Combinatorics
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Exponential Generating Functions

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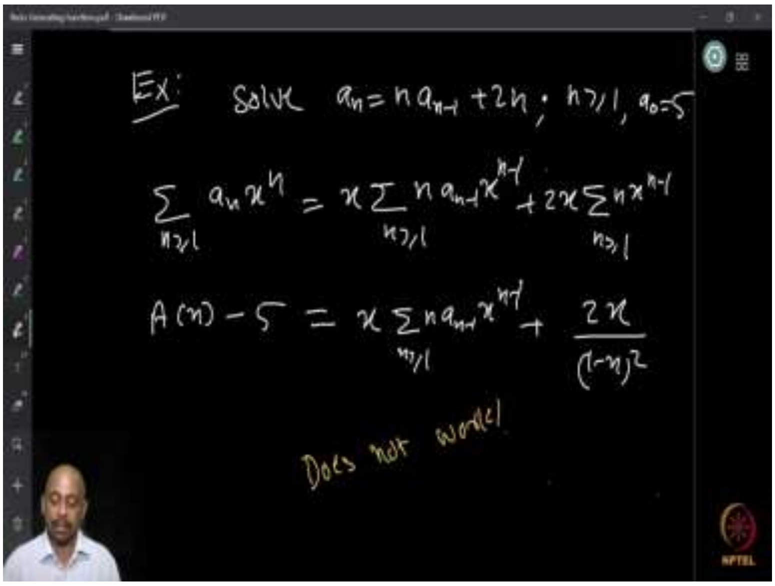

Welcome back. So as you remember in the previous lectures we were looking at generating function. And then we studied the ordinary generating function. So we said that we can use the ordinary generating function and solve recurrence relations and other questions about counting. And now we saw that it was very useful. And then we look at one more example and try to solve it using the method that we know.

Let us say that we are given the following recursion formula. So $a_n = na_{n-1} + 2n, n \geq 1$, $a_0 = 5$. So it is defined in terms of a_{n-1} and n. Now, as usual, when we have such a formula what we do is we multiply the recurrence relation both sides by x^n, and take the sum.

$$\sum_{n\geq 1} a_n x^n = x \sum_{n\geq 1} na_{n-1}x^{n-1} + 2x \sum_{n\geq 1} nx^{n-1}$$

$$A(x) - 5 = x \sum_{n\geq 1} na_{n-1}\, x^{n-1} \; + \frac{2x}{(1-x)^2}$$

But now the problem is that what do we do with this other term, the first term on the RHS?

So we have $x \sum_{n \geq 1} n a_{n-1} x^{n-1}$ and we do not know how to convert this to a nice formula, or write in times of A(x). How do you write it in times of A(x). Because of the n sitting there we will see that it is not easy to do that.

So now why it is, why did this not work? So one can show that this really does not work. This method does not work to get a generating function because, as one can verify that if you look at this a_n , this $a_n = n a_{n-1} + 2n$. So every time it is multiplied by n, and therefore you will see that the sequence grows faster than n!.

And because of this one can show that the ordinary generating function will not converge and then we will not get a nice function as we did in the previous case. So in that case what we will do? So if the function does not converge, we cannot really use the method of generating functions to do nice things. So in that case we will do some tricks. So we are going to learn something.

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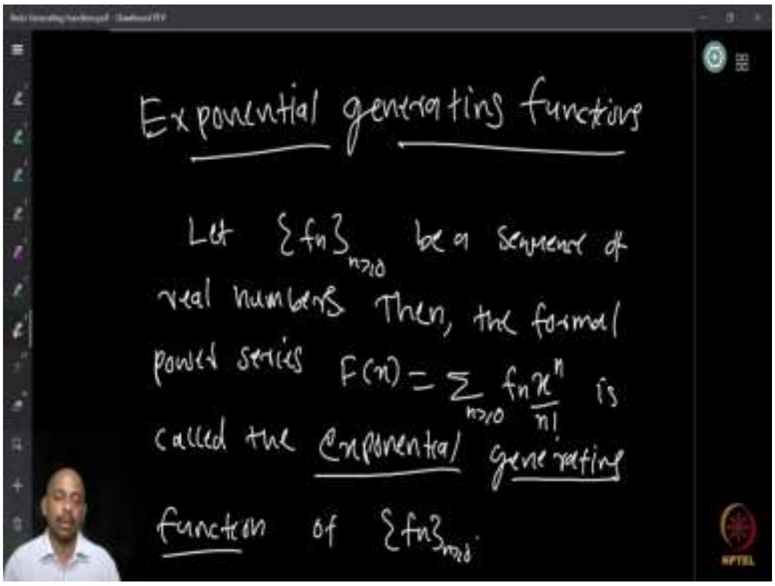

And one of the more standard way to deal with this kind of thing is to use what is called the exponential generating function. So what is the exponential generating function. Given a sequence $\{f_n\}$, a sequence of real numbers, then the formal power series $F(x) = \sum_{n \geq 0} \frac{f_n x^n}{n!}$. This is called the exponential generating function of sequence.

So what we have done here is that, we know that f_n is growing fast. So if f_n is growing fast, I want to make the function to be convergent. So to make the function convergent, the series to be convergent what I can do is to divide throughout by n! to make the coefficient of x^n to be small. So if the coefficient of x^n is small, then one can hope to make it convergent.

So there is a much better probability of converging. So if f_n is growing large, close to n!, one can try to do this and then it might work. Basically n! acts as a normalizing factor, to make f_n to be small.

Now how do you recover f_n? Well, all I need to do is to take the function, whatever we get, look at the coefficient of $\frac{x^n}{n!}$ in the series expansion. That will be equal to f_n.

So even though n! may not be present when we look at the coefficient of x^n, we get something but we have to look at the coefficient of $\frac{x^n}{n!}$. So if there is no n! present in the denominator, we multiply both sides by n! and then get the denominator to be $\frac{x^n}{n!}$.

And then we will get the coefficient of that will be precisely the f_n that we want. So this is something that we can do. So one can try to use the same method in the previous example and you will see that it will work.

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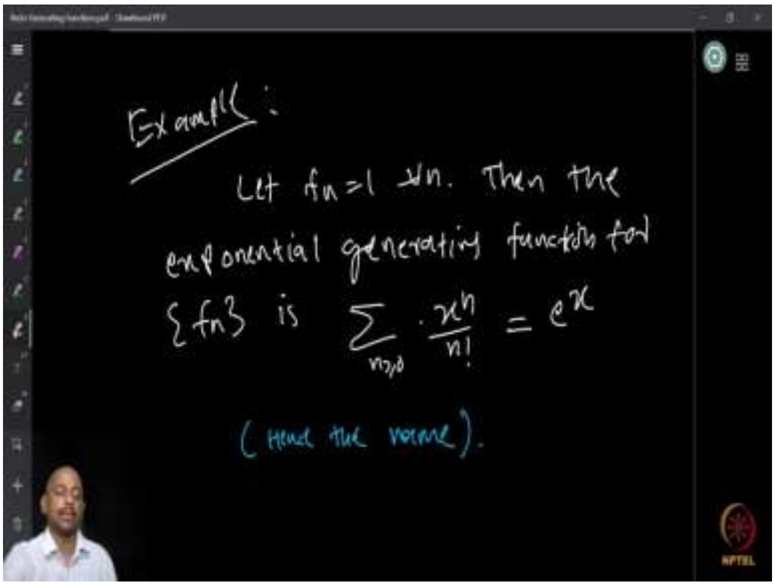

So let us look at a slightly different example to begin with. So we have the example $f_n = 1$ for every n. So we take the constant sequence even though it does not grow fast, I am going to just look at this example.

So I divide by n! everywhere, so I have the exponential generating function for $\{f_n\}$ is $\sum_{n\geq 0} \frac{x^n}{n!} = e^x$. So this, that is the example I wanted to present to you first.

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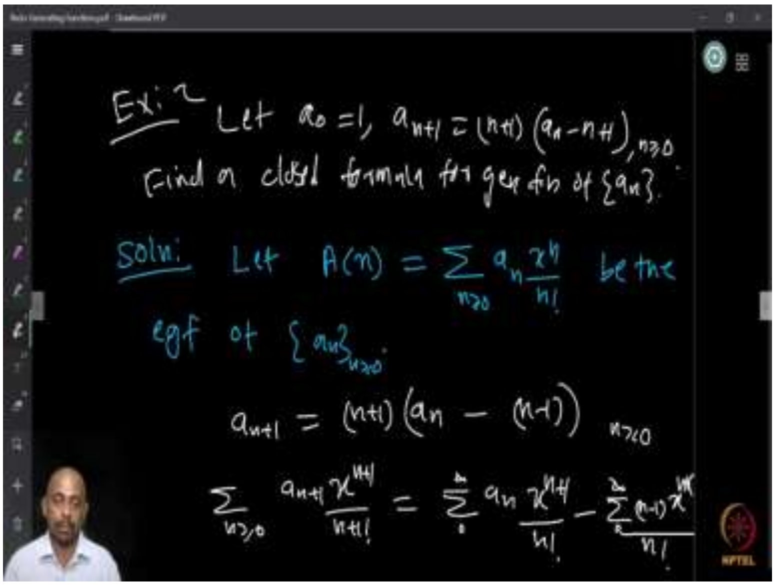

Now, let us take another example. So we have, let us say $a_0 = 1,$ and $a_{n+1} = (n+1)(a_n - n + 1)$ for n greater than or equal to 0. Find a closed formula for the generating function for the sequence $\{a_n\}$.

So one can check the ordinary generating function does not work because we have this n + 1 multiplying every time, so , the coefficient grows very fast. So we will try to use the exponential generating function. Let $A(x) = \sum_{n\geq 0} \frac{a_n x^n}{n!}$ be the exponential generating function for the sequence $\{a_n\},$ which is defined by the recursive formula.

So $a_{n+1} = (n+1)(a_n - n + 1)$. Take the summation by multiplying with $\frac{x^{n+1}}{(n+1)!}$ on both sides, I will get

$$\sum_{n\geq 0} \frac{a_{n+1} x^{n+1}}{(n+1)!} = \sum_{0}^{\infty} \frac{a_n x^{n+1}}{n!} - \sum_{0}^{\infty} (n-1) x^{n+1}$$

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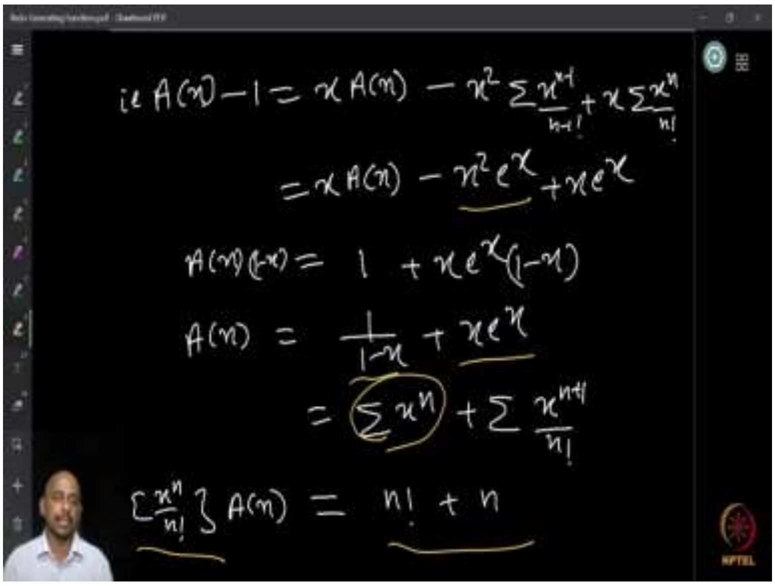

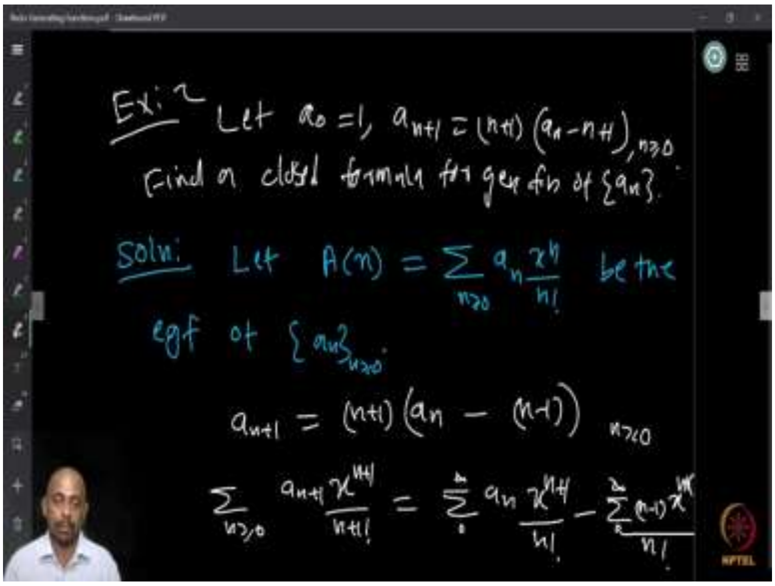

Now we can write the LHS, the first term $a_0 = 1$ is missing therefore it is A(x) - 1 we get

$$A(x) - 1 = xA(x) - x^2 \sum \frac{x^{n-1}}{(n-1)!} + x \sum \frac{x^n}{n!}$$

$$= xA(x) - x^2 e^x + xe^x$$

$$A(x)(1-x) = 1 + xe^x(1-x)$$

$$A(x) = \frac{1}{1-x} + xe^x = \sum x^n + \frac{\sum x^{n+1}}{n!}$$

And the coefficient of $\frac{x^n}{n!}$ in A(x) is n! +n

If you want you can go back and verify, whether it is true for the values. I will not go into the verification part, you can do it. This is how we can use exponential generating function to deal with things. The calculations are exactly the same, only thing is that instead of looking $\sum f_n x^n$, we are looking at $\sum f_n \frac{x^n}{n!}$,

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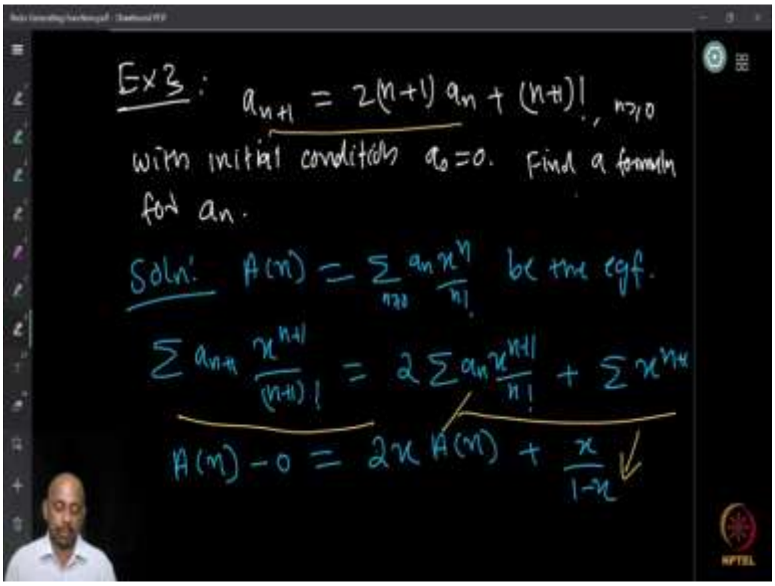

So another example, $a_{n+1} = 2(n+1)a_n + (n+1)!$, n greater than or equal to 0, with the initial condition that $a_0 = 0$. Find a formula for a_n. We have the EGF $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$, where a_n is the nth term of this, the series, that we have defined recursively.

So if you look at the exponential generating function, then, you can take the recursion relation and then multiply by $\frac{x^{n+1}}{(n+1)!}$ on the left side and on the right side you will get as usual the terms to be exactly as before

$$\sum a_{n+1} \frac{x^{n+1}}{(n+1)!} = 2 \sum a_n \frac{x^{n+1}}{n!} + \sum x^{n+1}$$

$$A(x) - 0 = 2xA(x) + \frac{x}{1-x}$$

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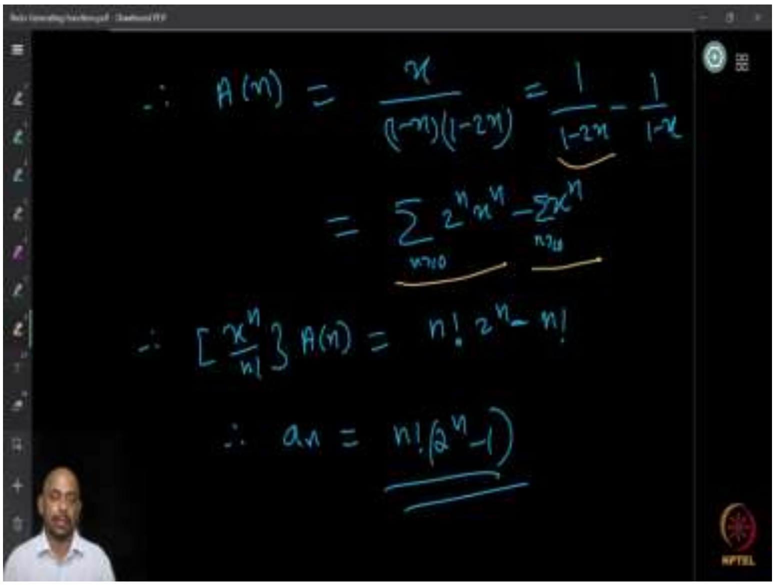

$$A(x) = \frac{x}{(1-x)(1-2x)} = \frac{1}{1-2x} - \frac{1}{1-x}$$

$$= \sum_{n\geq 0} 2^n x^n - \sum_{n\geq 0} x^n$$

But we want the coefficient of $\frac{x^n}{n!}$ in A(x).

$$\left[\frac{x^n}{n!}\right] A(x) = n!\, 2^n - n!$$

$$So, a_n = n!\,(2^n - 1)$$

So I can I solve questions using EGF very, similar to the one that we did before.

Now, you can take it as a homework to solve the first question that we started discussing and then failed, using the ordinary generating function. Now try to use it with exponential generating function. And once you have we can go to look at the meaning of the product.

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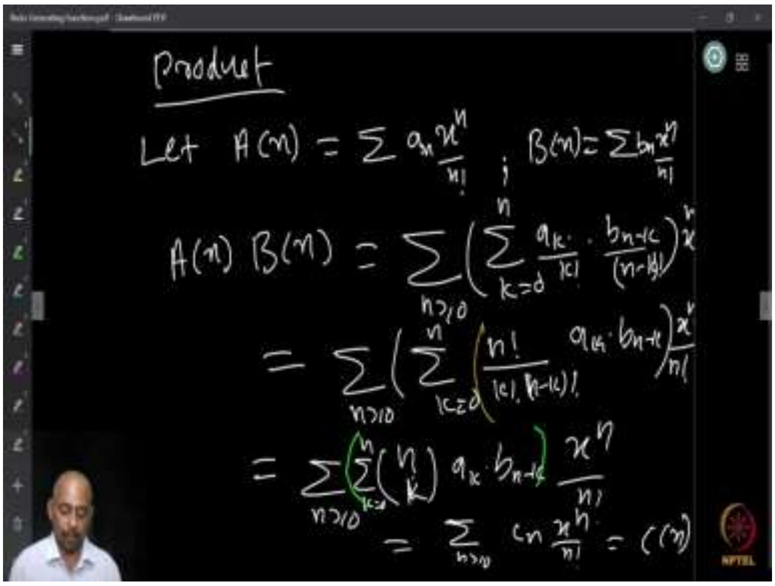

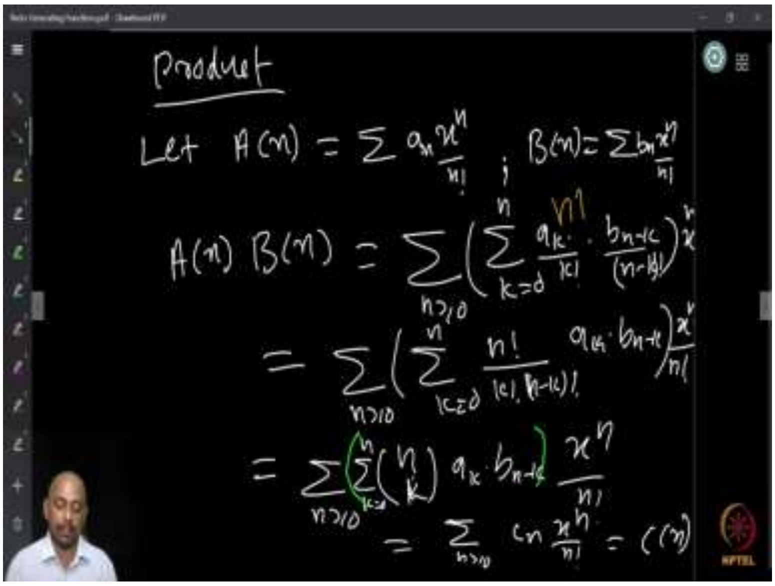

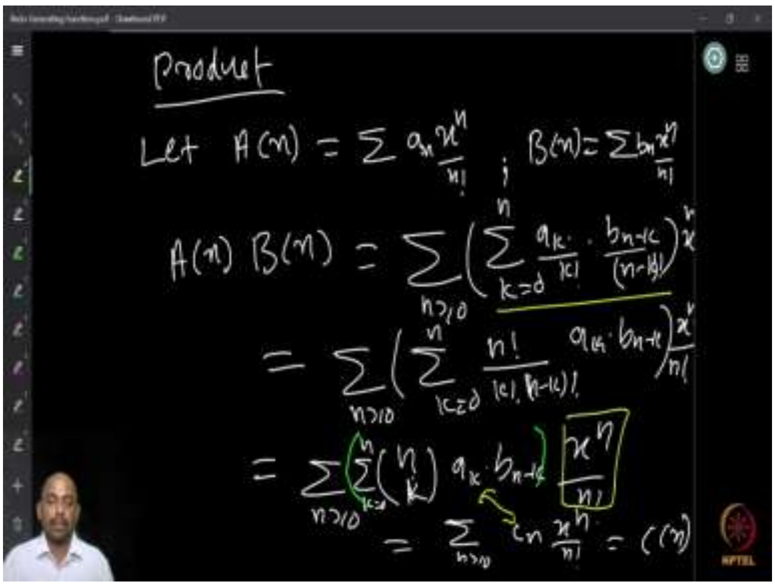

What is the product of generating functions? So how do you define the product? The product is of course defined as before but when it is exponential generating function how the product can be calculated? We want to see that. Let us look at the product of A(x) and B(x), where $A(x) = \sum a_n \frac{x^n}{n!}$ and $B(x) = \sum b_n \frac{x^n}{n!}$. Now the product

$$A(x)B(x) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} \frac{a_k}{k!} \cdot \frac{b_{n-k}}{(n-k)!}\right) x^n$$

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\; a_k b_{n-k}\right)\frac{x^n}{n!}$$

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right)\frac{x^n}{n!}$$

$$= \sum_{n\geq 0} c_n \frac{x^n}{n!} = C(x)$$

Now we have to see what this is. How did this come across?

Now this is easy to see because the coefficient of x^n comes from precisely $\sum_{k=0}^{n} \frac{a_k}{k!} \cdot \frac{b_{n-k}}{(n-k)!}$. that you take the coefficient of x^n in the kth term and (n- k)th term, which is $\frac{b_{n-k}}{(n-k)!}$.

And then their product, and sum over all k ranging from 0 to n. So therefore this is clearly our usual product of the series. So two series we have multiplied and then we get this. We are just writing it in a nice form by converting into this form we want, because we are looking at the exponential generating function, we want to have this as our defining term. So coefficient of $\frac{x^n}{n!}$ is what we want.

Therefore, we are converting into that form and then we will get $\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. Now we want to see how this can make sense so that c_n is well defined but what is the counting sequence for which object?

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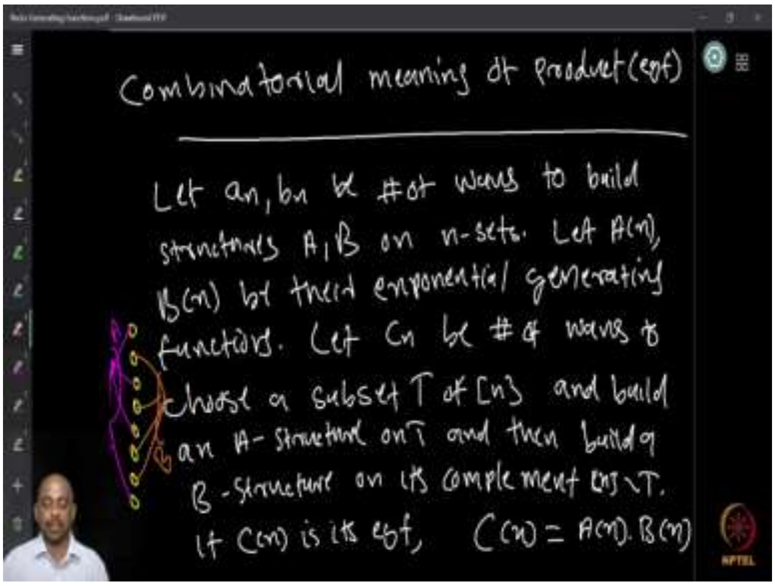

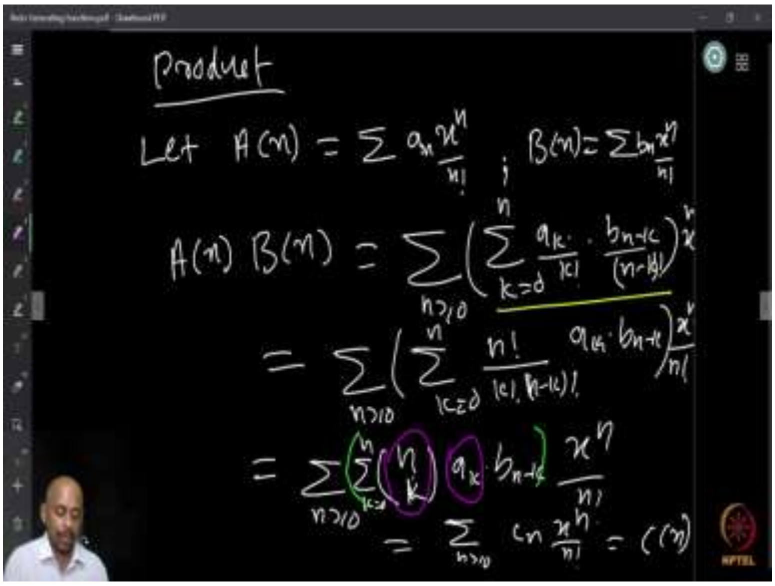

What is the combinatorial meaning of the product of exponential generating functions? We are describing the meaning as follows. Let a_n, count the number of ways to build type A structures, and b_n is the number of ways to build type B structures, on an n element set.

Now suppose A(x) and B(x) are the exponential generating functions and c_n be the number of ways to choose a subset that is the T of the set 1 to n, ([n]). So we have the set and we want to choose a subset. You should recall that in the previous case when we were looking at the ordinary generating function we were not using subsets we were just using sub intervals.

We, were not changing the orders, we were forced to choose in the particular sequence. If I have the set 1 to n, I have to choose from 1 to, let us say, i, i + 1 to, let us say j, j + 1 to etcetera.

So therefore there is a difference here, we are choosing a subset of 1 to n. So it can be any arbitrary subset.

So the number of ways of choosing a subset T, and then build a type A structure on this set T. Now once you do this then you build a type B structure on the complement of this set. And then, so let me try to put a nice figure. So we will see this later but this figure might be very instructive to see. So how do I draw this?

So what I want to do is that, I want to choose some subset of this, then I want to put a type A structure on it. So I am choosing some subset, and on this set I am going to put a type A structure on it. Then I take the complement, and on the complement I am going to put a B type structure. So given the set of n elements I choose a subset T, put a structure of type A on the set, then take the complement, and on this set I put a type B structure.

Now if c_n is the number of ways of doing this, then the claim is that C(x) is A(x).B(x), and this coefficient of x^n in this product count the number of ways of doing this.

So a_n is the number of ways to build a type A structures on an element set, b_n is the number of ways to build type B structures on this, and c_n is the number of ways to first partition the set, n element set into two parts like, one subset and its complement, and on the first set I am going to build a type A structure, and the second part I am going to put a type B structure. And this is counted by c_n, then C(x) = A(x).B(x),. That is that claim.

So how do you prove this? Well if you look at the, the definition it should be kind of clear because what we have here, definition of product. What does $\binom{n}{k}$? $\binom{n}{k}$ says that I can choose some k element subset of the n element set. So that is the number of ways of choosing the k element subset. Now once you choose the k element subset, then I have a_k ways of putting the type A structure on the k elements.

Because there is k element I have exactly a_k ways of doing that. So once you choose the subset I have $\binom{n}{k}$ ways of choosing this. And once you choose the subset, I have a_k ways of making the type A structure, and on the, once you choose the k element set its complement is unique. So the n - k element set, I can put a B structure on b_{n-k} many ways. That is the number of ways of doing this. And, that is what we want.

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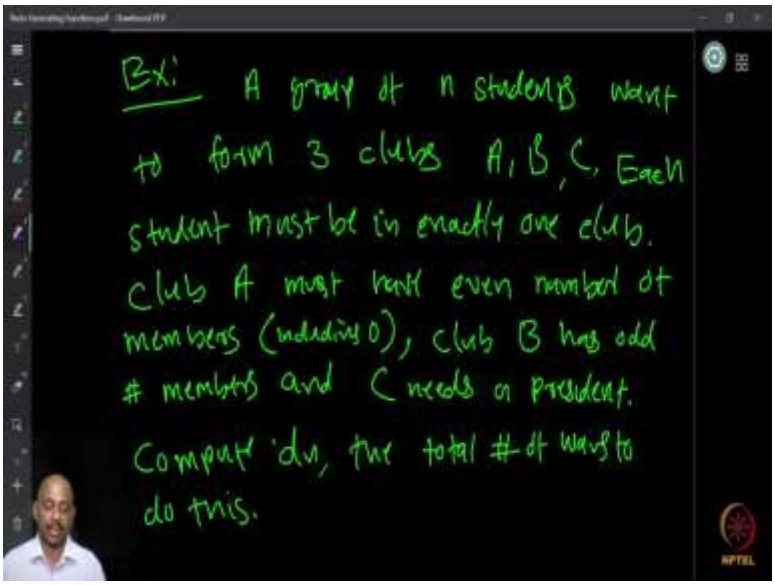

Here is an example. A group of n students want to form three clubs. So A, B, C are the clubs and each student must be in exactly one of the clubs. The club A must have an even number of members. I mean it could be 0, nobody can be there also. And club B must have an odd number of members.

On the other hand, club C says that I do not care whether I have odd numbers or even numbers, but there should be a president for the club. I mean we want clubs with presidents. So the club C requires that, by mandate, that there should be a president. Now compute, let us say d_n, which is the total number of ways to do this.

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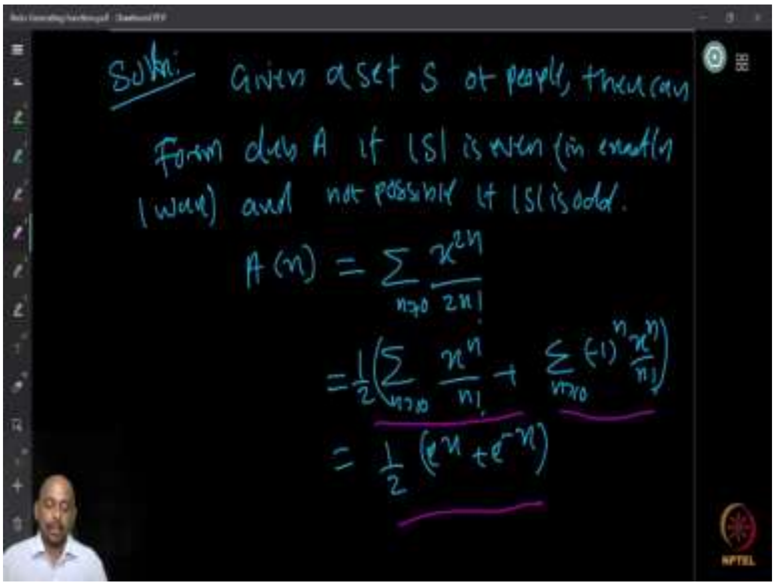

So given a set S of people, they can form a club A, if cardinality of S is even. So only when cardinality of S is even, we can form a club and there is precisely one way to do it. They know they all form a club from that. So once you say that these are the guys going to the club, if it is exactly even number of members they form the club A.

If cardinality of S is odd we cannot form a club of type A. So therefore the exponential generating function $A(x) = \sum_{n\geq 0}\frac{x^{2n}}{2n!}$. Because x^{2n} to say that we can only choose an even number of people. n can be 0, therefore n greater than or equal to 0. So I can choose 0 guys, 2 guys, 4 guys etcetera.

And what is this? I can write it as

$$A(x) = \sum_{n\geq 0}\frac{x^{2n}}{2n!}.$$

$$= \frac{1}{2}(\sum_{n\geq 0}\frac{x^n}{n!} + \sum_{n\geq 0}(-1)^n\frac{x^n}{n!})$$

$$= \frac{1}{2}(e^x + e^{-x})$$

This the generating function for A(x). So we get it immediately from this form, $\frac{x^{2n}}{2n!}$.

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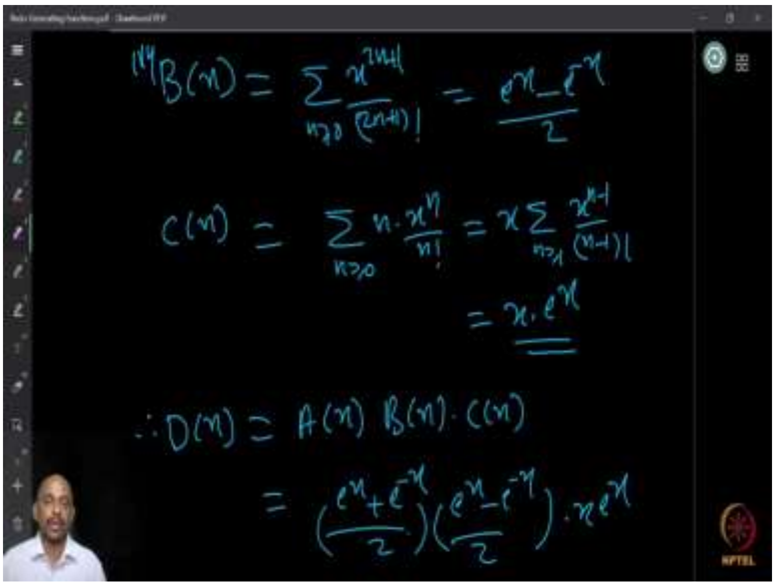

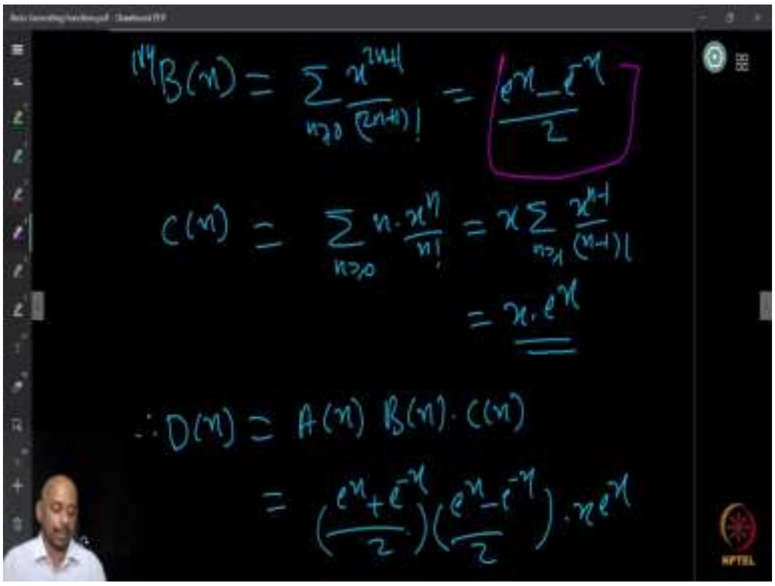

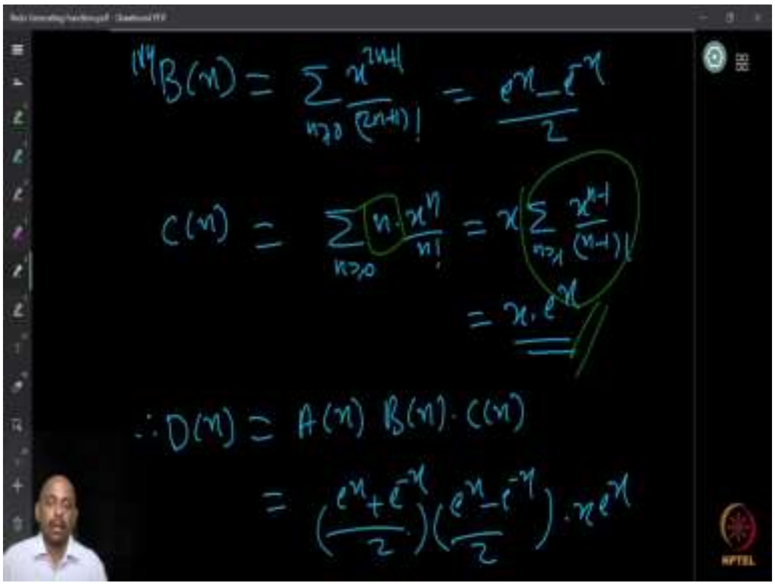

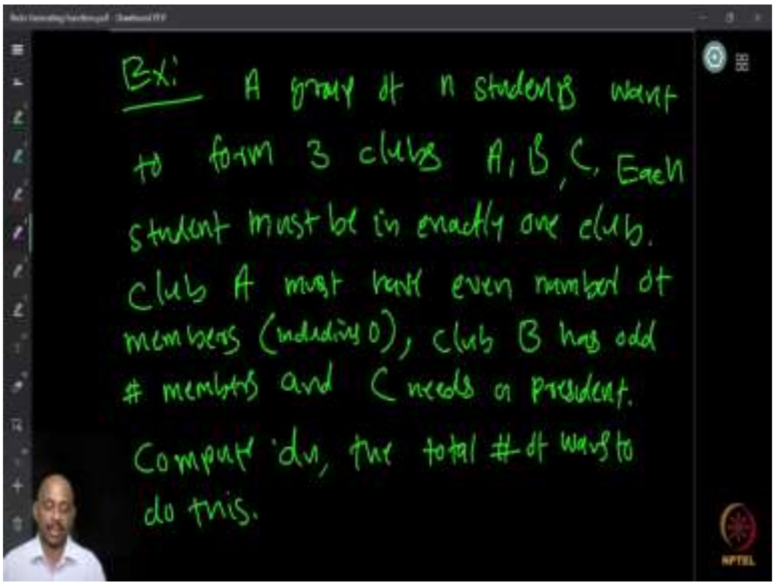

So yes, now let us solve for B(x).

$$B(x) = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!} = \frac{e^x - e^{-x}}{2}$$

And C of x is given by,

$$C(x) = \sum_{n\geq 0} n\ \frac{x^n}{n!}.$$

because once you have n people you have to choose one of the president in n possible ways. Any one of them can be a president. Therefore, there is a n possible ways to do when you are given a set of n guys. So I have $C(x) = \sum_{n\geq 0} n\ \frac{x^n}{n!} = x\ \sum_{n\geq 1} \frac{x^{n-1}}{(n-1)!}\ = xe^x$

And so we have all these things like, A(x), B(x) and C(x), and since we are looking at the number of ways of choosing a club A, B and C by partitioning into three parts, we can apply the product rule. So,

$$D(x) = \mathrm{A(x)\ B(x)C(x)}$$

$$= \ (\frac{e^x + e^{-x}}{2})(\frac{e^x - e^{-x}}{2})(xe^x)$$

So we get the generating function and then we can look at the coefficient of x^n.

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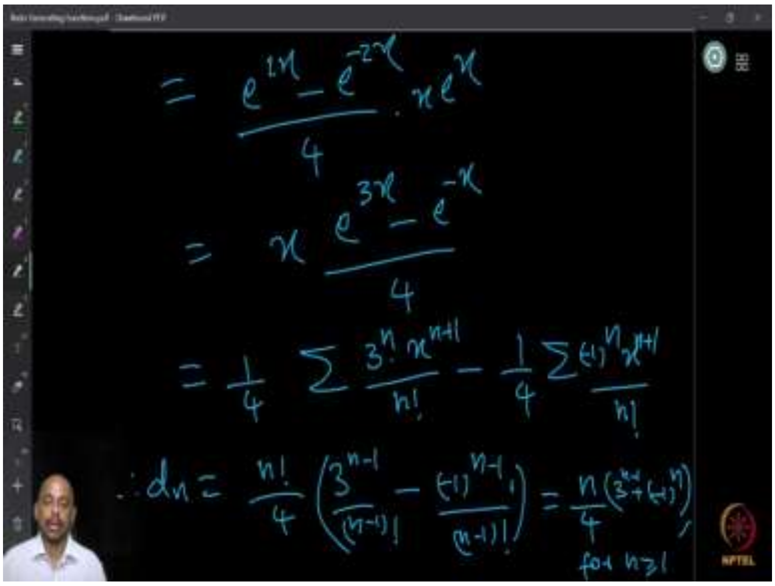

And for that we do some simplifications,

$$D(x) = \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)(xe^x)$$

$$= \left(\frac{e^{2x} - e^{-2x}}{4}\right)(xe^x) = x\left(\frac{e^{3x} - e^{-x}}{4}\right)$$

$$= \frac{1}{4}\left(\sum \frac{3^n x^{n+1}}{n!}\right) - \frac{1}{4}\left(\sum \frac{(-1)^n x^{n+1}}{n!}\right)$$

So the coefficient of $\frac{x^n}{n!}$ is,

$$d_n = \frac{n!}{4}\left(\frac{3^{n-1}}{(n-1)!} - \frac{(-1)^{n-1}}{(n-1)!}\right) = \frac{n}{4}(3^{n-1} + (-1)^n)$$

,for n a greater than or equal to 1. So this is a nice way to do it.