Combinatorics

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Composition of generating functions

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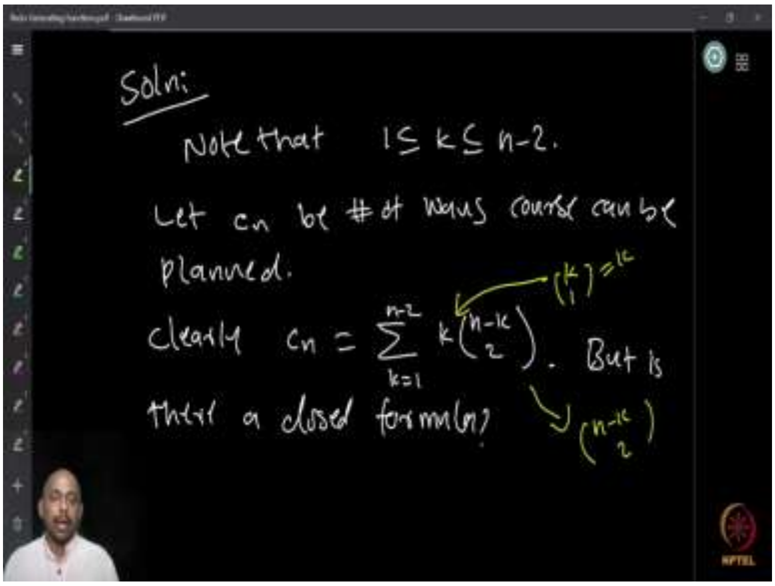

So therefore that is c_n. And the question is that is there a closed formula? So this is something we can see, can we, now, can we find a closed formula?

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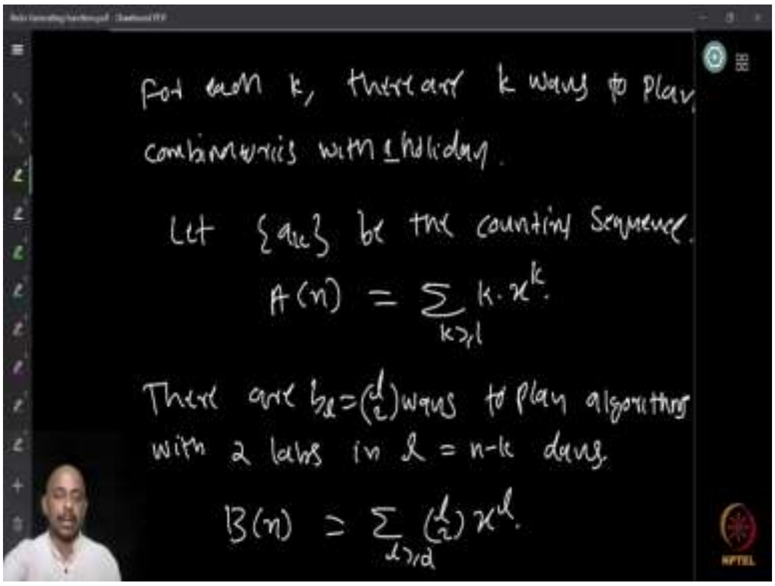

So we mentioned k ways to plan the combinatorics part with one holiday, and therefore the generating function that we can find separately for the counting sequence is $A(x) = \sum_{k\geq 1} kx^k$

which is the generating function.

And then similarly the generating function for the $\binom{l}{2}$ ways to plan two labs in l day is, l = n-k, and that $B(x) = \sum_{l\geq 2}\binom{l}{2}x^l$ because we need at least two days to choose for the labs.

So, we have these two generating functions, and we know the product of the generating functions and if you can solve independently for this, we can use that to get a closed formula for C(x). So that is the idea.

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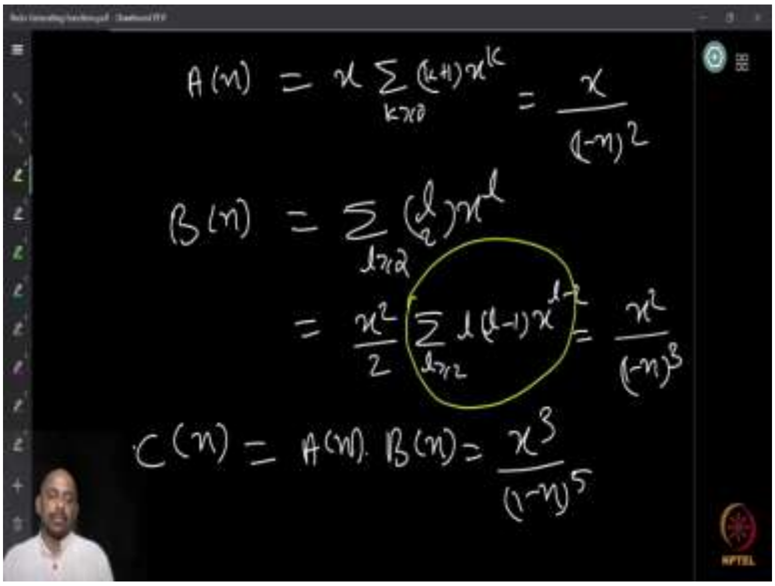

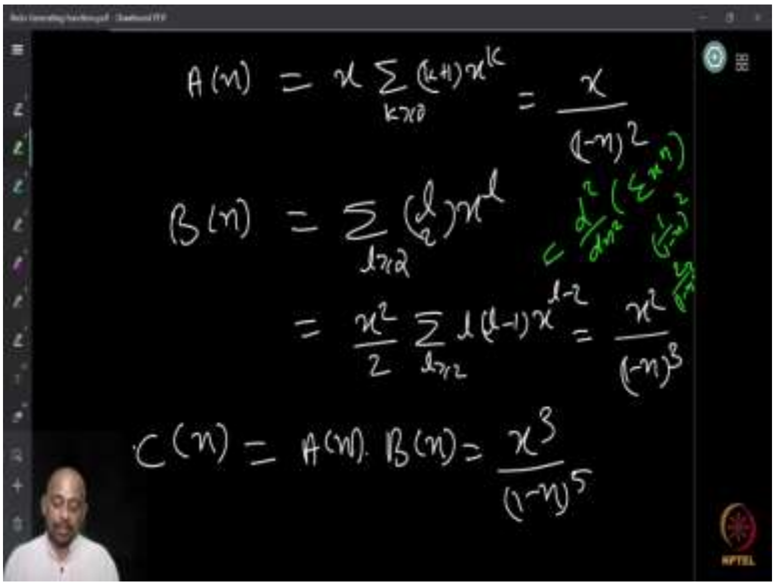

So what is A(x)?

$$A(x) = \sum_{k\geq 1} kx^k = x\sum_{k\geq 0}(k+1)x^k = \frac{x}{(1-x)^2}$$

Similarly,

$$B(x) = \sum_{l\geq 2}\binom{l}{2}x^l = \frac{x^2}{2}\sum_{l\geq 2} l(l-1)x^{l-2} = \frac{x^2}{(1-x)^3}$$

So then the, C(x) by the product rule is that A(x)B(x). So, $C(x) = \frac{x^3}{(1-x)^5}$. We can find out by Taylor series expansion, and then we need to shift the degree by 3, because there is an x^3.

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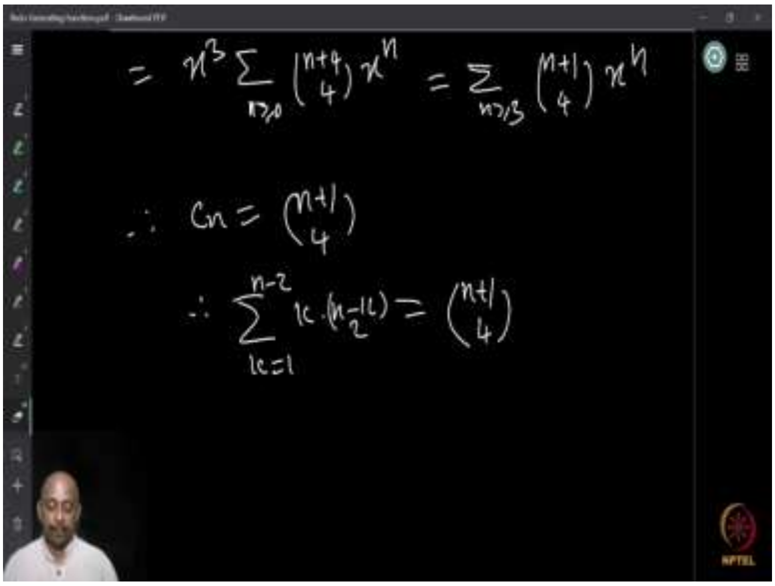

So, $C(x) = \frac{x^3}{(1-x)^5} = x^3 \sum_{n\geq 0}\binom{n+4}{4}x^n = \sum_{n\geq 3}\binom{n+1}{4}x^n$

Therefore the coefficient of x^n is, $\binom{n+1}{4}$, and therefore I have a formula for c_n. So $\sum_{k=1}^{n-2} k\binom{n-k}{2} = \binom{n+1}{4}$. I can solve this summation also. So Iget a closed formula for this summation, which otherwise would be much more difficult to come up with. So you can try this without using the generating function if you want to do this.

And similarly, the product actually helps to compute these things easier, that if you had this summation and then you use the generating function method it will be slightly more difficult.

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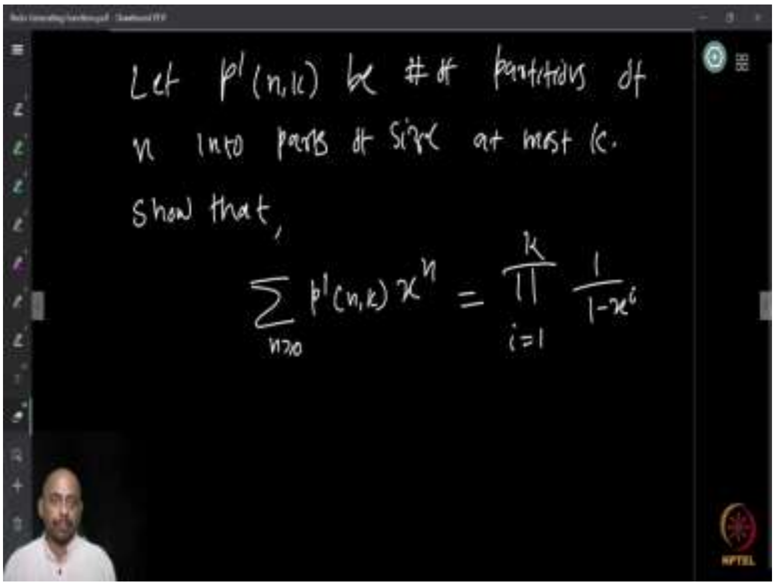

Now if you remember your partitions, let us define the number of partitions of the integer n into parts of size at most k. So I am denoting it by $p'(n,k)$ as the number of such partitions. So we want to partition but then each part has size at most k.

Now we want to show that the generating function for this sequence

$$\sum_{n\geq 0} p'(n,k)\mathrm{x}^{\mathrm{k}} = \prod_{i=1}^{k} \frac{1}{1-x^i}$$

So this is the generating function for the partitions of n into parts of size at most k. So let us try to do this.

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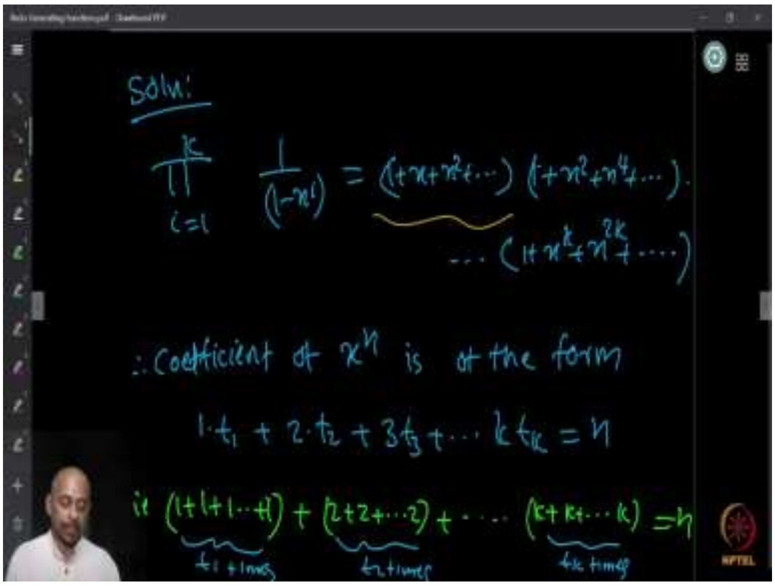

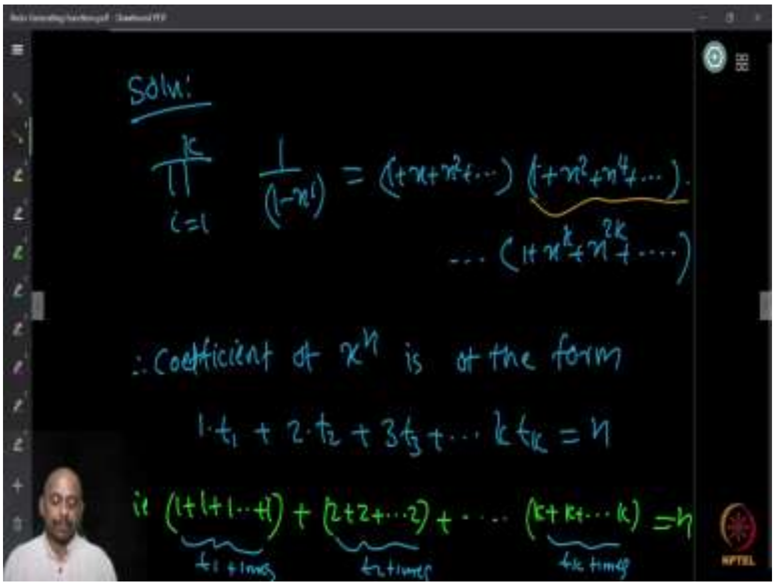

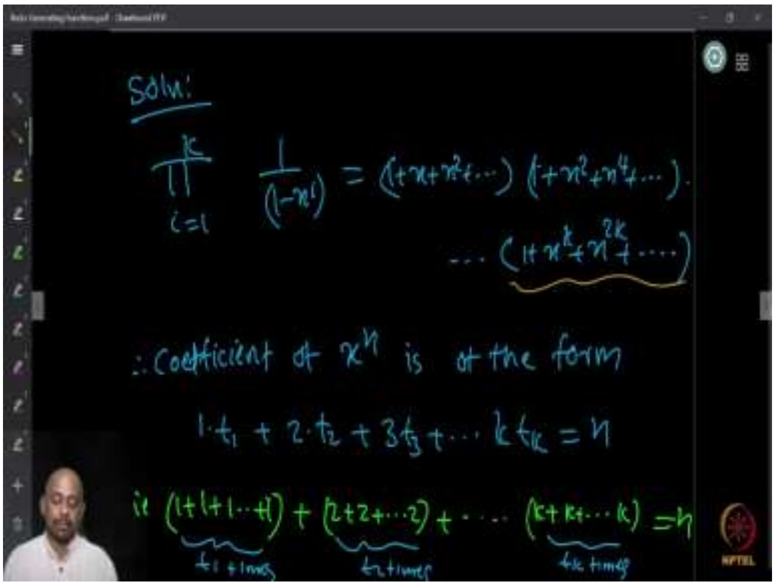

So let us write it as product of a series.

$$\prod_{i=1}^{k} \frac{1}{1-x^i} = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \dots (1 + x^k + x^{2k} + \cdots)$$

Now what is the coefficient of x^n from here? Coefficient of x raised to n is of the form,

$$1t_1 + 2t_2 + 3t_3 + \cdots kt_k = n$$

So, and sum of these exponents must be equal to n because we are looking at the coefficient of x^n. Now this is how I get the coefficient of x^n in this product, from anywhere I get x^n, this is how it is going to come.

Now, I want to see this in a different way. So this choice I can see as choosing t_1 copies of 1, t_2 copies of 2, and t_k copies of k. And all of them sum to n and the maximum value is k. So therefore this generating function actually represents the partition of n where each part is at most k.

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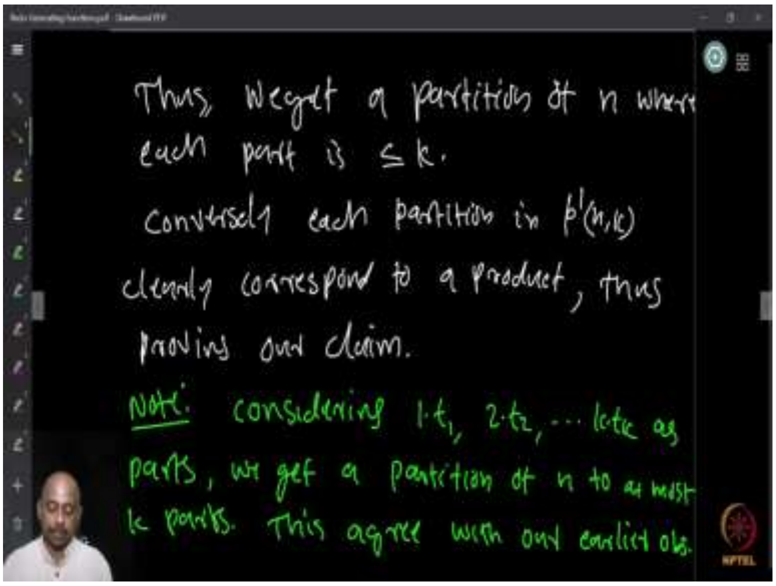

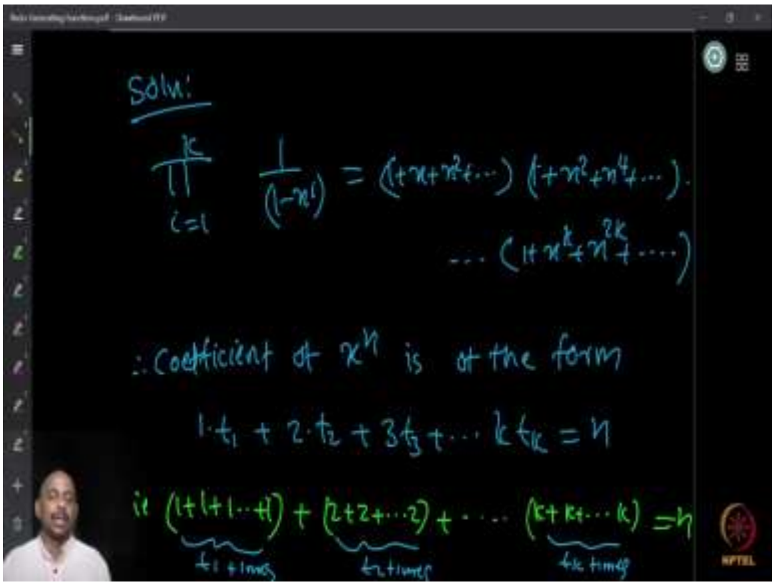

On the other hand, we can also see something else. So each partition, you can see that clearly correspond to a product, because, given a, partition we can say that like take the number of times i appears and this is a choice of x^i times that number. And therefore I get a one to one correspondence in between these two. So I get this.

Now, this is a more interesting part of looking at the generating function, this we observed earlier, but let us put it again.

I can also see the same summation as one times t_1 is a part, so it comes. And two times t_2 is a part, three times t_3 is a part, etcetera, k times t_k is a part. So I can also see this as, the number of ways of decomposing n into parts, at most k parts because there cannot be more than k terms here.

So therefore we get a partition of n into at most k parts, and this agrees with our earlier observation that the number of ways of partitioning the integer n into at most k parts is precisely the number of ways of partitioning, where each size is at most k.

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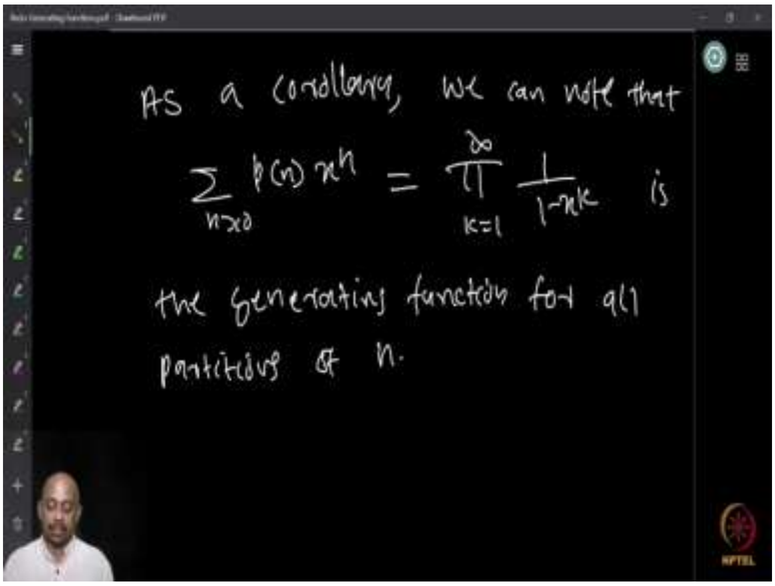

So as a corollary we can also get the generating function for p(n) which is the partition of n,

$$\sum_{n\geq 0} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

because here we are not putting any restrictions on the number of parts. So our k can vary from 1 to infinity. It can be anything, but of course once the k exceeds n you will see that this will not give you any coefficient of x^n therefore we do not have to worry about that.

But for every n we have this. Therefore, I can define the $\sum_{n\geq 0} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$. So this is a nice way to come up with the generating function for related things.

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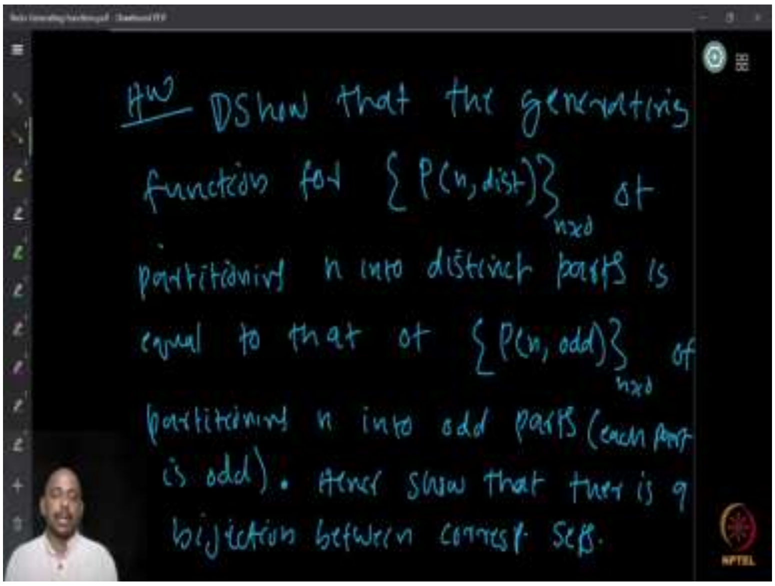

Now, a homework question. So, show that the generating function for a partition of n into distinct parts $\{P(n, dist)\}_{n\geq 0}$ is equal to the number of ways of partitioning n into odd parts. P(n, odd) is denoting the number of ways of partitioning n into odd parts. So each part is odd. And, P(n, dist) is the number of ways of partitioning into distinct parts. So each part is distinct.

And these two are identical. So this is what I want to show. The generating functions are the same. And using this show that there is a bijection between the corresponding things. Of course that comes immediately. So we need to show that the generating functions for these two are the same.

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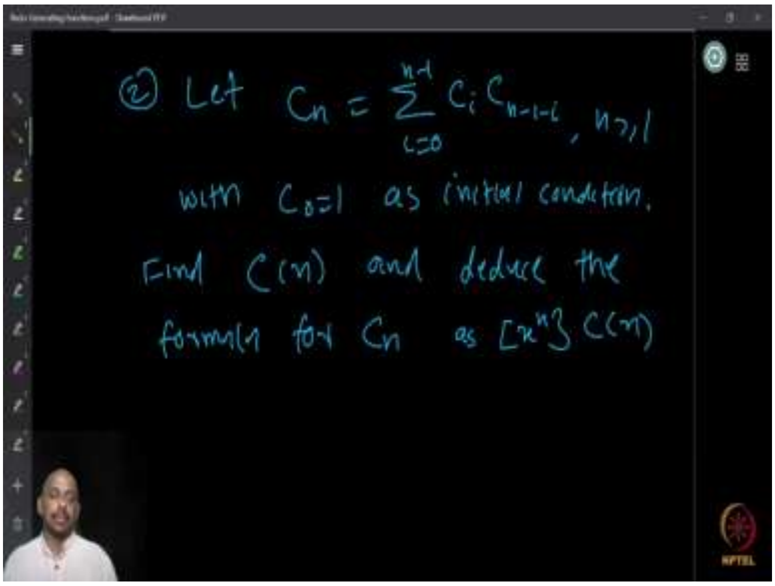

Second question ask you to solve the recurrence relation $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$ where n is at least 1, with the $C_0 = 1$ as the initial condition. So find C(x) and then deduce the formula for C_n as the coefficient of x^n in C(x). So this is what you should do.

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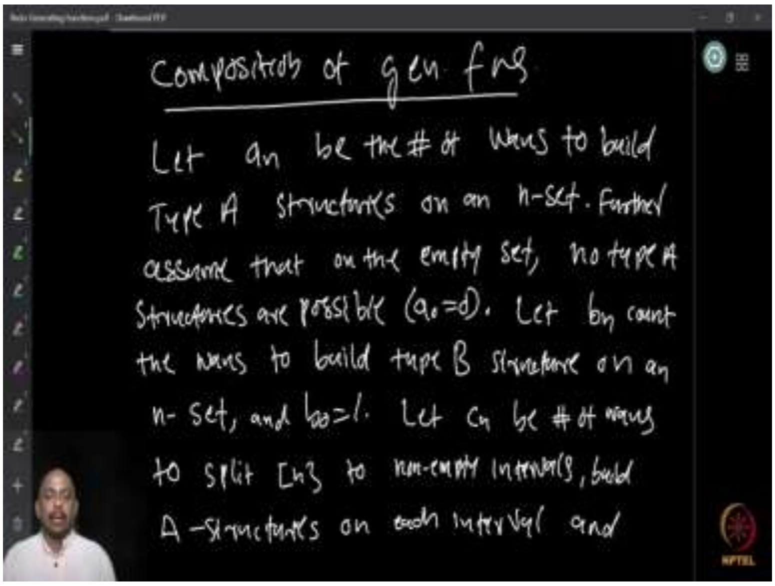

Now we are going to look at the composition of the generating functions. So what I mean by composition is that I can substitute one function instead of the variable of the other function. So if I have A(x) and B(x) then I can talk about A(B(x)). So x is replaced by B(x). So wherever x appears in A, I replace it with the B(x).

So then what happens according to the combinatorics and that is the question that we want to look at. Let a_n be the number of ways to build a type A structure on an n element set. So you have an n element set, and I want to look at the number of ways to build a type A structure on this.

Then on the empty set we assume that there is no type A structures that you can build because this is a necessary assumption because as you will see when we are going to substitute this inside, you will see that if you can, you can make empty structures out of a set, then you have a way too many objects that you can make up. And then it will not become a converging sequence.

So what we will do is that we will assume that $a_0 = 0$. So on empty set you cannot make any structure of type A. Let b_n denote the number of ways to build type B structure on an n-element set, and $b_0 = 1$. So this assumption is not strictly necessary, but it will be good for us in this type. Now let c_n be the number of ways to split the set 1 to n into non-empty intervals, and then build let us say A type structures on each interval.

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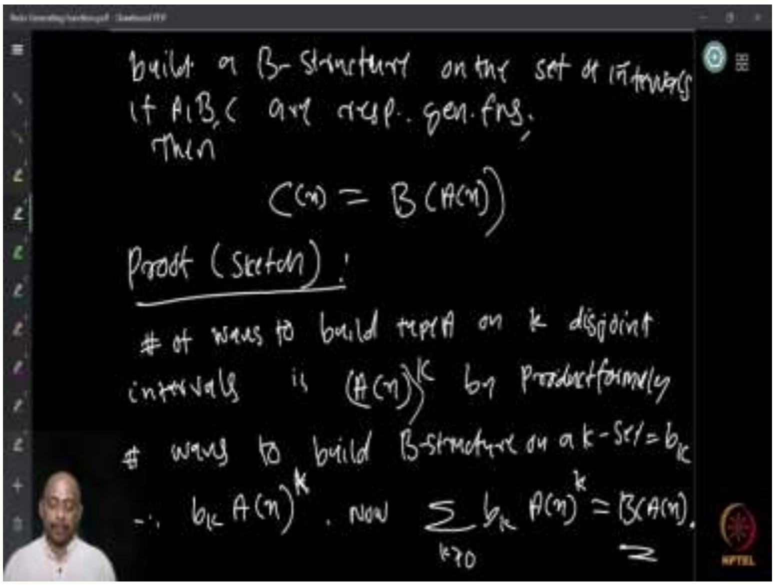

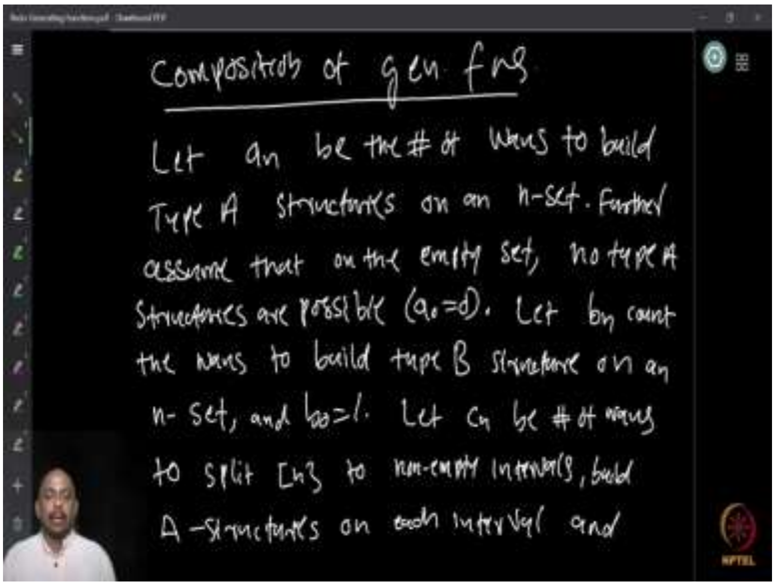

And build B structures on the set of intervals. If A, B, C, represent the generating functions for a_n, b_n and c_n , then we have C(x) =B(A(x)). So let us look at it again. What we are going to do is that given an n-element set, I partition it into non-empty intervals. So, the set 1 to n is partitioned into non-empty interval. So let us say 1 to something, 1 to 5, and then 5 + 6 to let us say 9, and then 10 to 21, etcetera. So we put it into non-empty intervals.

Then for each part, 1 to 5, I will put an A structure on this set. Then I put another A structure on 6 to 10, and similarly I put A structure on the remaining part. Now on the set of intervals I put a B structure. So there is how many ways I have partitioned, that many elements will be there. So each interval becomes an element of the set. And then on these intervals I put a B structure.

If I am doing this then the generating function for the number of ways to build this structure, which is C_n, is let us say C(x) is the composition of B(A(x)). That is, you substitute for x in B(x) with A(x). So there is a bracket missing, I think. B(A (x)).

Now, I will give you a sketch of the proof. So let us see why this must be the case. So the number of ways to build type A structure on let us say k disjoint intervals is $\left(A(x)\right)^k$.

The number of ways to build a type A structure on k disjoint intervals is the product, for each interval we want to take the product. But then since we have exactly k intervals, if you assume, k to be fixed then it is basically the coefficient of $\left(A(x)\right)^k$ because for each of the k intervals, I am building A structure. So whatever is the number in that interval, I have the generating function will be A(x). So A(x) is the generating function for building type structure A for any thing.

Then, if you have i elements in that particular interval, it happens to be i elements, then the coefficient of x^i will give you the number of ways to build it. This is the nice thing about generating function.

So the number of ways to build the type A structure on the k disjoint interval is the product of the generating functions, each of them. So A(x). A (x)…. k times. But then if I start with an n element set, if I start with an element set then the number of ways of doing this is the coefficient of x^n in $\left(A(x)\right)^k$.

And why is this? Because, if I look at the coefficient of x^n how the n came about? So if n has come out from this product of k terms, it must have chosen some values like n_1, n_2 etcetera from each of the k terms. So that is how I get coefficient, the summation of $n_1 + n_2 + \cdots = n$. The coefficient of x^n comes in that way.

So if I have taken from the first part and the second part and then the third part etcetera up to kth part, then coefficient of x^n precisely says that number of ways of doing that when I have exactly, let us say, two elements here, five elements here, seven elements here, etcetera, for a particular choice, and for all possible such ways.

But then we are going to build B structure on the k set. Now how many ways we can do that, that is precisely b_k ways, on a k element set. So because we are partitioning into exactly k intervals, and each interval I am considering as an element, and then I am going to put a B structure on this interval. So therefore there are precisely b_k ways to do that.

But for, for all possible choices for the, the splitting of A into this, we get it from $\left(A(x)\right)^k$. to divide into exactly k parts and doing this.

And therefore it is basically $\sum_{k\geq 0} b_k\left(A(x)\right)^k$. Of course we are looking at the coefficient of whatever in the appropriate value. But now, which is the definition of B(A(x))? This is just a sketch of the proof.

We can formally prove this, but I do not want to go into the details of this. But this should tell you enough to understand why this, and now of course you have to think about this a little bit to see how the generating function plays. But think about it and then get back.

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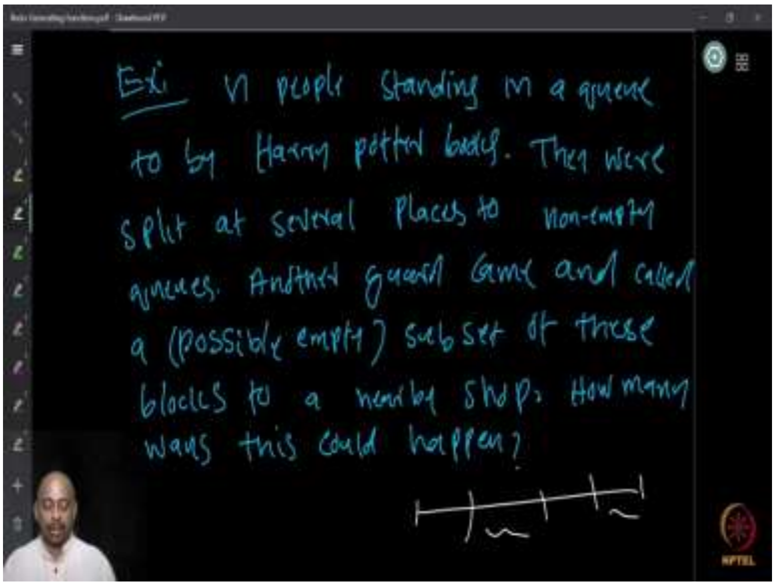

Now an example. So we have n persons standing in a queue to buy, let us say, some books, Harry Potter books or something. People stand in queue to buy this. Now they were split at several places to non-empty queues. Some person comes and then says that okay, you are standing in queue, let us split you into several parts by putting some barricades or something in between.

Then another guard comes and then says that okay, they call some subset of this blocks of people. So they have several blocks of people, the queue is partitioned into several blocks, and then some, another guard comes and then say that okay, some subset of these guys, and say that okay, let us all go to my shop. I will give you faster access to the book. Here you are waiting in the queue, let me take some of you guys from here and here. Then I will take you to my shop, I will give you a better deal there. Now how many ways this can be done. So let us solve this.

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Soln:

$a_k = 1 \ \forall k \geq 1, \ a_0 = 0$

$\therefore A(x) = \frac{x}{1-x}$

$b_k = 2^k \ \forall k, \ B(x) = \frac{1}{1-2x}$

$\therefore B(A(x)) = \frac{1}{1-2\frac{x}{1-x}} = \frac{1-x}{1-3x}$

$\therefore [x^n] B(A(x)) = 2 \cdot 3^{n-1}, \ n \geq 1$

Ex: n people standing in a queue to buy Harry Potter books. They were split at several places to non-empty queues. Another guard came and called a (possibly empty) subset of these blocks to a nearby shop. How many ways this could happen?

Composition of gen fns

Let a_n be the # of ways to build Type A structures on an n-set. Further assume that on the empty set, no type A structures are possible ($a_0=0$). Let b_n count the ways to build type B structure on an n-set, and $b_0=1$. Let c_n be # of ways to split $[n]$ to non-empty intervals, build A-structures on each interval and

So how do you solve this? So we know number of ways of choosing this first type of things. Like that is basically split into several places, and what does that guard do? That guard does nothing.

So what the first guard does is to basically just split the queue into several parts, but he does nothing else. So he does not modify it any way. And he do not put any structures. The queues are standing as it is. So there is exactly one way to do nothing. Once you split it several places you have the queues and we do nothing.

So when we were defining this what we said is that a_n be the number of ways to build type A structures on n element set, and then c_n was the number of ways to split n into non-empty intervals. And then build A structures on each of the interval.

But here what is the A structure? There is nothing because we are just splitting the interval and doing nothing, which is basically, the coefficient is 1. There is precisely one way to do nothing with it. So, I can say that a_k is 1 for every k because after splitting, I do not do anything. For whatever k, I get k intervals and that is it. So, $a_k = 1,$ for all $k \geq 1, a_0 = 0$

So, the generating function for $A(x) = \frac{x}{1-x}$. And $a_0 = 0$ because, if there is 0 guys there is no way to split. We do not allow non-empty structures.

Now, $b_k = 2^k$ because, why is that? Like we have to choose some subset of k blocks. There is k blocks and I want to choose some subset. There is, even the subset can be empty. He may not be able to call anybody, because maybe the other guard will come and say that you cannot steal our customers.

So anyway, so there is precisely 2^k subsets possible from the k blocks. We are just choosing the subsets and then taking into its place. Now $B(x) = \frac{1}{1-2x}$.

And now we apply the composition rule.

$$B\big(A(x)\big) = \frac{1}{1 - \frac{2x}{1-x}} = \frac{1-x}{1-3x}$$

So therefore from this I can get 1-3x will give you $3^n x^n$, and then you multiply with 1-x and look at the coefficient of x^n. That is, 2.3^{n-1}. This is something you can verify. For n greater than equal to 1, I have this. So we can verify that there is precisely 2.3^{n-1} ways to do this.

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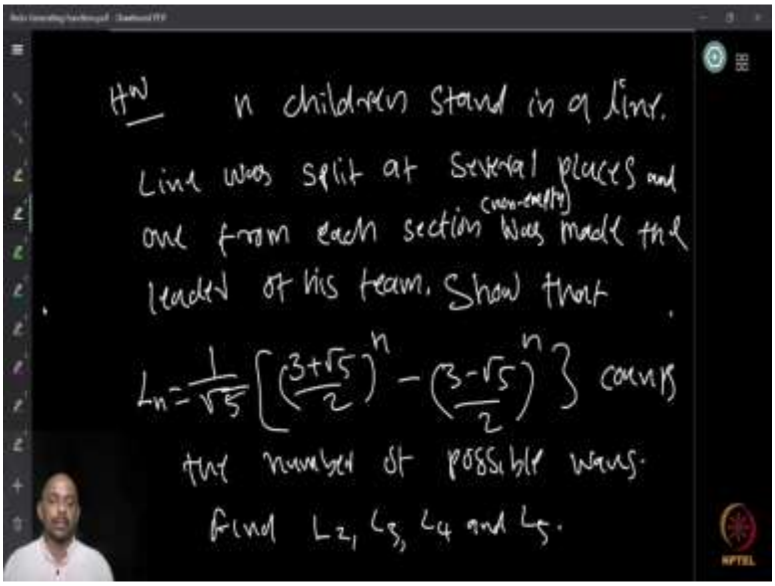

So here is a homework question. We have n children who is standing in a line, and the line was split at several places. And one from each non-empty section, who was the leader of his team. Because each section is non-empty, I can choose one person and then make him the leader of his team.

Then show that, $L_n = \frac{1}{\sqrt{5}}[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n]$ counts the number of possible ways of doing this. And then once you find this formula, use it to find L_2, L_3, L_4 and L_5.

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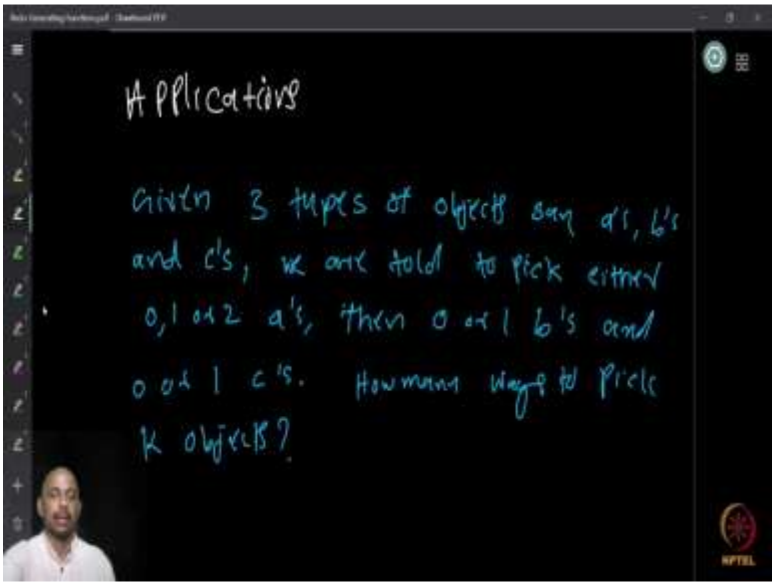

Let us look at a couple of more applications. So let us say that we have three types of objects are given, let us say a, b and c. Now given these three types of objects, we are told to pick, let us say, either 0, 1 or 2 a's. Then 0 or 1 copies of b's, and 0 or 1 copies of c. Then you can ask how many ways to pick k objects from here.

So whatever could be the object. So three types of objects are there, type a objects, type b objects and type c objects. And you can pick either 0 or 1 or 2 a, 0 or 1 b and 0 or 1 c. So how many ways one can do this, for, to picking k objects.

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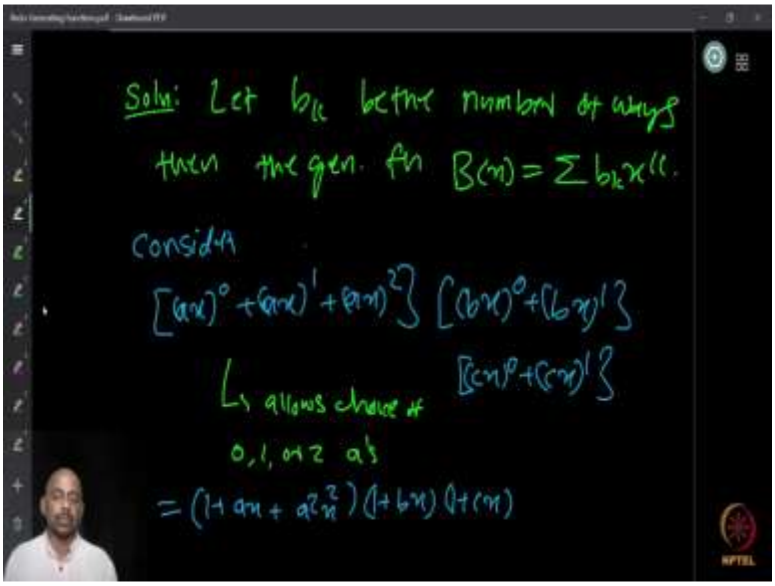

Let us denote by b_k, the number of ways of doing this. So the generating function for this sequence b_k is $B(x) = \sum b_k x^k$. Now what is precisely B(x)? To count B(x) I am going to use a slightly different method.

So if you look at the following, so I want to choose a. So I am going to look at the function $(ax)^0 + (ax)^1 + (ax)^2$. Then I take the product with $(bx)^0 + (bx)^1$, and then take product with $(cx)^0 + (cx)^1$. That is consider $[(ax)^0 + (ax)^1 + (ax)^2][(bx)^0 + (bx)^1][(cx)^0 + (cx)^1] = (1 + ax + a^2x^2)(1 + bx)(1 + cx)$.

Now, when I take the product of this, and look at the coefficient of x^k for some number. Where is x^k 's coefficient coming from. So if I choose x^k as, let us say $(ax)^1$, then $(bx)^0$ and then $(cx)^1$, then the n will be 2. And then what does this choice represent?

This choice represents that I have taken 1 a because $(ax)^1$, then I have taken 0 b's, $(bx)^0$, then I have taken exactly one type C objects x, which is c x whole raised to 1. I could get two in another way, maybe. I can choose a x raised to 2, b x raised to 0, and $(cx)^1$. So this way I can get two. Similarly, I can just take two A type objects. Similarly, I can pick one B type objects and one C type object and choose nothing from the first part $(ax)^0$.

So this tells you the choices of a are either 0, 1 or 2 because ax have power 0, ax have power 1, ax have power 2. Similarly, bx power 0, bx power 1, and bx power 2. So this will tell you how many a's are chosen, how many b's are chosen, how many c’s are chosen.

So I will get x^4 with a^2bc. So I have to choose 2 a’s, 1 b and 1 c.

Now this is one possibility, though I have several possibilities. But now how do you put them together to find B(x). So observe that like $(ax)^2$ says that the, the choice is of a, and then I am taking two of them. x square tells it is 2 and then a says that it is choosing a. But now it does not matter, we want to find how many total ways to do this. So it does not matter, maybe I have taken two but I just need to make sure that I get k only.

So it does not matter whether I take two a's, one b and one c or like one a, two b and like, in this case there is no two b, but, or one c, but as far as the total is k, I am fine. So therefore, I can substitute for a to be 1. I take the product $(1 + ax + a^2x^2)(1 + bx)(1 + cx)$. And in this product let me put a equal to 1, b equal to 1, and c equal to 1.

That says that I am choosing two from the first part, one from the second part, one from the third part, or two from the first part, zero from the second part, and like zero, copies of b from the third part etcetera, which is choosing one. This will tell you the number of choices.

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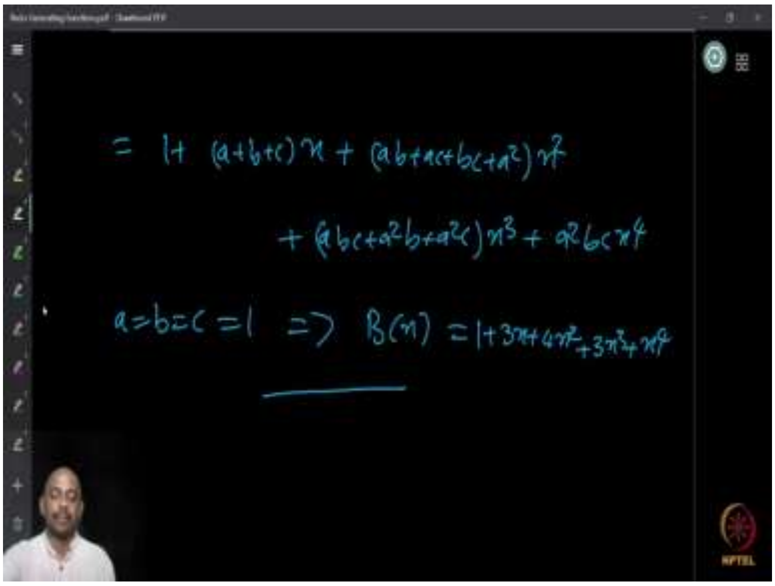

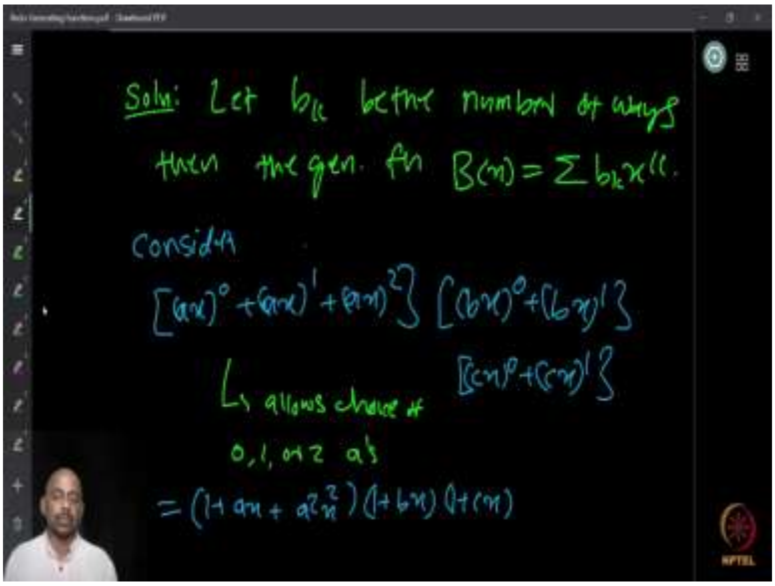

So the generating function, now, I can get by substituting for a = b = c = 1 in this, and then I will get it is $B(x) = 1 + 3x + 4x^2 + 3x^3 + x^4$. And what does this generating function tells us?

It says that I have the choice of objects, at most four of them, I mean k cannot be more than 4, and if k is equal to 4 there is only one way to do that, and if k is equal to 3 there are 3 ways to do it, if k is equal to 2 there are 4 ways to do it, and if k is equal to 1 there is 3 ways to do it, and if k is equal to 0 there is only one way to do it, choose nothing.

So this tells us, the solution for this. So let us verify that. We can verify that also, for such an equation. So how many ways to choose? There are four different ways to choose two objects. So let us see how this can be. I can choose two type a objects and nothing from b or c, that is one way.

Then I can choose one type a object and one type b object, that is the second way. Then I can choose one type a and one type c object, which is the third way. Then I can choose nothing from a type but one b and one c, that is the fourth way. So there are the four different ways. So I get coefficient of x^2 as 4. Similarly, one can verify for other values. So this is another nice way to use generating functions.

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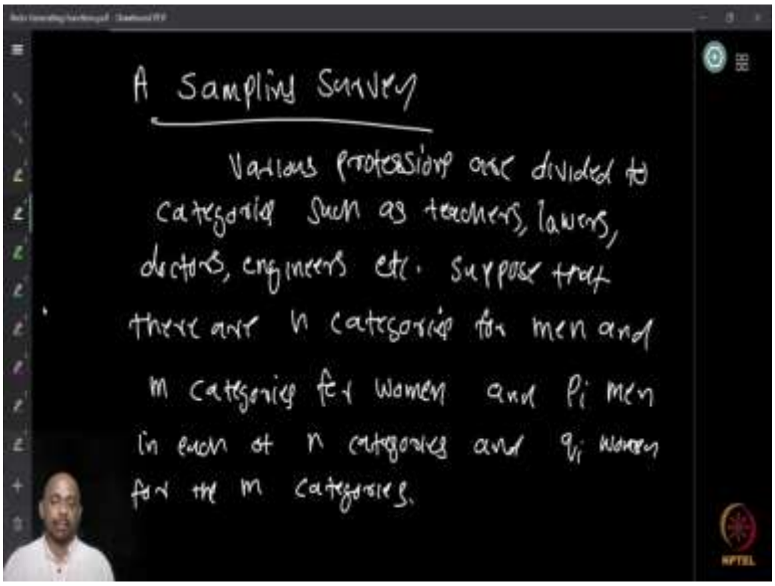

Here is another example. So here you have an example of a sampling survey. So what is a sampling survey? So there are various professions of people. We have divided the professions to different categories such as, like, we have teachers, we have lawyers, we have doctors, we have engineers, so, farmers, all these kind of professions are listed.

And then from each of these n categories of people, we have for men as well as for, m categories for women. So out of this different professions, let us say that we fix n of them, some n of them for men and some m of them for women.

Then there are p_i men available in each of the n categories. So category i, there is p_i people. And q_i ladies are available in each of the m categories. So category 1, I have q_1 people, category m I have q_m ladies. And from this we need to choose people for survey.

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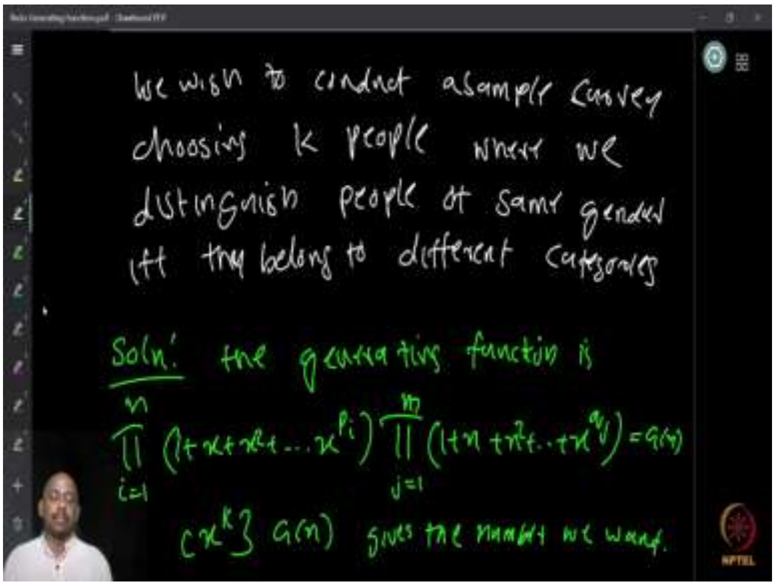

So we wish to conduct a survey by choosing totally k people, where we distinguish people of the same gender, men and women are to be the same gender, if and only if they belong to different categories. I do not really want to distinguish in the same category, I do not want to distinguish between men and ladies. So how many different ways to do this?

So if you have really understood the generating function topic that we have covered so far, you can immediately see the generating function for this, you can come up with very clear, very easily, because from the previous example you get an idea about this also.

So if I look at the function $1 + x + x^2 + \cdots + x^{p_i}$, that tells there is, for the ith category I have precisely p_i persons, and then I can choose some of this like, I can choose either one of them, none of them, two of them or p_i of them. At most p_i are there, so that is the choices from the first category.

Similarly, second category I have $1 + x + x^2 + \cdots + x^{p_2}$. So similarly, $1 + x + x^2 + \cdots + x^{p_n}$. So for the n categories, I have these choices, and each one, from the different categories I can choose them to be in any way, and the product will tell you the number of ways of doing this.

So, $\prod_{i=1}^{n} 1 + x + x^2 + \cdots + x^{p_i}$ tells the choices for the men from the category. And then $\prod_{j=1}^{m} 1 + x + x^2 + \cdots + x^{q_j}$ j gives the choices for the ladies from each of the m categories.

And the generating function $\prod_{i=1}^{n}(1+x+x^2+\cdots+x^{p_i})\ \prod_{j=1}^{m}(1+x+x^2+\cdots+x^{q_j}\ \ )=G(x)$.

So the coefficient of x^k from this generating function will tell you the number of ways we want, and for, what you can do is to take some numbers, fix the numbers m and n, and then p_i's accordingly, and then take the product and find out the coefficient of x^k for some k and then see whether it actually matches with your manual computation. So one can try this.

So with that, we will now, we will change to a different topic, now. In the next class onwards we will look at a slightly different type of generating function. So at the moment we will stop and continue in the next class.