Combinatorics
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Euler's Pentagonal Number Theorem

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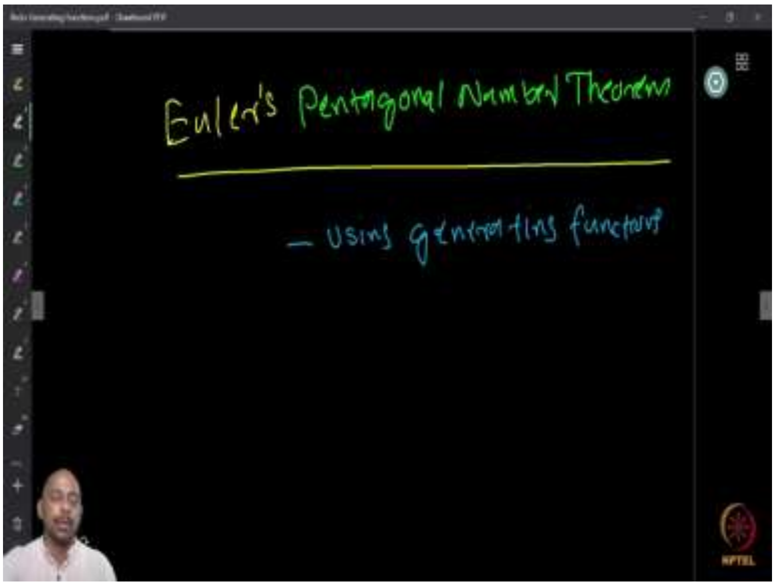

So finally we are going to prove a famous and beautiful theorem. And the proof is very nice. You will enjoy it, which is the Euler's pentagonal number theorem. We are going to use generating functions to prove this. There are several proofs but this one is in particular very beautiful.

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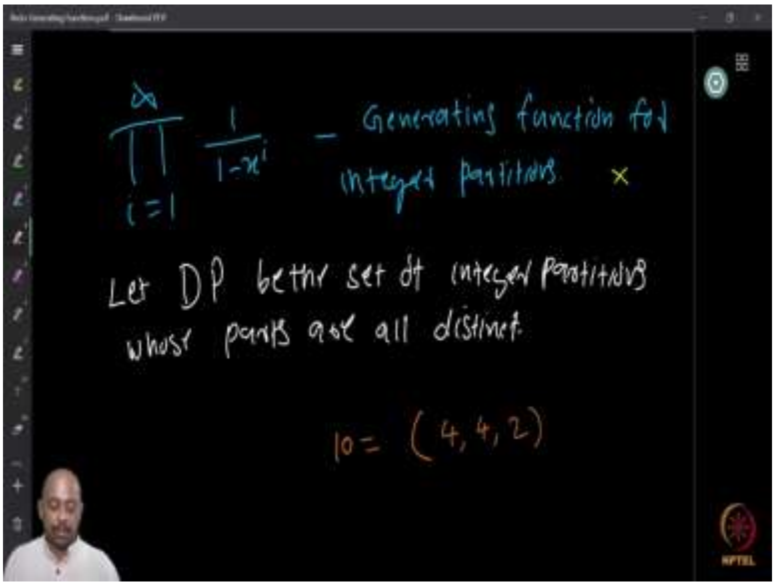

We start with something that we have already seen, the generating function for integer partitions. We proved that it is , $\prod_{i=1}^{\infty} \frac{1}{1-x^i}$. Now let us define DP to be the set of integer partitions whose parts are all distinct.

So each part must be distinct. I do not want a partition of integer. So if I am looking at the partition of, let us say, partition of 10, I can write it as (4, 4, 2) but this is not allowed to be in DP because the parts are not distinct. So I do not want such partitions. I want to avoid these kind of things where parts are repeated.

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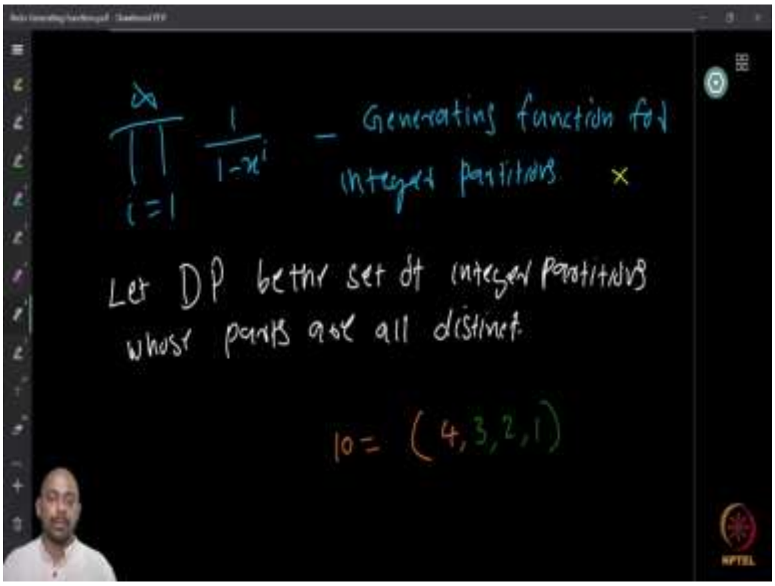

On the other hand, (4,3,2,1) is a partition where the parts are distinct, therefore this is allowed. We want to look at the set of all integer partitions where the parts are distinct for any i. So therefore, look at all possible integers, for each of them look at the set of partitions and in which the parts are distinct and collect them together. So that is the set DP.

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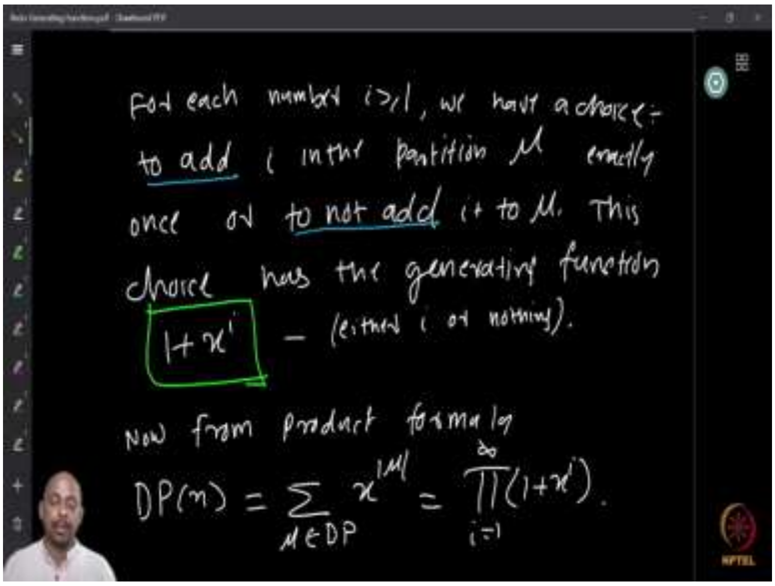

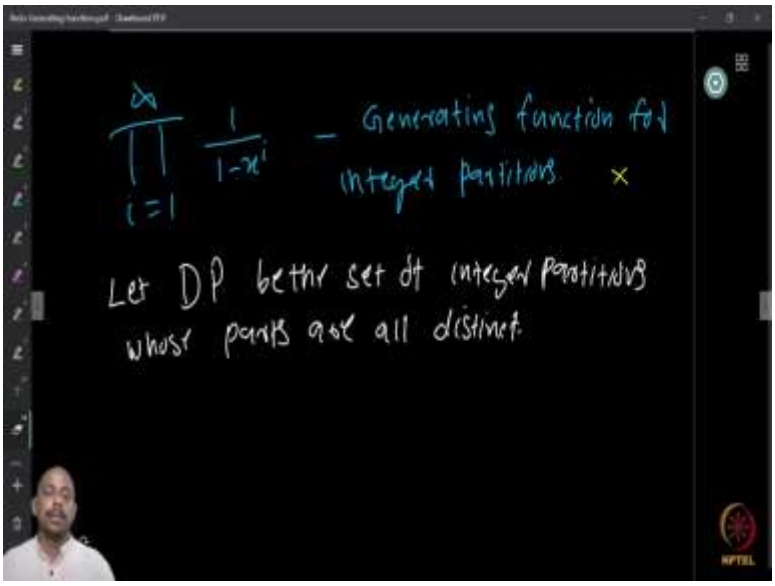

Now we want to look at the generating function for DP. So let us see what is this. First we observe that for each number i, when we are deciding the partition, what, as we did in the case of 10 for example, write each number 1, 2, 3, 4, 5 etcetera, we have a choice whether we want to add i into the partition μ exactly once or not to add it. So for every i we have a choice either add it once or not at all. Then that is okay.

But if I add I can only add it once. Now as we just noticed this choice is given by the generating function $1 + x^i$ because if I choose i, I say that I have chosen i to be in the partition. And if I choose 1, then I say that i is not in the partition.

So then if you look at the product $\prod_{i=1}^{\infty}(1 + x^i)$, that tells that I am choosing either exactly one copy of i for any integer, I am going to choose it either exactly once or nothing. And therefore this is the generating function for DP(x), that is something very clear to us. Now that we can get by product formula.

But on the other hand, what is DP(x) by the definition of DP? DP says that DP is the set of all partitions where each part is distinct. Now because of this how do I get the generating function? So because each part is distinct and I sum over all the elements in DP, so what number it is representing. So if the partition is representing, let us say, n the partition of n, then x^n is what we want to say.

So because we want to find the coefficient of x^n to count exactly the partitions of n which belongs to DP, and another way to say that it is $\sum_{\mu \in DP} x^{|\mu|}$. So whenever the size of μ is n we will look at the partitions μ which belongs to DP and how many are there, that is precisely the number of such partitions.

So therefore this is the definition of DP(x), in another way. So we know that $DP(x) = \sum_{\mu \in DP} x^{|\mu|} = \prod_{i=1}^{\infty}(1 + x^i)$. So we got the generating function for DP. Now, so we started with the generating function for the integer partition which is $\frac{1}{1+x^i}$.

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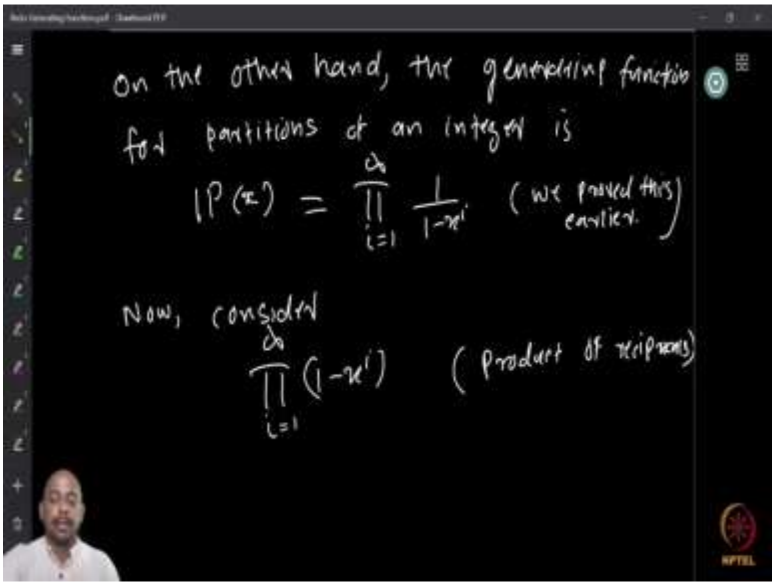

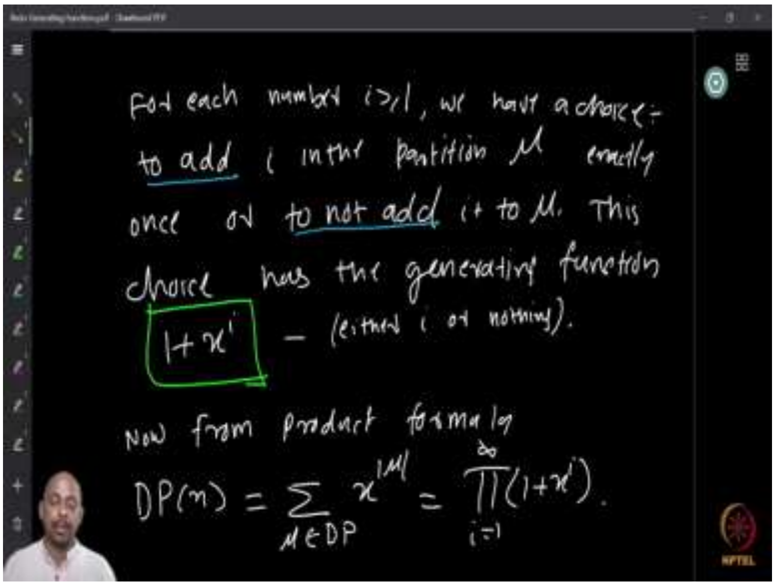

So there is some kind of similarity between these two things, especially if I, instead of looking at $\frac{1}{1-x^i}$, suppose I look at the product of the reciprocals $(1-x^i)$. So if I look at $\prod_{i=1}^{\infty}(1-x^i)$, it is just the reciprocals in the generating function of the partitions of an integers.

Then it is very closely related to $\prod_{i=1}^{\infty}(1+x^i)$ because I am just replacing x with -x. Because of this similarity I am doing this. And, the fact that the Euler's pentagonal theorem is about the expansion of this product. What will be the terms in the expansion, that is what the theorem is about.

So let us now look at this product which is very much related to the both, in two different ways, this product is related to the distinct parts generating function by changing x to -x and also related to the partition for integers as reciprocals of the factors. So therefore this is related to both, and we are going to use this relation.

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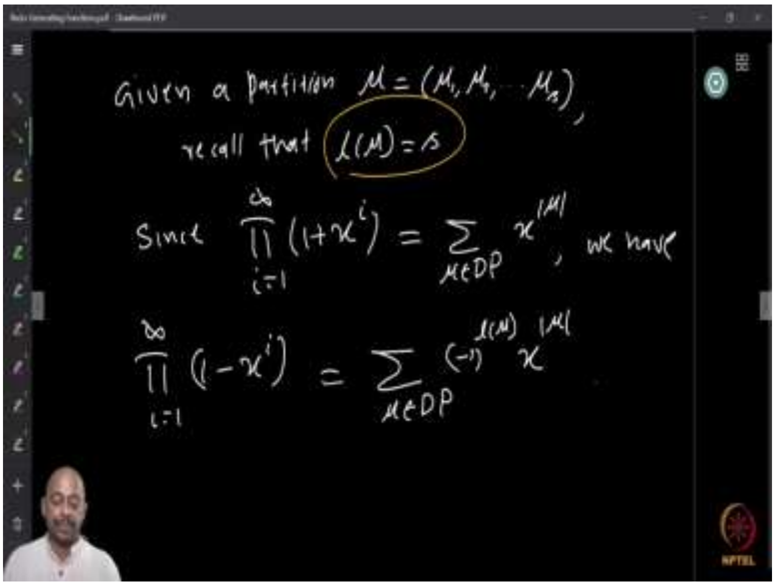

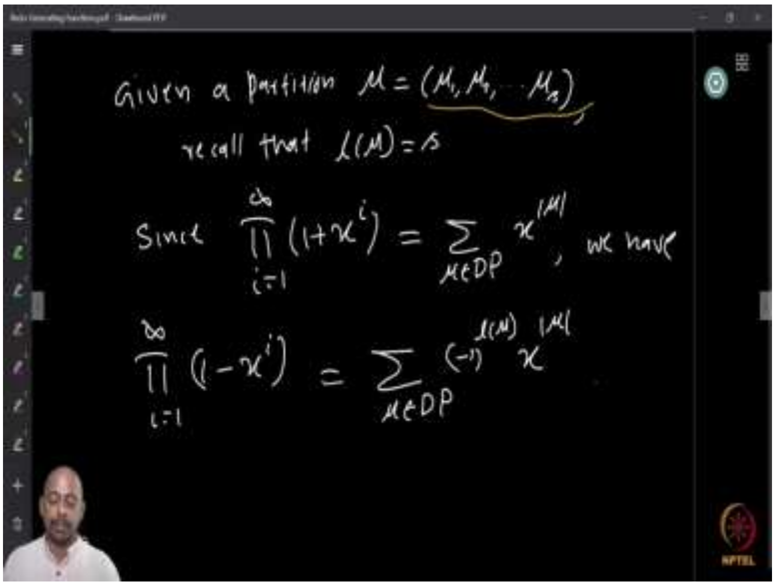

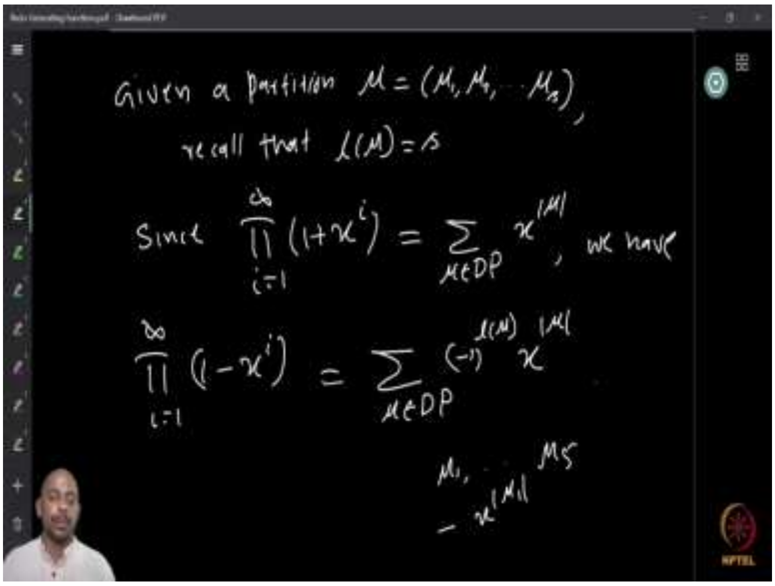

Now, suppose we are given a partition of some number, say $\mu = (\mu_1, \mu_2, \dots, \mu_s)$. So that are exactly s parts. So the length of $l(\mu) = s$. So if you recall when we studied integer partition, we said that length of the partition denoted $l(\mu)$ is the number of parts. So given a partition $\mu = (\mu_1, \mu_2, \dots, \mu_s)$ the length of μ is precisely s.

By definition, $\prod_{i=1}^{\infty}(1 + x^i) = \sum_{\mu \in DP} x^{|\mu|}$.

Now when I change the sign of x to -x, then what happens?

It is not changing x to -x, sorry, it is changing the sign from $(1 + x^i)$ to $(1 - x^i)$. So I have, $\prod_{i=1}^{\infty}(1 - x^i)$. So how can I represent this in terms of the summation that we have? I claim that, $\prod_{i=1}^{\infty}(1 - x^i) = \sum_{\mu \in DP} (-1)^{l(\mu)} x^{|\mu|}$

Now can you think why this is precisely the case? Let us say μ is the partition of n, then if you are looking at x^n, how do I get the term x^n? So x^n comes from the partition μ, and the partition μ is let us say $(\mu_1, \dots, \mu_s)$. Now how did this , $x^{|\mu|}$, x^n came about?

Well it came about because in the product on the left side we have $(1 - x^i)$. And what is $(1 - x^i)$ saying? That I am choosing $(-x^i)$ if whenever I am choosing i. So when I choose μ_1, I am basically choosing minus of x raised to size of mu 1. Similarly, for every i I am choosing $-x^{|\mu_i|}$.

And then the size of μ_i's add up to n, then we get μ. If there are odd number of negative terms then the product is odd, and otherwise it is even. So therefore if I look at $(-1)^{l(\mu)}$ it accounts for the sign, my length of μ is odd then I will get negative, and length of μ is even I will get positive term.

So that explains why the $\prod_{i=1}^{\infty}(1 - x^i) = \sum_{\mu \in DP} (-1)^{l(\mu)} x^{|\mu|}$

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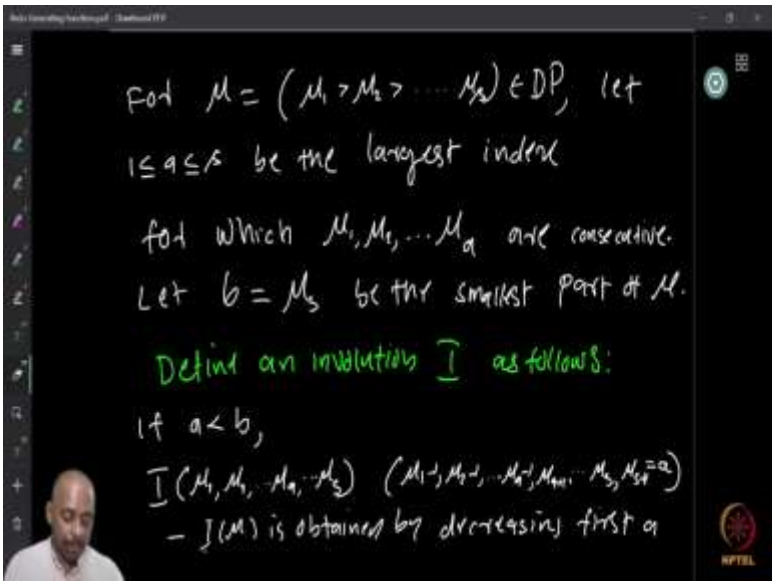

Now we are going to use some trick. We have learned about involutions earlier. So given a partition $\mu = (\mu_1, \mu_2, \ \dots, \ \mu_s)$, of course, we will assume that $\mu_1 > \ \mu_2 > \dots > \ \mu_s)$, because in a partition we order them according to the size and since the parts are distinct we know that it is never an equality.

So μ is in the distinct parts. Let us say that the largest index a, for which $\mu_1, \mu_2, \ \dots, \ \mu_a$, are consecutive numbers. So let us say that μ_1 and μ_2 are such that μ_2 is exactly $\mu_1 - 1$, μ_3 is exactly $\mu_2 - 1$, etcetera, up to, let us say, a.

So we are looking at these specific type of partitions, not specific type of partition we are looking at the partition then deciding up to which number we can have this consecutive numbers. So let us see.

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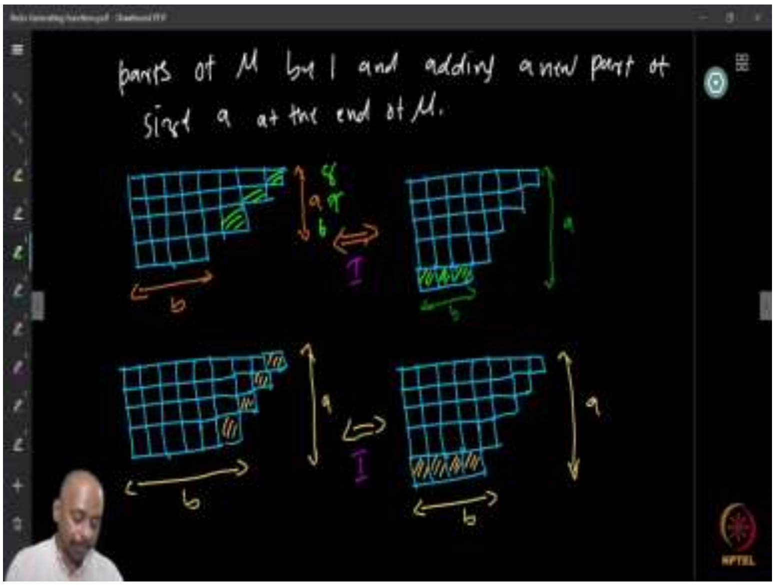

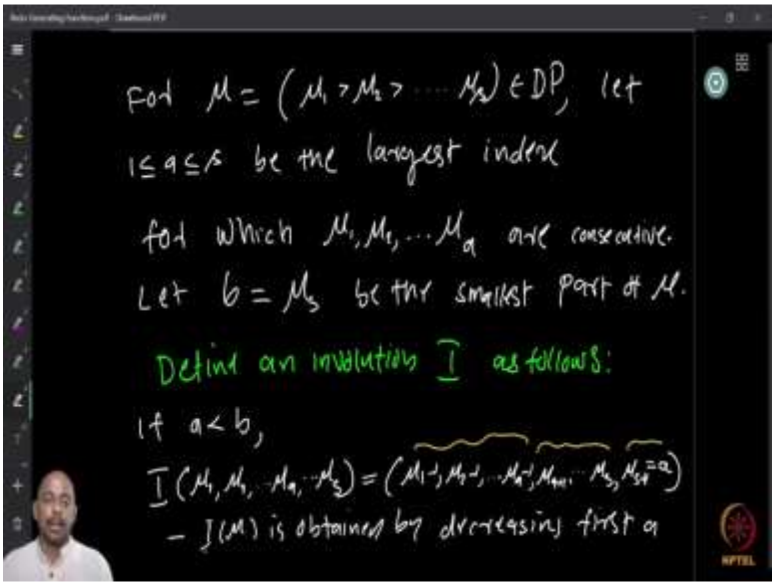

Here is an example. A partition, the ferrous diagram. If you look at this you will see that here I have 1, 2, 3, 4, 5, 6, 7, 8, the first line, there is 8. So if the partition with the first term is 8, μ_1 is 8, then we have 7, which is μ_2, then we have 6, which is μ_3, but then μ_4 is 4, it is not consecutive. So 8, 7, 6 is consecutive but the next one is not. So therefore my a is now 3.

So this is how I, I choose a. So a is the largest index for which $\mu_1, \mu_2, \ldots, \mu_a$ are consecutive numbers. Then we define b to be, b is equal to μ_s, which is the smallest part of μ, the last part is b. So in this case what is last part? Last part is 1, 2, 3, 4. So last part is b.

So given any partition in DP we know that $\mu_1 > \mu_2 > \ldots > \mu_s$ and we have a non-zero index because μ_1 is always there so a is at least 1, so it will be between 1 and s. And therefore it is well defined, and we have a and we have b, b is the last part, so therefore a and b are well defined.

Now we are going to define an involution which takes partitions to partitions. So I am going to transform the partitions by defining an involution. So if a is less than b, so we have defined a and b, if a is strictly less than b, then I define the involution $I(\mu_1, \mu_2, \dots, \mu_s) = (\mu_1 - 1, \mu_2 - 1, \dots, \mu_a - 1,\ \mu_{a+1}, \dots, \mu_s, \mu_{s+1} = a)$.

I include a new part, the number of parts is now increasing, and I am adding $\mu_{s+1} = a$. So because a is strictly less than b, we know that this is definitely a partition of μ, again. So, the involution does not affect the total. So it keeps the size of the partition or area of the shape of the figure. The area is fixed because I am just rearranging the cells.

And we know that when I subtract, these numbers will never be larger than μ_a. So therefore, this will be still a non-increasing sequence. So therefore I have defined a partition by doing this. So $I(\mu)$ is another partition. So $I(\mu)$ is obtained by decreasing the first eight parts in μ and by adding the part with size a to the last part of μ.

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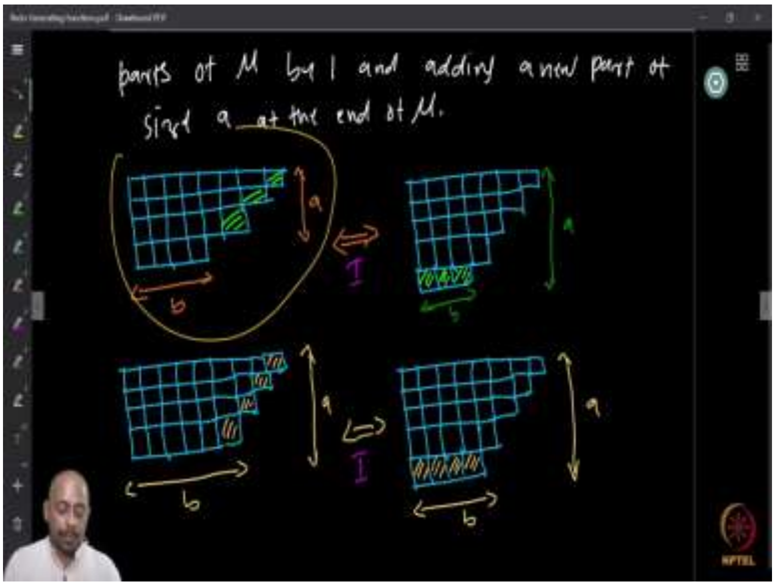

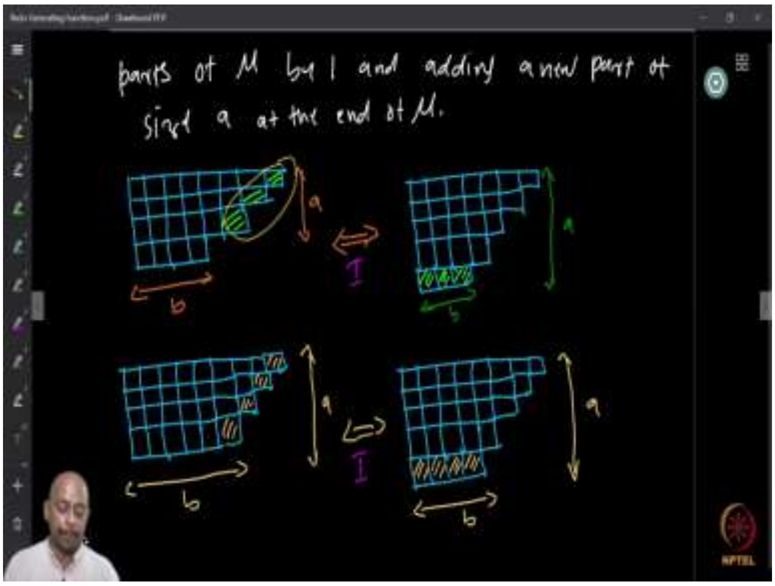

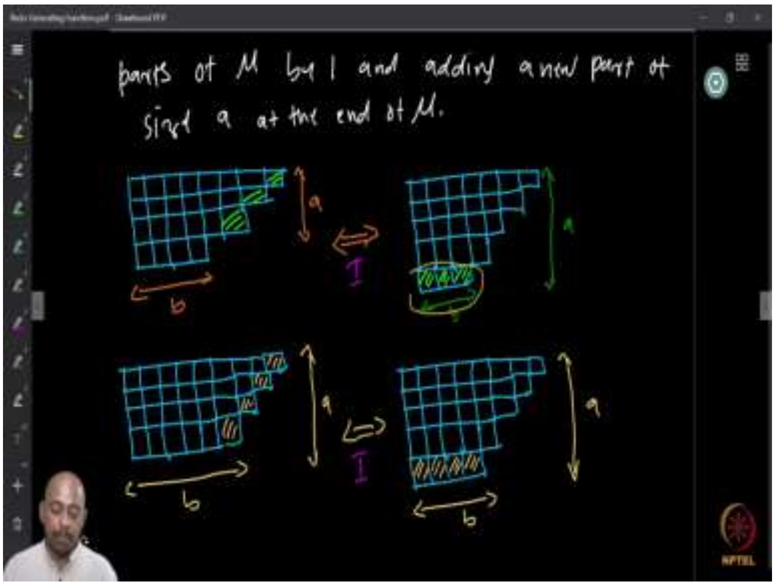

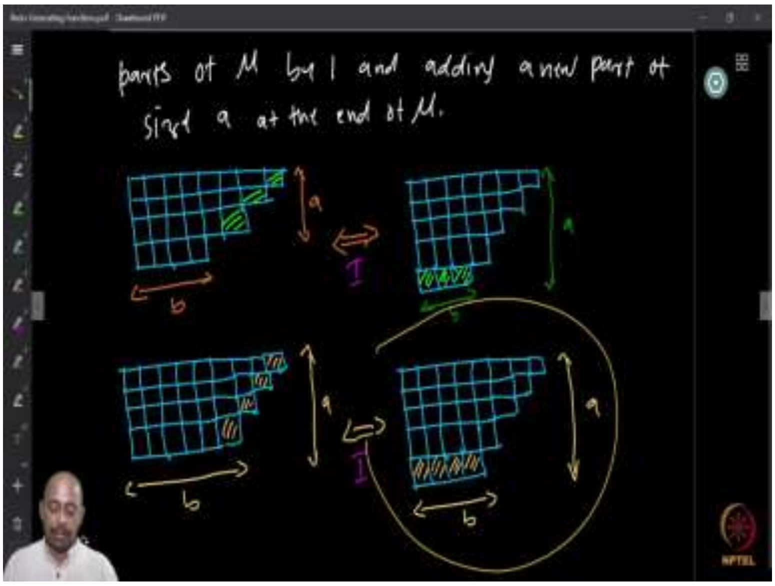

So here is the pictorial representation of the same. I start with this partition where I have these numbers, and a and b are here, a is less than b because a is 3 and b is 4. So therefore I put this a, the three elements, I am moving them from there, I have reduced the numbers from here, then putting at the last part, which is b, below the b that we have.

And now for this, again now we can look at what is a and b. So if you look at the a and b you will see that b is now 3, a is 1, 2, 3, 4 and 5. So therefore now a is larger than b. So we will define the involution for that also, and we define in such a way that its involution will be the original image itself. And similarly here is another example we have a and b, where this a is reaching here, I again put it here I get another map, so I have, I have the involution of the first figure.

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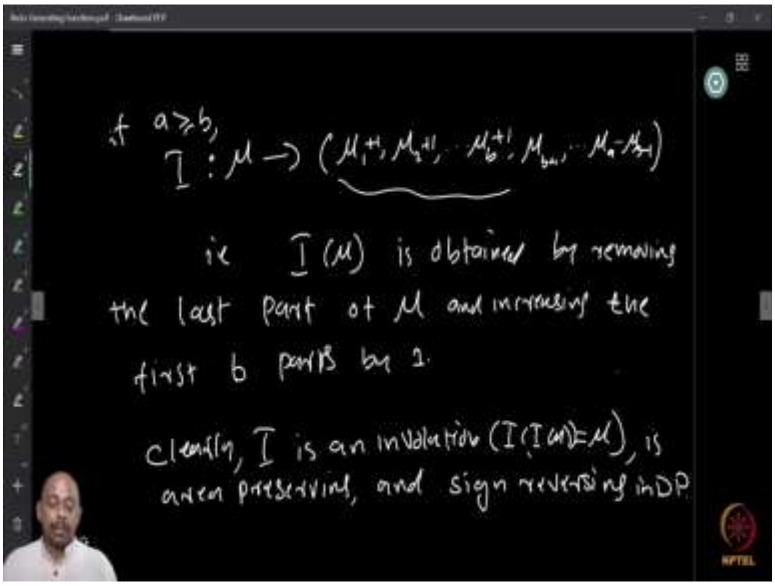

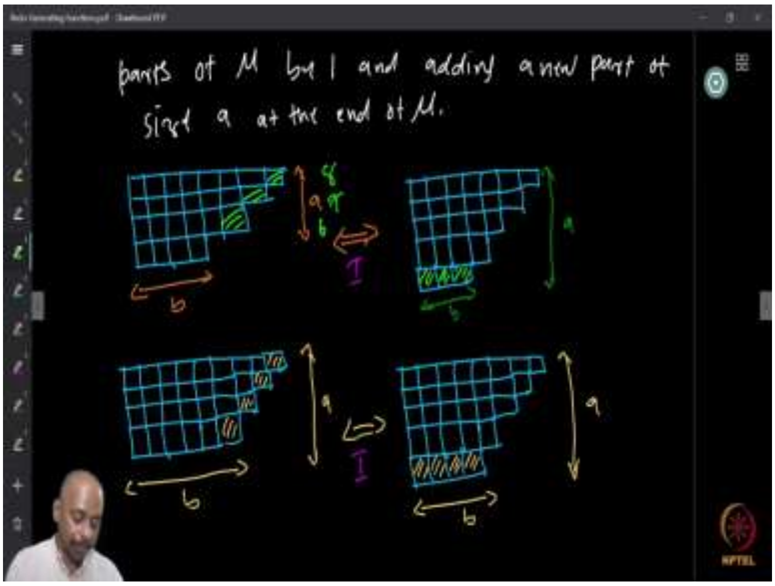

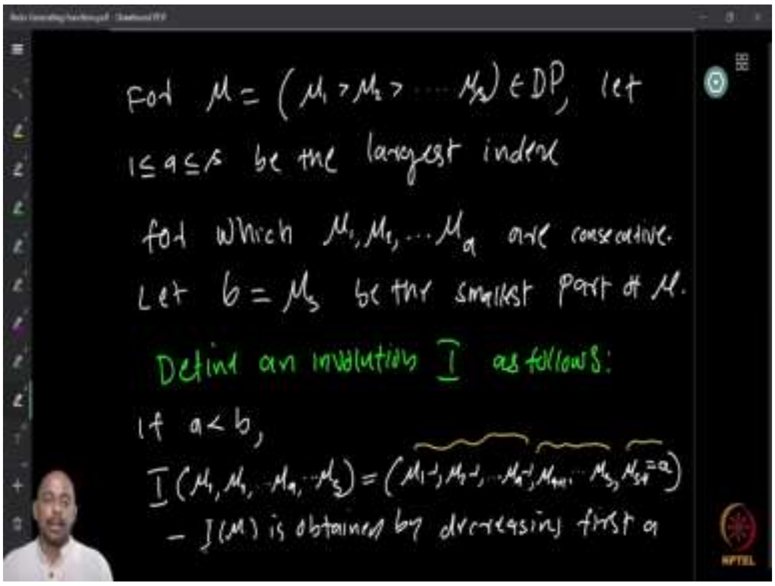

So the, what is the other case? If a is greater than or equal to b, then, I takes μ to $(\mu_1 + 1, \mu_2 + 1, \dots, \mu_b + 1, \mu_{b+1}, \dots, \mu_a, \dots, \mu_{s-1})$

That is $I(\mu)$ is obtained by removing the last part of μ and increasing the first b parts by 1. Since a is greater than or equal to b, I have the leverage to do this. So I add these parts and I get the involution. So $I(\mu)$ is obtained by removing the last part of μ and increasing the first b parts by 1.

Now the claim is that I is an involution because $I\left(I(\mu)\right) = \mu$. And this is something we can verify. It preserves the area because we do not remove any of the cells. We are just moving around the cells. So the total is going to be the same. But on the other hand it is sign reversing in DP.

Why is it sign reversing? Because if you are going to get distinct parts in the new μ, if the new partition, that $I(\mu)$ is having distinct parts, then the number of parts has changed by 1. So the length of μ is changing now. The length of μ and length of $I(\mu)$ differs exactly by 1.

So in this summation the coefficient $(-1)^{l(\mu)}$, that sign has changed for this particular partition. So it basically is a map from partitions to partitions, where for the first partition and second partition, if both of them are in D P then the sign of them are opposite. And they are they are different partitions, and the signs are different.

So if I can match up this, those terms will cancel out in the summation. So if I am looking at the summation here, the corresponding terms will cancel out which means that in this product, the product is basically the expansion of the product is precisely the way we are obtained here, so we need to figure out which terms are going to cancel out in this product. And that is what our aim is.

So we have define this involution and see whether there is any problem with this involution, or, if any of the terms are remaining. So there will be some cancellation because whenever μ is going to $I(\mu)$, and μ and $I(\mu)$ are distinct partitions we know that those two will cancel out because they have different signs. The involution is a sign reversing involution.

So we now observe that, not all involutions of μ are going to be in DP. So only those in D P will be counted by the summation, therefore only those pairs will cancel out. So we need to figure out which are the ones which are not counted in D P.

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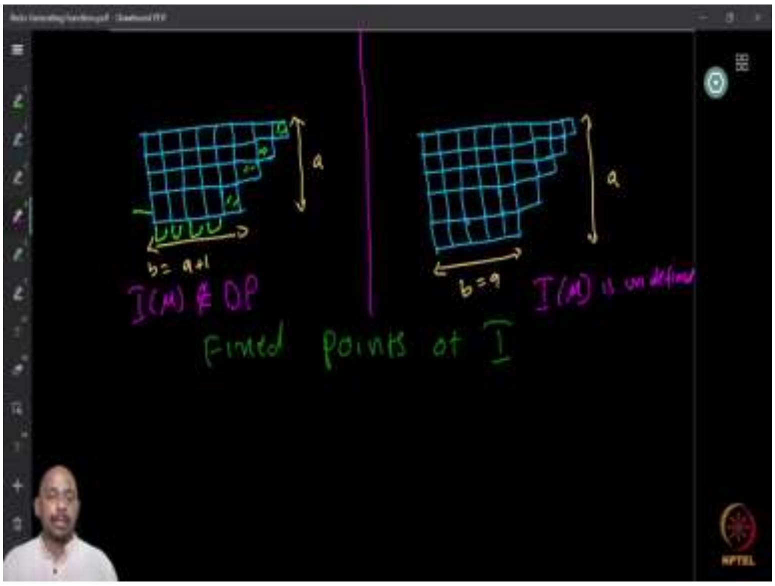

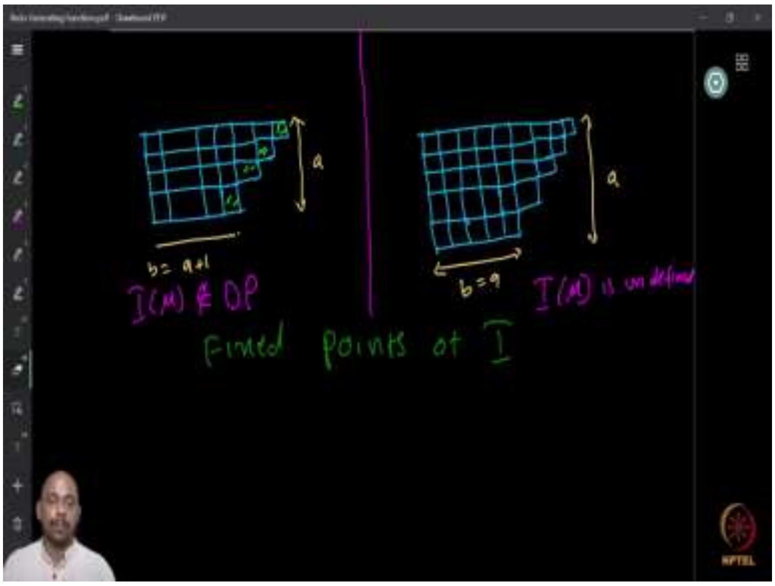

So there are two cases. One is that, one is that like, if b is equal to a plus 1, so if exactly a plus 1 in that case what happens? So b is here, 5 and a is 4, and if b is actually equal to a plus 1, when I remove these four parts, the last four cells, and we are going to put it below, below b, what happens? I am going to put four cells below b, but now b has decreased by 1, so and therefore we have four new parts coming here where these parts have disappeared.

So we are getting a partition but the last two parts are the same, because this, there is 4 here, b becomes b minus 1 and a is also equal to b minus 1. So therefore these two parts are having exactly the same size. So same numbers are appearing and therefore it is not in D P. So $I(\mu)$ is not in DP for this case. Therefore, this term will not be cancelled. So we can say this is a fixed point.

Now there is another case, then also we have this issue that, that is when b is equal, equal to a. So if b is equal to a, then we cannot define $I(\mu)$ because the when I take away the a parts, then a becomes larger than b - 1, b, we had how many terms, the last part is also removed.

So therefore if b is equal to a then what happens is that when you remove the last part it becomes b -1. I cannot put a below it. So I cannot really define $I(\mu)$. So $I(\mu)$ is undefined. So therefore I can say $I(\mu)$ is μ itself. So this is again a fixed point. So whenever $I(\mu)$ does not belong to DP we will say $I(\mu) = \mu$.

So now our involution is now defined well, that whenever we can define $I(\mu)$ to be in DP, I define $I(\mu)$ in the previous, as in the previous example, and these two cases I will define $I(\mu)$ to be μ itself. So therefore I have the involution and we know that $I(I(\mu)) = \mu$ itself.

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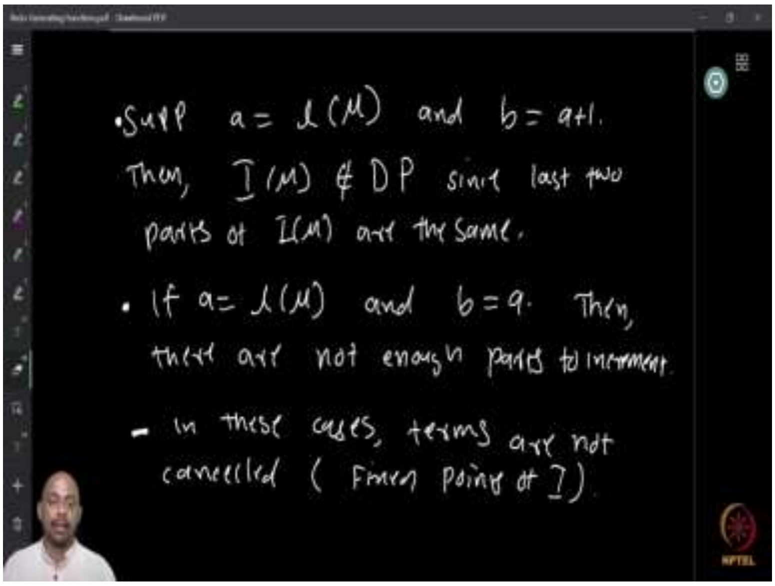

And once we have this, so here is the written explanation that if $a = l(\mu)$ and $b = a + 1$, then as we observed $I(\mu)$ is not in DP, and in that case we have to set $I(\mu)$ to be μ itself. And similarly if I cannot define this because the length of the part that I am going to add is more than the last part in μ we have to define $I(\mu)$ to be μ itself.

So, these two terms are fixed points, and all other terms are not fixed, and therefore they will cancel out. So when exactly this happens? So when exactly are the fixed points there? And fixed points are the only ones going to survive in the product. So therefore, and in the sum.

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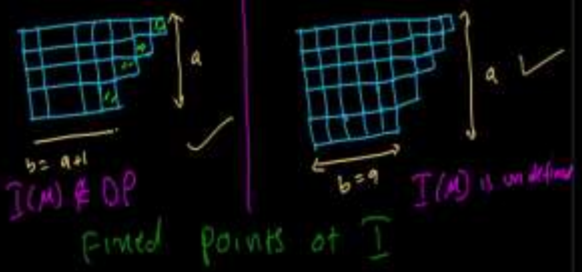
These happen when μ is of the form,

$\mu = (2k, 2k-1, \dots k+1)$ of length k and area $|\mu| = \frac{k \cdot (3k+1)}{2}$, $k \geq 1$,

or $\mu = (2k-1, 2k-2, \dots k)$ of length k and $|\mu| = \frac{k \cdot (3k-1)}{2}$ $k \geq 1$.

Further, empty partition is also not cancelled.



• Supp $a = \ell(\mu)$ and $b = a+1$. Then, $I(\mu) \notin DP$ since last two parts of $I(\mu)$ are the same.

• If $a = \ell(\mu)$ and $b = a$. Then, there are not enough parts to increment.

– In these cases, terms are not cancelled (Fixed points of I).

So let us analyze this. So we can see that whenever μ is of the following form, this will happen. Because we want the numbers to be consecutive, so if $\mu = (2k, 2k-1, \dots, k+1)$ and it has exactly length k. Then of course what is the area of this? $|\mu| = \frac{k(3k+1)}{2}, k \geq 1,$

Or if $\mu = (\ 2k-1, \dots, k+1, k)$, where again, length k and $|\mu| = \frac{k(3k+1)}{2}, k \geq 1$. In these two cases we can verify that case a, and these two previous cases will appear precisely in this case.

So this is something one can manually verify because we are partitioning a number with this property. The, the sizes are now well-defined because here everything must be decreasing, here also everything must be decreasing by 1 exactly, consecutive. And these two cases will happen precisely when the numbers are either of this form. And area is precisely the sum of the numbers, so therefore that is also clear.

So these are the fixed points one can verify. And then the empty partition of course is also fixed point. It is not cancelled, and therefore we can use this information to define the product from the summation formula.

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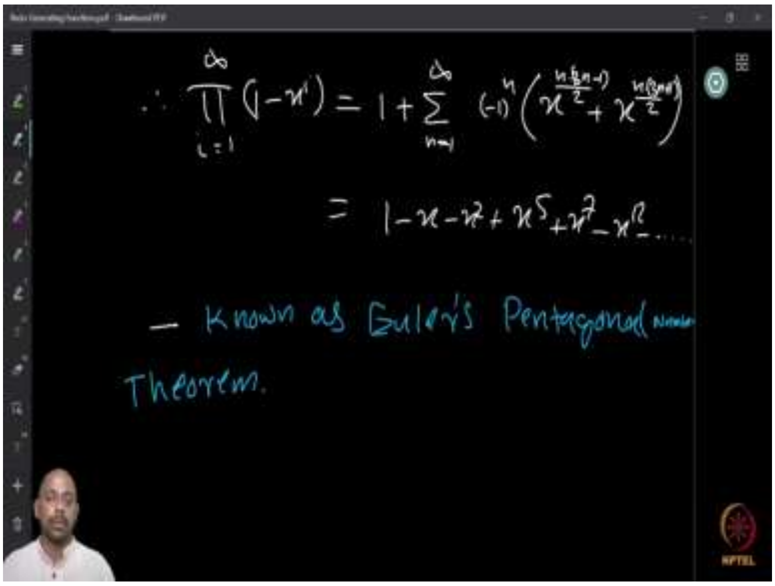

So what are the terms that is going to survive are precisely when the power of x is either, the size is $\frac{n(3n-1)}{2}$ or $\frac{n(3n+1)}{2}$. And in this case the sign will be $(-1)^n$ which is the number of terms. So n is the number of terms and the area is $\frac{n(3n-1)}{2}$ or $\frac{n(3n+1)}{2}$. So

$$\prod_{i=1}^{\infty}(1-x^i) = 1 + \sum_{n=1}^{\infty} (-1)^n \, x^{\frac{n(3n-1)}{2}} + x^{\frac{n(3n+1)}{2}}$$

$$= 1 - x - x^2 + x^5 + x^7 - \cdots$$

So these numbers which are appearing in the exponents are called the pentagonal numbers. And this result is known as the Euler's pentagonal number theorem, pentagonal number theorem.

And this is a very nice and well-known result, something comes a lot in mathematics, and very beautiful result. And we have a very elegant proof of that. And I think with this application we can conclude our visit to generating functions. And there are more techniques one can learn but we will keep that for an advanced course. And here I will leave you with some few exercise questions which you can try out.

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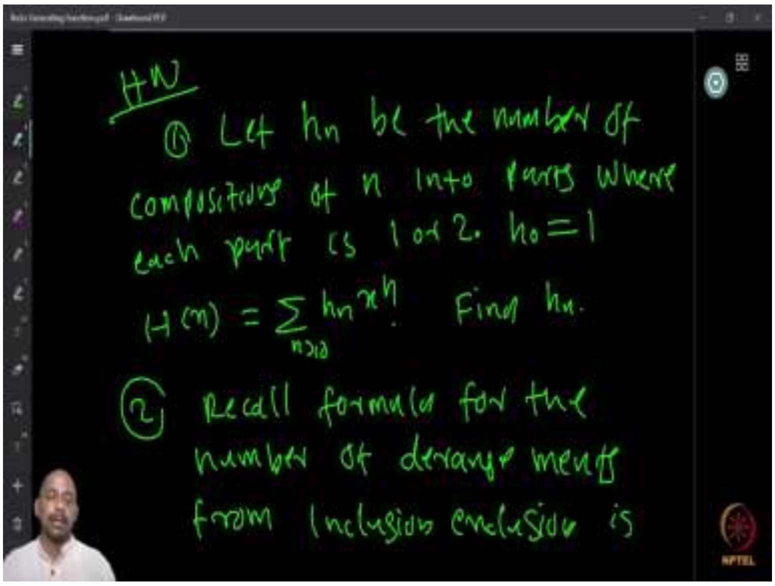

So the questions include, the first one is let h_n be the number of compositions of n into parts where each part is either 1 or 2. If $h_0 = 1$, and $H(x) = \sum_{n\geq 0} h_n x^n$ is the generating function, find out h_n. Second question calls to look at the derangements. We know that we have solved the derangements using the inclusion-exclusion. So formula for derangements.

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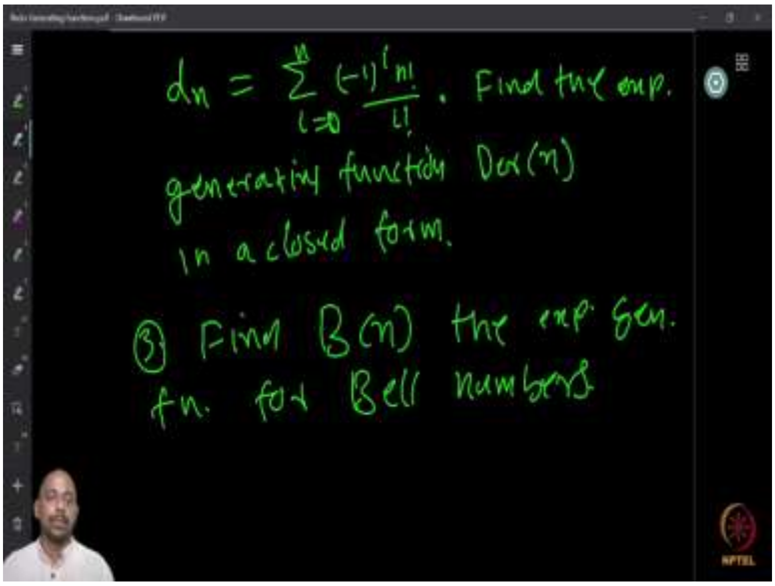

And we have shown that the formula is $d_n = \sum_{i=0}^{n} \frac{(-1)^i n!}{i!}$. So this is something that we have proved. Now find the exponential generating function for the derangements. Let us say Der(x), and try to find it in a closed form. Then find the exponential generating function for the bell numbers that we have discussed in, when we were looking at the partitions.

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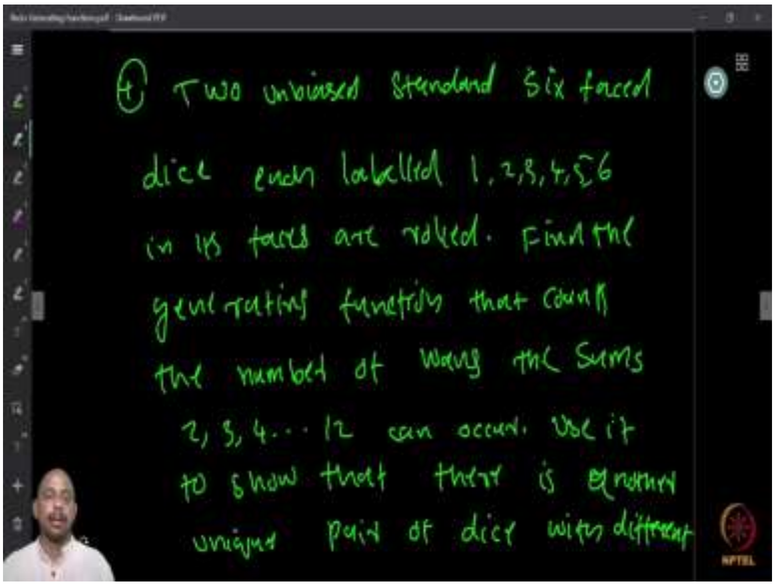

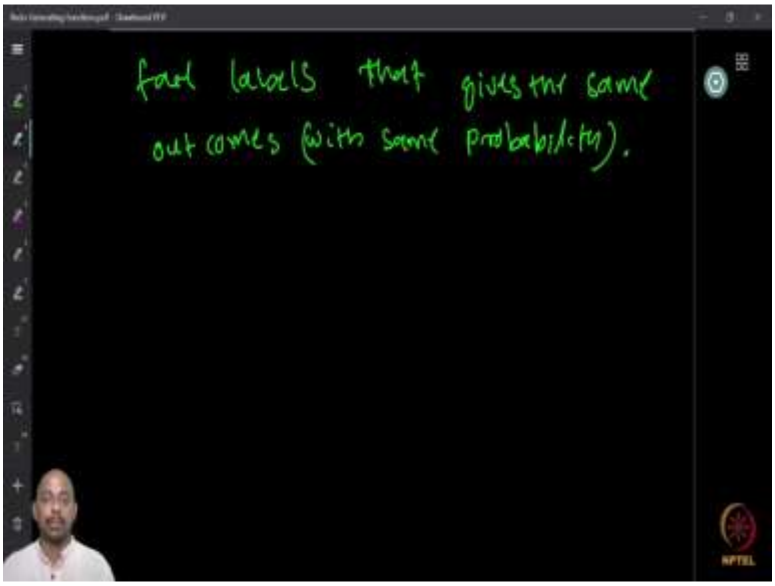

And finally, here is another interesting, very interesting question. Suppose, you are all familiar with the, the dice, the six-faced dice with the numbers, the dots 1, 2, up to, let us say, 6. So there are the 6 faces. Each face has distinct number of dots 1 to 6. Now let us say that this dice is picked up and rolled. So there are two of them. So I rolled them. Now the outcome of a roll is basically the sum of the numbers on the top. So take the numbers on the top, or number of dots on the top, count them.

Now because it is 1 to 6, sum of these two will be ranging from 2 to 12. when 1, 1 is happening it is 2, 3 and 4, or 5 and 2 are happening, or 6 and 1 are happening, you will get 7, etcetera up, to 12 is possible. And, and these are the only outcomes that we can have from this rolling. Now what we want to do is to use generating functions. So find the generating function for counting the number of ways, the sum 2 to 12 can occur. So whenever 12 is happening, the coefficient of x^{12} must be the number of ways that can appear.

Now use the generating function and try to play with it, and show that there is another unique pair of dice that we can come up with. So the two dice, but the numbers are going to be different. Not 1 to 6, it will have some other numbers. So can you show that there is another set of six-faced dice where the numbers of the sides are slightly different from 1 to 6, but with the same outcomes, with the same probability, that the number of ways each outcome, the number 12 or 5 or 3 or 2 comes, is the same.

So this type of dice have a special name. That is something we can find out later. So try to solve this. This is a nice question. It might take a little bit of work but it will be worth the exercise. So with that we conclude this introduction to generating functions. And we will see more about it in advanced courses. So with that I wind up, and we will continue with a different topic in the next lecture.