Combinatorics
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Lecture 03
Ramey theorem as generalization of PHP

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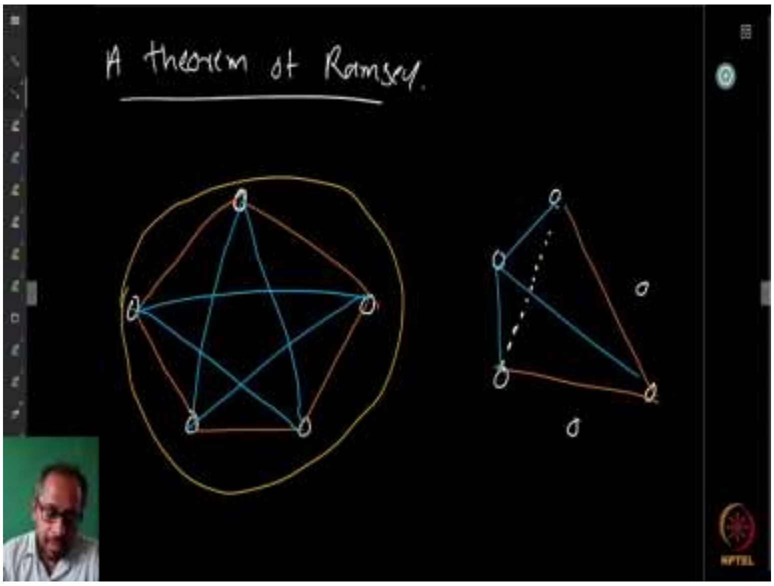
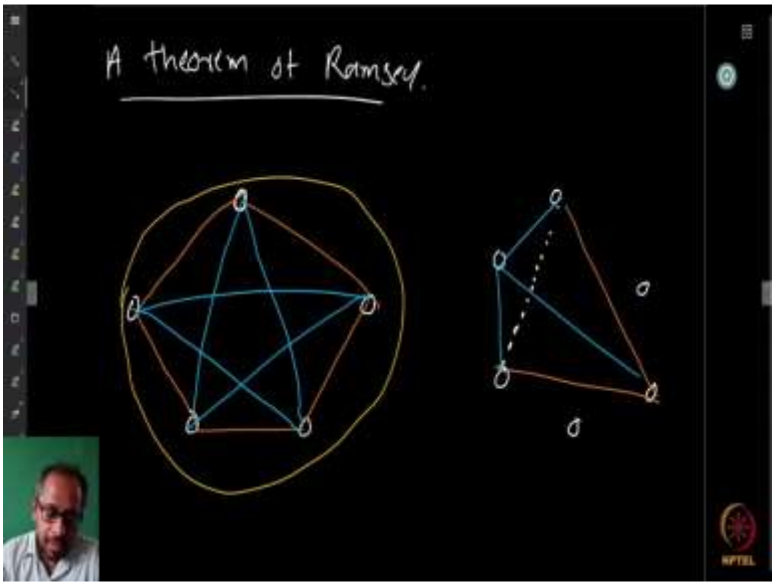


Hello, welcome back. In the previous lecture, previous 2 lectures we were looking at the applications of pigeonhole principle. And then while we were doing this we came across a small application where we showed that, if you take 6 or more people then among these people you can always find either 3 people who know each other, who has met each other or 3 people who are strangers, who have never met each other.

Now, this result is one of the starting points in a huge area in combinatorics called Ramsey theory. And there can be many, many the generalization of this and generalizations are kind of also related with the pigeonhole principle in some sense. We will see that in a moment and try to look at today's lecture with the more general theorem of Ramsey and then we will see that how it is a generalization of our pigeonhole principle.

So, the theorem that we proved was saying that, if we have a graph and the graph has 6 or more vertices, we will take a 6 vertex graph for example, and you color and you put all possible edges, you consider the complete graph and then you color all the edges with just 2 colors, then we were able to show that you will find in either a red triangle or a blue triangle.

A complete graph on 3 vertices with all edges having red color or blue color, mutual strangers or mutual friends. Suppose instead of 6 suppose we had only 5, then we use the fact that we had 6 vertices at least to apply the pigeonhole principle. Now, but that does not necessarily mean that in a smaller graph you cannot have such a property. But let us now show that if you have only 5 vertices in the graph, then we need not have the property. So, I am going to give a 5 vertex graph, so here is a 5 vertex complete graph.



And I have given a 2 coloring of the edges, all the 10 edges are here. And then you have these 5 edges, 5 cycle like you give red color and then in the inside we look at these 5 edges and then also give blue color. So, this colors all the edges, but the point is that you cannot find a triangle of the same color, you cannot find a single color triangle. If you take any of the blue edges, they never form a triangle, they will form a 5 cycle and outside also it forms a 5 cycle.

But we needed at least 6. So, 6 is the smallest number with this property. Now, if you have anything larger, we already saw that we always have the property because once 6 will do this, you can always take a subset with 6 and that has this property, the entire set has that property. So, and we use the general principle for the 6 case like this, you had at least three of the same color and then we look at the edges between those three neighbors.

And those neighbors you can see that whichever way you color it either creates a triangle of either red, which is or blue. Now, so this talks about coloring the edges of a triangle with 2 colors. What if we allow more colors? What essentially the Ramsey theorem was saying? We said that, when you try to make things more chaotic, as far as the set that we are considering is really huge. Then you can still find some order within that.

So, here is the question. That suppose I increase more colors, then what can you say? Suppose I say that I use instead of 2 colors, I use k colors. So, if you take k colors and color the edges of the complete graph with k different colors, what is the smallest number of vertices, which can guarantee there will be a complete graph of let us say, either triangle or something larger, like 4 vertex complete graph, 5 vertex complete graph or whatever where all the edges have the same color.

You can ask this, now one question is does there exist such a small number? Such a not small, such an integer, such that after that, you can always guarantee this. Now it turns out that there exists always, but this needs proof. And we are not going to prove it at this moment. But, I will let you think about it and maybe try to prove for some special cases of just 2 colors with a larger number of cliques. So, here is the general question or the theorem of Ramsey in a slightly more general form.

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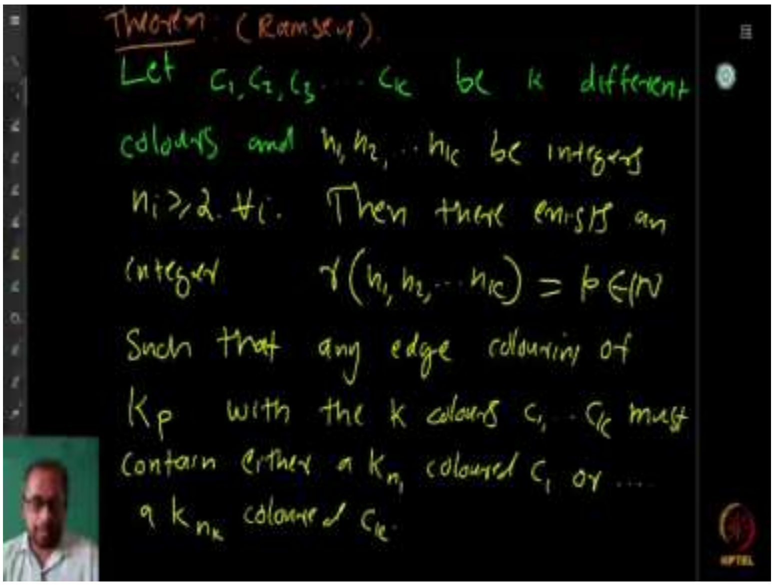

You will find even more general form later. Let $C_1, \dots, C_k$ be k different colors and $n_1, \dots, n_k$ be integers, where $n_i \geq 2$. Then you can find some integer $r(n, \dots, n_k)$. So, $n_1, \dots, n_k$ are the number. So, in the problem that we looked earlier we were looking at, we need a complete graph on three vertices which is all blue or three vertices which is all red. Here we were saying that you have n_1 vertices n_2 vertices etc. n_k vertices, then you are asking for either a complete graph on n_1 vertices where all the edges have colour C_1 or a complete graph on n_2 vertices all the edges have colour C_2 or a complete graph on n_k vertices where all the edges have colour C_k.

So, the theorem say that then there exists an integer $r(n, \dots, n_k) = p$, such that any edge coloring of the complete graph on p vertices K_p with the k colors $C_1, \dots, C_k$ must contain either a K_{n_1} colored C_1 or a K_{n_2} colored C_2or … or a K_{n_k} colored C_k. This is something which you cannot avoid. So as far as the number is large enough, as far as the number is larger than $r(n, \dots, n_k)$.

So, when p is larger than $r(n, \dots, n_k)$ or whatever something, then, you can always consider this some subgraph and then that subgraph has this complete subgraph of same coloring. And, the entire graph has this also. So that is okay, we do not have to worry about writing in a different way. So, this is the general form of Ramsey theorem. You have a k coloring of the edges of the complete graph. And then you are asking, can you find smaller complete graphs of certain orders, where all the edges have the same color.

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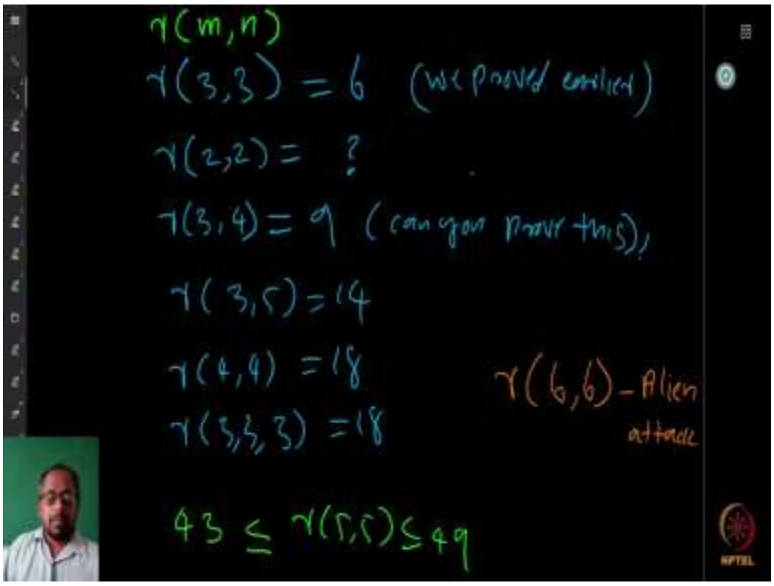

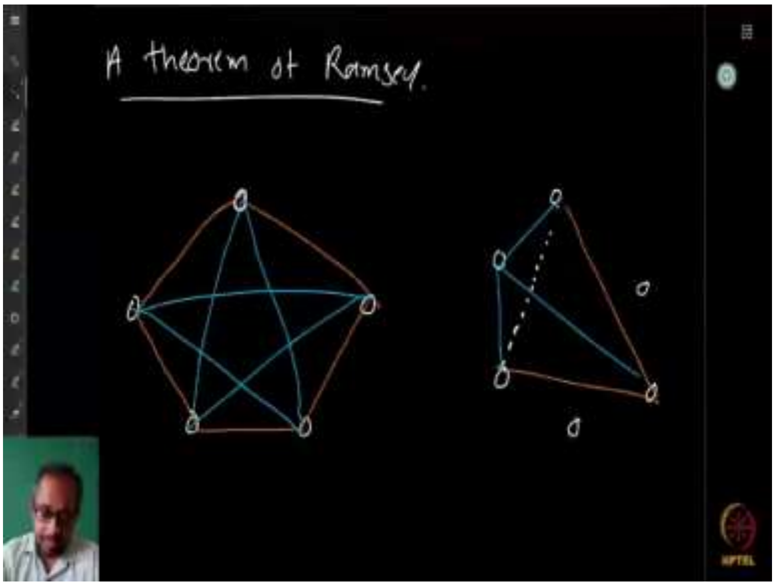

Now, so in using same notation, what is $r(m,n)$? $r(m,n)$ says that you are looking for a complete graph. So, you are looking at 2 coloring, because there are only 2 arguments here namely m and n. And you are looking at a complete graph on m vertices as a subgraph with all the edges having the first color or complete graph on n vertices where all the edges have the second colour. So, this is $r(m,n)$. Now, what we proved earlier was that using pigeonhole principle $r(3,3) = 6$.

If you have 6 or more vertices in the graph, then you can always find a triangle of the same colour. Now what is $r(2,2)$, can you think of this and try to solve it? Very easy I think, you can do it. What about $r(3,4)$? So $r(3,4) = 9$. Now the question is that can you prove it? Can you come up with that proof that $r(3,4) = 9$. Again, these things I do not know. I mean maybe it is not much difficult to prove $r(3,5) = 14$, $r(4,4) = 18$ and $r(3,3,3) = 1$, this you have 3 coloring now.

Then show that if you have 18 vertices or more, then you will contain a triangle of first colour or second colour or third colour. So, these numbers we already know for these small values. Now what about $r(5,5)$? So, $r(5,5)$ ask for a complete graph on 5 vertices, but all the edges have the same colour.

A complete graph on five vertices, where all the edges have the first colour red or a complete graph on five vertices, where all the edges have the colour blue. Now, what is the smallest such number with this property? Well, we still do not know exact value, what we know is that it is between 43 and 49. That is $43 \leq r(5,5) \leq 49$

Now, what is $r(6,6)$?. Now, I want to tell you a story. So, there is a story of Paul Erdos, very famous mathematician, many of you might have already heard about him. So, when he talks about problems related to Ramsey number, he will say the following joke. But half serious, not just a joke. The joke is the following one. He believes that $r(5,5)$ is very difficult to compute, we still do not know what its exact value.

But he says that suppose some alien forces, alien forces come like from some other galaxy, they come here or some other star system from our galaxy, but they are vastly more superior to us, they have all kinds of technology, they have something like what you see in these kind of sci fi movies, you can just destroy a planet with some weapon. Now they are so powerful that they have such weapons.

And they come here and then they tell us that well we will let you live if you tell us the value of $r(5,5)$, then he says that maybe we can put all our forces together, put all the computers to work for the same question, do some parallel computation, all the mathematicians are on their own or work together to find out this value, maybe we can save our planet by finding the value and telling them the correct value.

But suppose they asked for $r(6,6)$, just kill them before they even think of attacking us, because they say that it is not valid, at least he believes that it is not possible. Not with the current knowledge and techniques and all we have that is what he says. So, yeah, finding Ramsey numbers precisely is a very, very difficult problem. People try to find bounds like upper bound lower bound etc, we will not go into any of those things, not in this course.

Or maybe at end we can try to prove some lower bounds or something or even upper bound, maybe using some arguments. Now, some new techniques if you will learn maybe. Now, on the other hand we can find some not necessarily great looking but some kind of upper bound without much difficulty.

So, for that what I want to do is I want you to show that $r(m, n)$ exists. So try to show that $r(m, n)$ exists by showing an upper bound for this. What we are saying is the if a number is larger than $r(m, n)$ then of course, we can guarantee that there will be a m complete graph or an n complete graph of red colour or blue colour. But when we say we find an upper bound, if is we are saying that okay, we do not know present value of $r(m, n)$.

If you make sure that the number is maybe much, much larger than the actual value of $r(m, n)$, you go is much larger than, let us say $20{,}000 m.n$ for something or like, $20{,}000^{\{mn\}}$ or something like that, whatever some number, or m^2 , 2^mor 2^n or something. And then can you say, at least in that case, that anything larger than that will contain a m complete graph.

So this kind of some, some yet much larger number can be an upper bound saying that anything larger, so, therefore, it says that it exists, because we know that, as far as you go above this, you are guaranteed. So it means that it exists definitely we do not know, what is the smallest one, but we know still such thing exists. So, prove that $r(m, n)$ exists by showing some argument, maybe induction, something that you know. Well, anyway, think about it and try. I am not giving it as homework, but it is not part of this work that we are looking at, but it will be instructive and interesting..

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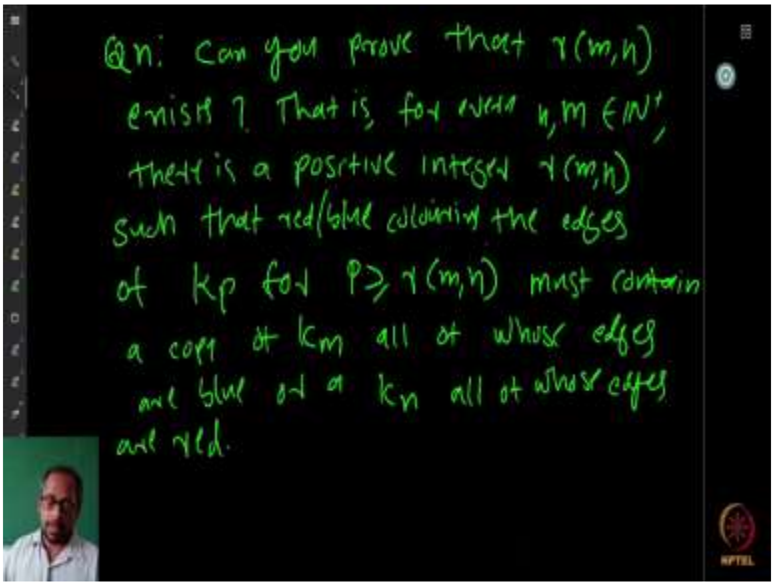

So here is the question. Can you prove that $r(m,n)$ exists? That is for every $n,\ m$ belongs to $\mathbb{N}^+$, there is a positive integer $r(m,n)$ such that red/blue coloring of the edges of complete graph on p vertices, for $p \geq r(m,n)$ must contain a copy of K_m all of whose edges having blue colour or a K_n where all edges are red colored. So, one of these must be present.

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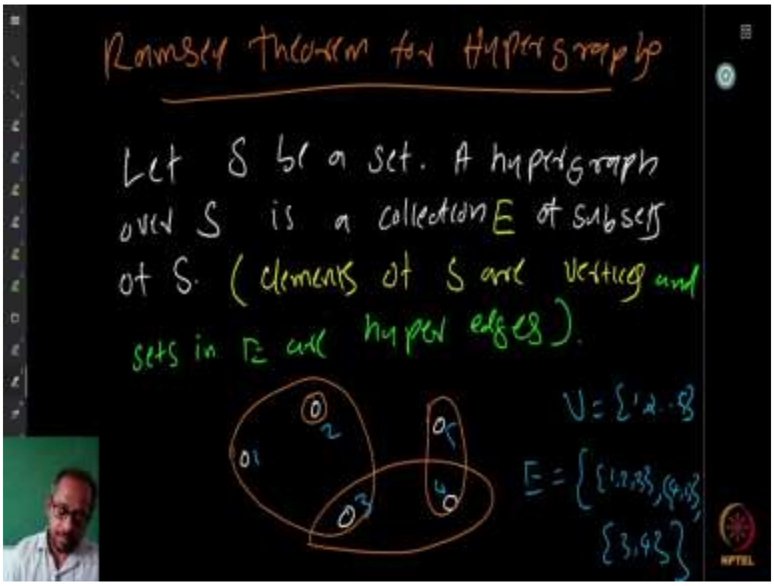

Now, an even more generalized question that we can ask. So, what is called Ramsey theorem for hypergraphs. So, when we talked about graphs, what the graph was? The graph was basically a set, we had a set of vertices and then some 2 element subsets. We were restricting ourselves to binary relations. Now, suppose we know instead of binary relations we look at arbitrary relations or talking about arbitrary subsets. Then you have what is called hyper graphs.

So a hyper graph is basically a set together with a collection of subsets. So E is just a collection of subsets. We do not even say what kind of sets are there. So, elements of S are called vertices and the sets are called hyper edges. So here is an example. You have this 5 vertex, 5 vertex graph, hyper graph, and where now I have these edges. So, this 2-element subset is an edge, this 2-element of set is an edge. Then this three elements of set is also an edge.

So, this you know, so what if you give numbers like 1, 2, 3 etc upto this, let us say this is 1, this is 2, this is 3, this is 4, this is 5, then our vertex set is the set 1, 2, …, 5. Then what are the edge sets the edges basically 1, 2, 3 is an edge, then 4, 5 is an edge and then 3, 4, is an edge. So, 1,2, 3; 4, 5 and 3, 4, so these are all edges of this hypergraph. So, this is an edge 1, 2, 3. So, in earlier we had only 2 elements subsets, but now you can have any kind of subsets. We can even just have 1 element set, that is also allowed, when you let us say that you have this is, I can say that this is also an edge, so this is okay.

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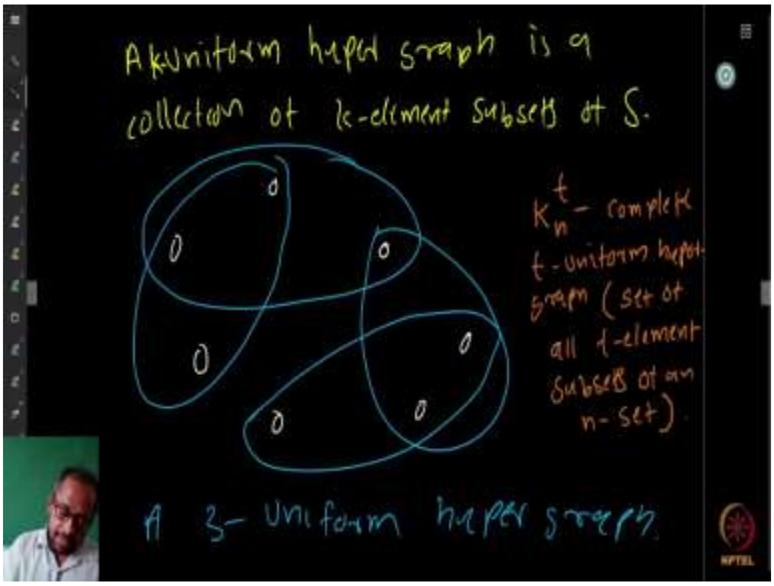

Now, when we have these kind of arbitrary subsets, even more difficult to deal with. So we will often look at slightly more restricted version what we call uniform hyper graphs. So, a k -uniform hyper graph is basically a collection of k -element subsets of S. We only look at k for some fixed k, like when it is 2-uniform hyper graph, it is just the graph. When it is a 3-uniform hyper graph, then you have all the edges have 3 vertices inside, all the edges have 3 vertices inside. So, here is an edge, here is another edge, here is another edge, then here is another edge.

So, yeah and then when we have all possible 3-element subsets part of these edges, then it is called a complete - uniform hypergraph denoted by K_n^t . A complete t-uniform hypergraph, set of all t- element subsets of an n- element set. So, the subscript n says the number of vertices and the superscript t says the cardinality of each the edges. So here we have given a three uniform hyper graph, not a complete one.

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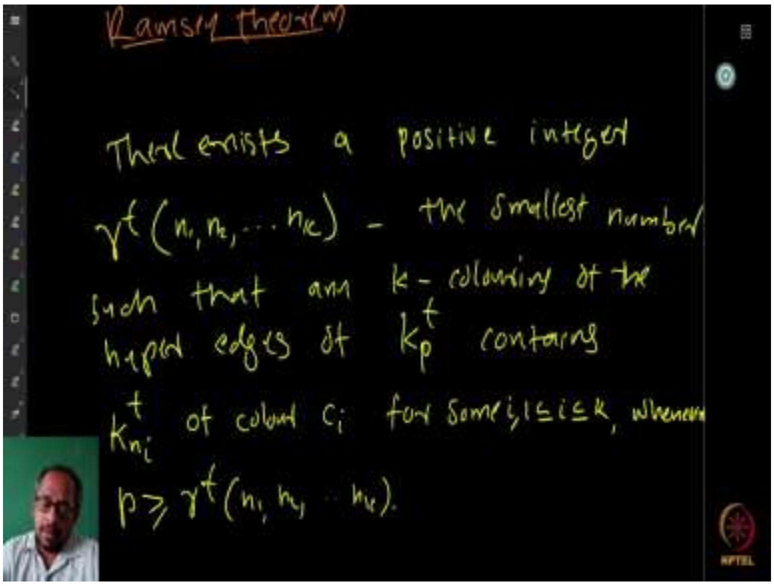

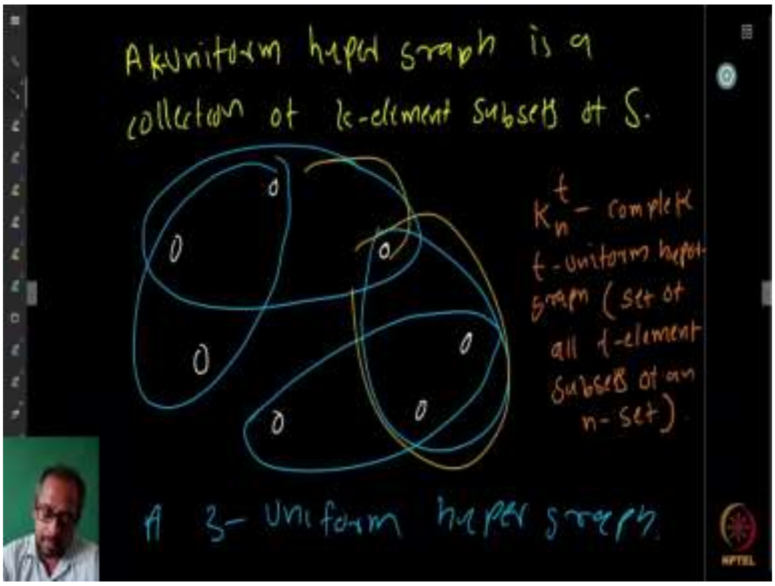

So, now, here is the general form of Ramsey theorem. So, it again, it says the existence of a positive integer r^t now, because we are talking specifically about t- uniform hypergraphs, $r^t(n_1, \ldots, n_k)$, the smallest number such that any k coloring of the hyper edges. So, now instead of coloring the edges we are coloring the hyper edges. All these, now this thing I will color with color let us say red, this I will color with blue and then red maybe this also red, something like that.

So, any k-coloring of the hyper edges of K_p^t contains $K_{n_i}^t$ of color C_i for some i, whenever $p \geq r^t(n_1, \dots, n_k)$. So, in this k- coloring I will assume that the colors we have used are $C_1, \dots, C_k$. We will assume that the k -coloring means that colors $C_1, \dots, C_k$, just some index here. So, that is the generalized form of Ramsey theorem.

Now, we want to see why this is a generalization of pigeonhole principle, that is what we started with. We started by saying that we are going to look at Ramsey theorem and say that it is a generalization of pigeonhole principle. Now, how do you say that? Can you think about a way to see this and relate it with the pigeonhole principle? Now, if you want, think for a few minutes by pausing the video and then continue later.

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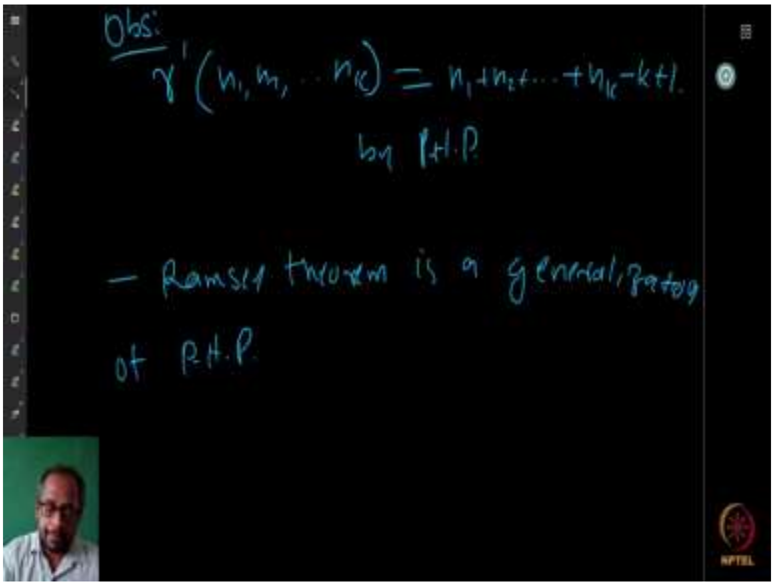

So, first observation is that suppose $t = 1$, so when $t = 1$ what happens? Then we are looking at 1- uniform hyper graph, which means that we have just vertices and every vertex you have an edge by itself, which means that there is nothing happening here, just set itself, there is nothing else really there. We can say that which of them belongs to edges, but that is. But if you are looking at complete graph, then of course, all of them are basically of this.

So, I am looking at 1- uniform hyper graph $r^1(n_1, \dots, n_k)$, which says that I want n_1 vertices of the color 1 and n_2 vertices of the color 2 or n_k vertices of color k . But, this is something that we already saw. What we saw was that like if you are coloring let us say $n_1 + n_2 + \cdots + n_k - k + 1$ vertices with the colors $C_1, \dots, C_k$, then at least one of them like will have this property that n_1 with color 1 or n_2 with color 2 or n_k with color k.

Basically, coloring and putting in the boxes are the same thing. Now let us say there are labelled k boxes, with colors $C_1, \ldots, C_k$. When I put balls into boxes one by one, that is by saying that okay I am coloring these balls or the elements whatever it is with colors. I have a k -coloring now, then we are saying that okay but we have by PHP is that if you have $n_1 + n_2 + \cdots + n_k - k + 1$ total vertices in the graph then or total elements in the set and we have total balls that you are going to put into k boxes.

Then one of the boxes will contain at least n_1 of the same first color, n_2 of the second colour, etc.. n_k of the last colour. These are all generalized form of PHP. So, we can see that even for just $t = 1$ what we get is something in the form of generalized form of pigeonhole principle. So now, we can see what happens now in terms of t, it is not exactly in the same statement, but you can see that why it is a generalization. So, that is it, so Ramsey theorem is a generalization pigeonhole principle.

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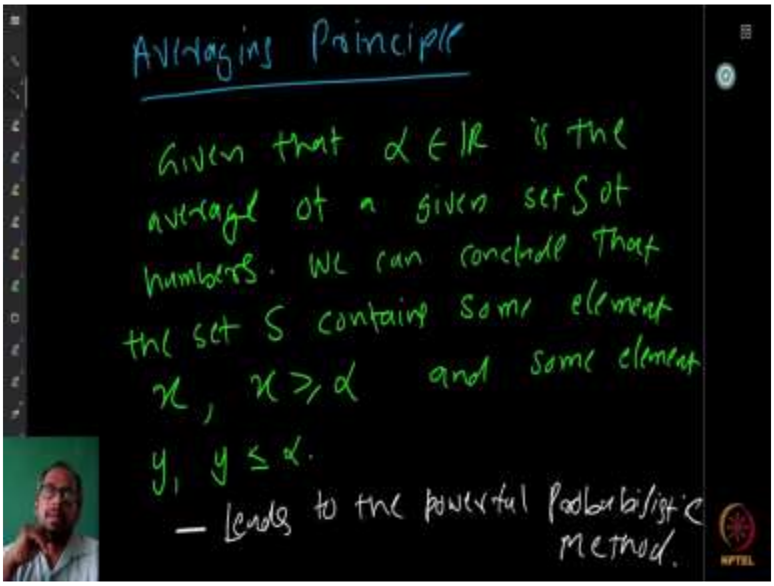

So, we want to wind up this section by stating something which does not look like pigeonhole principle but has some flavors of it. So, this is called averaging principle. This is also a very useful tool, we will not use it at this moment, but we will see that it can be used to show some very amazing result by generalizing into what is called probabilistic method, the same idea but slight improvisation. So, here is the averaging method. Suppose I give you numbers, let us say, I do not know these numbers, I do not tell you, I tell you that I give you that I have some collection of numbers I do not even tell you how many numbers.

But I tell you that the average of these numbers is something like let us say alpha or like 27, suppose I tell you that average of set of numbers that I have is 27. What can we say from this information that I give you? If I give you that the average is 27, then you know that there must be at least one number whose value is greater than or equal to 27. Because if every number is strictly less than 27, the average will be strictly less than 27.

Similarly, you will also know that there is always some number whose value is less than or equal to 27, because if everything is strictly greater than 27, then also average will be strictly greater than 27. Without knowing anything about how many numbers are there, what kind of numbers we have there anything, we can still say this information from just the average. So, this is called the averaging principle.

So, given that α belongs to real numbers $\mathbb{R}$ is the average of a given set of numbers, we can conclude that the set contains some element x whose value is greater than or equal to alpha and some element y whose value is less than or equal to α. Of course, they can all be equal to α, that is all. Now, as I told you this can lead to some powerful techniques like probabilistic method.

Now, suppose I tell you that, I give you a little more information. I tell you that the average of a set of integers is 7.5, so average of a set of integers is 7.5. Then you know that there is at least one number whose value is greater than or equal to 8 and one number whose value is less than or equal to 7. This is also something that you can say. So, with that, we finish this part.