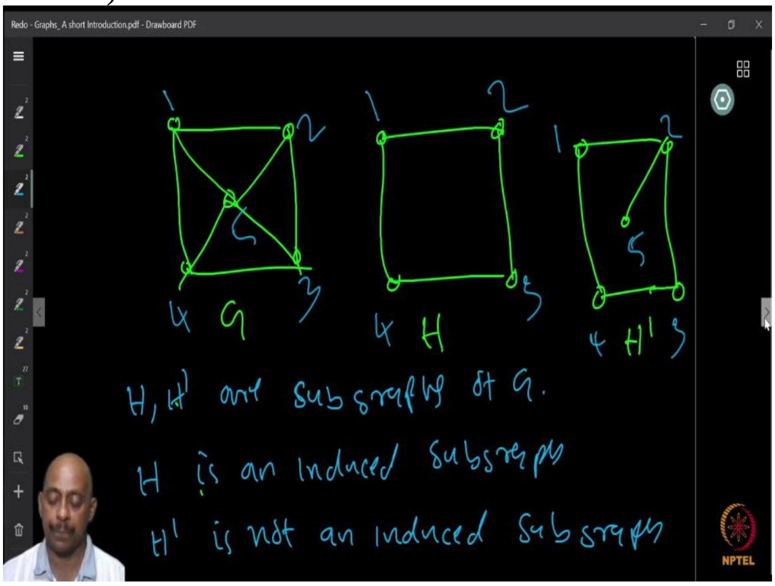
So, the vertex set is a proper, not necessarily proper, it could be any subset of the vertex set. Then but the edges for an induced graph, we cannot choose the edges, the edges come automatically from the graph G. All the edges which are present in between the vertices will automatically come.

To see an example, let us look at the graph G here, with vertex set {1, 2, 3, 4, 5, 6} and I select a subgraph let us say there is a subgraph with vertex set let say {1, 2, 4, 6}. So I select 1, 2, 4 and 6, now since I am looking at induced subgraph, the edges will come directly from G itself. Since I look at these vertices, I let us say mark 1 here, vertex 1, and then I take 2, and then I have 4 and 6.

But now I notice that there is no edge connecting 1 to any other vertices 2, 4 or 6. Therefore, those edges are not there. Then {2, 4} is an edge so I select 2,4 and {4,6} there is an edge, therefore, I also choose 4, 6, those edges will come. I cannot say that I do not want edge {2, 4}, because if I do not choose edge {2, 4}, then what I get is not an induced subgraph, it is just a subgraph. It is a subgraph, but it is not an induced subgraph.

So, this one is a subgraph of graph G, but it is not an induced subgraph. On the other hand, if I add it, all the edges that are present in G between these vertices, then it is an induced subgraph. I can denote this fact that I am looking at induced subgraph by writing the graph within square bracket the subset of vertices that we are looking at. Because that is clearly now defined, like what are the edges. So therefore, I can just write the vertex set and say that this is the subgraph, the induced subgraph that I am looking at.

(Refer Slide Time: 36:00)



Here are some more examples, you have the graph G. The graph H and H′ are subgraph of the graph G. H is a subgraph, because you can see clearly that the vertex set is a subset, if I want, I can also name it. Now, with this, one can say that H is a subgraph of the graph G, H′ is also a subgraph of graph G.

On the other hand, H is an induced subgraph, because if I select all the vertices {1, 2, 3, 4}, all the edges between those vertices in the graph G are present in H. On the other hand H′ is not an induced subgraph, because H′ has all the five vertices, but then it should have all the edges between them also, for example, {3, 5} is an edge in the graph, but 3 and 5 are vertices here but {3, 5} is not an edge, and therefore, H′ is not an induced subgraph. This is what one should be clear. We will continue with topics in the next class.