Combinatorics
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Application of Ordinary generating functions

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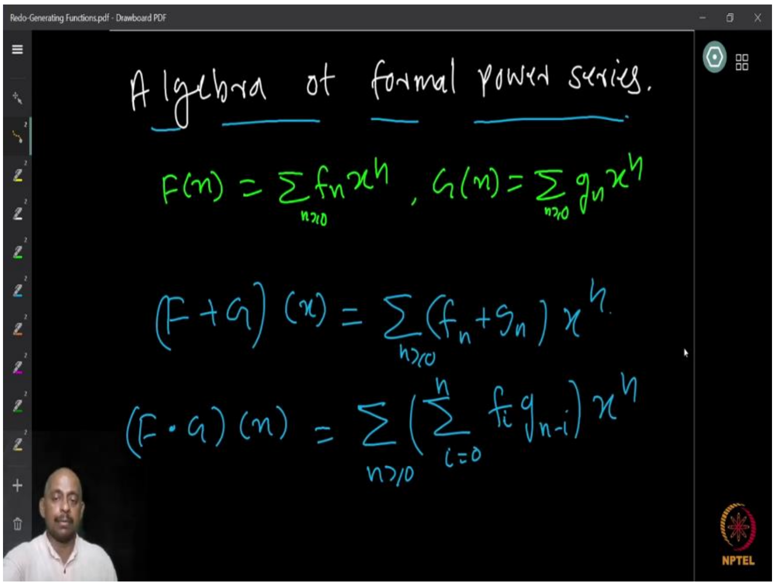

So, the associated prominent structure, because when we talk about this series, the series corresponding to the counting sequences of objects, combinatorial objects, then we have to also give some meaning to what happened to the sum what happened to the product etcetera. So, we will do that in soon enough, but for the time being, so, we have this as a sum and then similarly, we define the product.

So, the product of two generating functions F and G is

$$(F.G)(x) = \sum_{n\geq 0} (\sum_{i=0}^{n} f_i g_{n-i}) x^n$$

So, the coefficient of x^n is $\sum_{i=0}^{n} f_i g_{n-i}$. Now, if you think about it, it will be kind of clear, because if you take the product of this infinite series, one can see why this should be the coefficient. So, let us see why.

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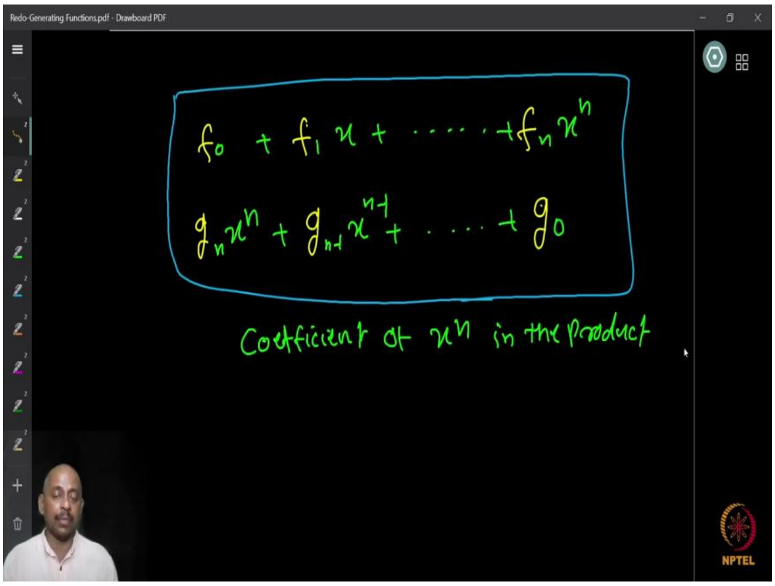

Let us say that we are multiplying these two series, but now, since we are multiplying series which are like kind of extensions of polynomials, we have only non-negative powers. Exponents are non-negative. If I have a non-negative exponent, When I take the product, the exponent can only increase it cannot decrease.

So, therefore if I am going to look at the coefficient of x^n for fixed at n. I never have to look at terms whose exponent is going to be larger than n because, it is never going to contribute to the coefficient of x^n in the product. So, what we can do is to just look at what are the terms whose degrees less than or equal to n. So, we have the first $n+1$ terms which is $f_0 + f_1 x, + \cdots + f_n x^n$ these are the terms where the degree is less than or equal to n.

And then similarly, the second series again the degree is less than or equal to n. So, I have $g_n x^n, g_{n-1} x^n, \ldots, g_0$ x and these are the n terms. Now, when we take the product, of course, we are going to multiply each term with each of the other term in the second sum, second series. So, we are going to look at in which cases, we will get x^n from this product.

So, we can clearly see that if I take x^k from one of these terms. Let us say I take $f_k x^k$. Something like $f_k x^k$ I take then since I have $x^k$to make it x^n, I have to take x^{n-k} and the only term that I can multiply it with is in the second series, which is $g_{n-k} x^{n-k}$.

So, now, we can see why the previous one makes sense. So, if I have f_0 then I have to take g_n because I need x^n, if I take f_1, I have to take the g_{n-1} because I have to get $x. x^{n-1}$ which will be x^n. Similarly, I have to take $f_n x^n$ and g_0 because the degree must be equal to x^n.

So, therefore, I have this $n+1$ terms in the sum which are going to be $\sum_{i=0}^{n} f_i g_{n-i}$. So, these are the n plus 1 terms and their sum is going to give me the coefficient of x^n in the products and that is what we have written. So, that is our product of formal power series. Of course, we can extend this to more number, you can just take the thing and do it easily.

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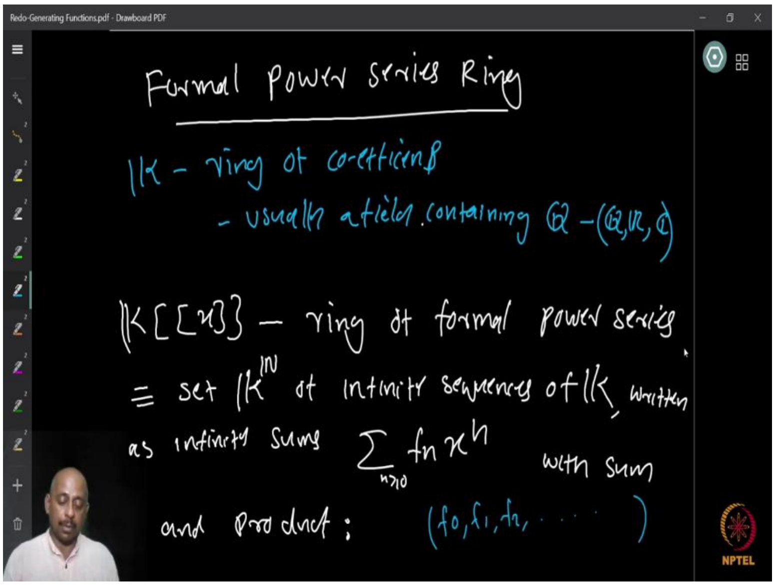

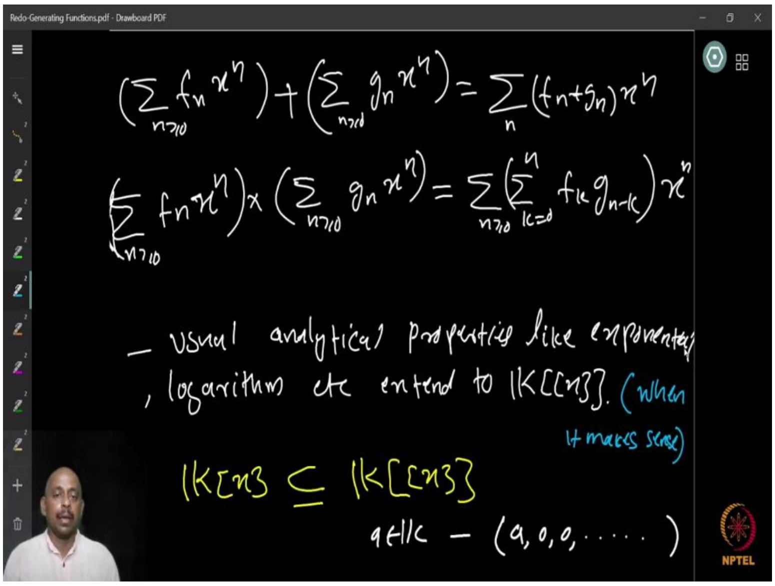

Now, as we mentioned the formal power series has a ring structure. So, the k from which we take the coefficient can be a ring of coefficients. Now the ring of coefficients we usually consider it to be a field because it will give better nice property for us. And for most of the purposes we will assume that it contains $\mathbb{Q}$.

So, we will take only $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. When it does not contain $\mathbb{Q}$, we have to assume some other or we are looking at other type of fields, we have to be more careful that is it, we will not go into details. Now, we denote by $\mathbb{K}[[x]]$, the ring of formal power series because $\mathbb{K}[x]$ we use to denote the polynomial ring over $\mathbb{K}$.

So, $\mathbb{K}$ is a field, $\mathbb{K}[x]$is usually used to denote the one variable polynomial ring over the field $\mathbb{K}$. Now, what is this ring of formal power series? So, as we know the power series is in one-to-one correspondence with an infinite sequence. So, basically a power series is a sequence of elements from $\mathbb{K}$ itself. So, therefore, what we are going to look at is the infinite sequence $\mathbb{K}^{\mathbb{N}}$, which is the functions from $\mathbb{N}$ to $\mathbb{K}$.

So, $\mathbb{K}^{\mathbb{N}}$ is the set of all functions from $\mathbb{N}$ to $\mathbb{K}$ and this is precisely what we are talking about when we say $\mathbb{K}$ double bracket x, $\mathbb{K}[[x]]$. Now, to represent this what we are going to do is to this take the infinite sequence we are going to write it as $\sum_{n\geq 0} f_n x^n$. This is only a way to represent what we really want to work with this.

So, this is what we are going to really work with. This is basically the function from $\mathbb{N}$ to $\mathbb{K}$ and likewise, given any such infinite sequence, we associate the series and then this series will give you the variable x and therefore, it will give you the ring of formal power series. So, what we are working with is the infinite sequences of $\mathbb{K}$.

Now, so as we mentioned before the sum and product of this infinity sequences or the ring elements is as before $\sum_{n\geq 0} f_n x^n + \sum_{n\geq 0} g_n x^n = \sum_{n\geq 0} (f_n + g_n) x^n$

and product of these two is $(\sum_{n\geq 0} f_n x^n) \times (\sum_{n\geq 0} g_n x^n) = \sum_{n\geq 0} (\sum_{k=0}^{n} f_k g_{n-k}) x^n$.

Now, one thing to note is that the usual analytical properties like exponentiation, logarithm all these can be extended to the formal power series ring also, whenever it makes sense. We will not go into the details of this at the moment for advanced courses we can think about this.

Then another point I want to note is that, suppose you look at the field element. So, some element in $\mathbb{K}$, a is an element in the field $\mathbb{K}$. Now, suppose I represent this field element as again an infinite sequence. So, for example, I can represent this as (a,0,0,…), the first element is a and everything else is zero. So, then I will get the constant because first term is the x raised to zero which is constant and then all the other powers of x has coefficient 0.

Therefore, I get the constant a. So, if I represent 'a' as this infinite sequence, where every other term other than the first term is zero, then I can see that, the polynomial ring $\mathbb{K}[x]$ is contained in the ring of formal power series $\mathbb{K}[[x]]$, because we can just associate now the polynomial ring. So, a polynomial ring, we have a finite terms. So, this finite set what we do after the remaining we can put zeros to make an infinite sequence.

And then these vectors can be infinite vectors now, and therefore, we can see that they are sitting inside the larger formal power series ring and if you just look at the addition and scalar multiplication. So, I can define now, the scalar multiplication as multiplication by the field element where the field element is represented by this sequence (a,0, 0,…). So, I have the corresponding series, summation over all things a plus everything else is 0.

So, then I can define the product of these two, power series and this is the product by the field element and this will be the scalar multiplication. But now, I have the scalar multiplication and the sum one can show that there is a vector space structure over $\mathbb{K}$ for the power series ring and then once you extend it with the product, it will loose some nice properties and it will become just adding. So, well we will not discuss any of this further. So, let us try to see how to use these things in combinatorics.

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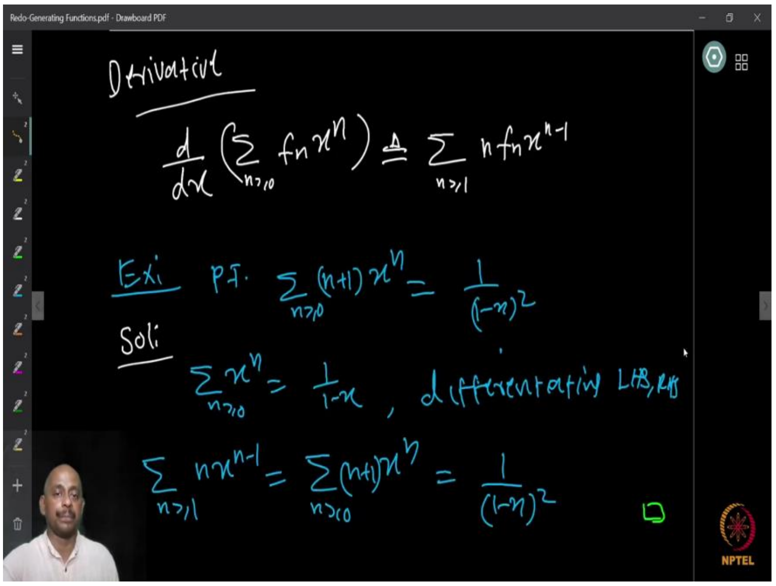

So, a final word that we can define other operations like derivative and all, so, how do we define the derivative. So, the formal derivative of the formal power series. Again keep in mind that we are looking at formal derivative not the actual derivative, we will not worry about whether it is continuous or like whether we can differentiate or differentiable etcetera when we try to look at this kind of thing.

So, therefore, we will define the derivative the usual way without worrying about the analytical property. So, $\frac{d}{dx}(\sum_{n\geq 0} f_n x^n) = \sum_{n\geq 1} nf_n x^{n-1}$, which is the usual term by term derivatives of the power series. So, this we define here also but we say it is a formal derivative.

So as an example, to the things that we learned so far, I am going to prove the following result:

$$\sum_{n\geq 0}(n+1)x^n = \frac{1}{(1-x)^2}$$

Now, I am going to give three different proofs for this. So, we will use three different methods to do this. First, let us use the thing that we just learned immediately before, that is the derivative. So, I know that the derivative $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$ is whole square.

So, I know that $\sum_{n\geq 0} x^n = \frac{1}{1-x}$. Now, let us take derivative on the left as well as on the right-hand side. We get, $\sum_{n\geq 1} nx^{n-1} = \sum_{n\geq 0}(n+1)x^n = \frac{1}{(1-x)^2}$

So this is the first proof. Now, let us look at another way to prove this.

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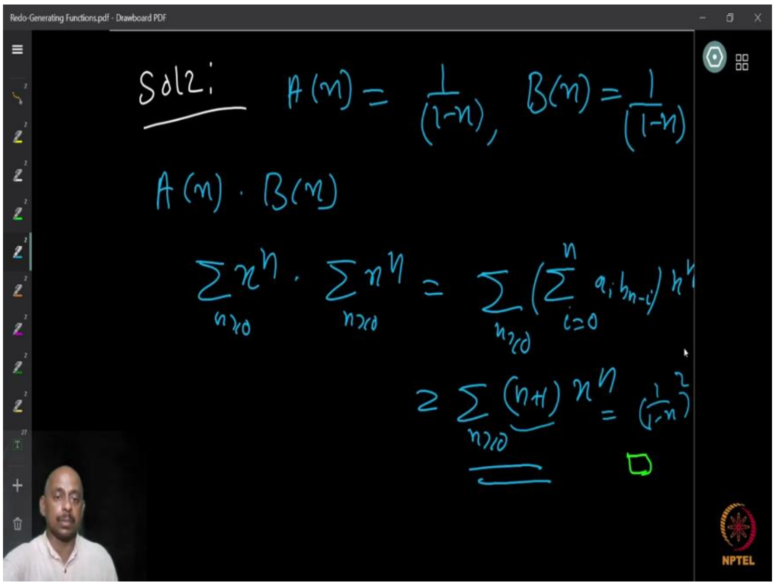

So, we have defined the product of the analytic function before. Let us try to use that. So, I know that $\frac{1}{(1-x)^2} = \frac{1}{1-x} . \frac{1}{1-x}$.

We know one by 1 minus x is the series summation x raised to n ($\frac{1}{1-x} = \sum_{n \geq 0} x^n$) with the generative function let us say A(x). And similarly, I say B(x) is the same series $\sum_{n \geq 0} x^n$.

So, the product of these two series, is by definition,

$$A(x).B(x) = \sum_{n \geq 0} x^n \ . \sum_{n \geq 0} x^n = \sum_{n \geq 0} (\sum_{i=0}^{n} a_i b_{n-i})\, x^n = \sum_{n \geq 0} (n+1) x^n$$

So, what is the coefficient of x^n here? Well, a_i is coming from the constant sequence x^n, when the coefficients are all one. So, therefore, a_i is one. And similarly, b_{n-i} is also one. So therefore, $a_i b_{n-i}$ is one into one which is one.

So, I am going to add i equal to 0 to n everything is one. So, how many terms are here? There are exactly n + 1 terms. So, n + 1 ones if I add I will get n + 1. So, the coefficient of x^n is n + 1 so, I get $\sum_{n \geq 0} (n+1) x^n$. This is the product and that product is precisely one by $\frac{1}{(1-x)^2}$ because it is A(x) B(x).

Now, I am going to give you the third proof and this time we want to use the generalized binomial function. So, $\frac{1}{(1-x)^2} = (1-x)^{-2} = \sum_{n \geq 0} \binom{-2}{n} (-x)^n$

$= \sum_{n\geq 0}(-1)^n(n+1)(-x)^n$, Since ($\binom{-2}{n} = \frac{-2\times -3\times \ldots \times -(n+1)}{n!} = \frac{(-1)^n(n+1)!}{n!}$

$= \sum_{n\geq 0}(n+1)x^n$

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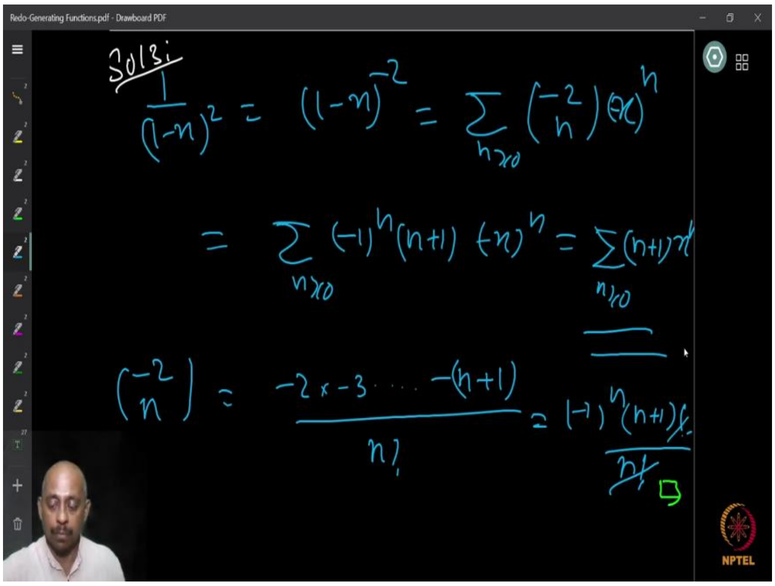

So, these are three different proofs of the same identity which we proved using the techniques that we have learned.

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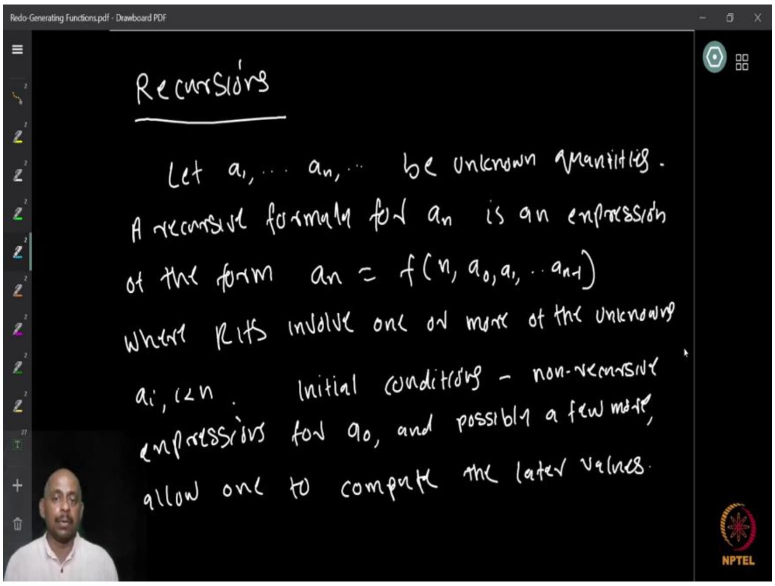

Now, we want to look at some applications of these formal power series that we have learned. And one of the places where we can use is to use this to solve recursion relations or recursive equations. So, what is a recursive relation? So let us say we have some unknown quantities say a_1, a_2 etcetera it is a sequence of unknown numbers.

Now, when we say a_n is given as a recursive formula, it is an expression of the following form that a_n is written as a function of the previous values of a_i which is $a_0, a_1, \ldots, a_{n-1}$ and possibly also of n. So, a_n can be written as a function of n and the values that has appeared before and you may not have to use all of them. Maybe some of them is fine. I can write a_n as a function of n-1, a_{n-1}, or I can just write as a function of n and a_{n-1} and a_{n-2}.

So, these are posibble, so if I can write a_n as a function of these previous values, then we say it is a recursive formula for a_n. We usually will also be given some initial condition, because when we have a_n is written as, in terms of the previous terms, we need to see what are these unknowns in the at least the beginning few terms.

Like for example, what is a_0, if I do not know a_0 or a_1 whatever, then when I go back, I will have a trouble because once I reach a_0, a_n in terms of let us say a_0 and something. If I do not know what is a_0, I cannot do anything. So, therefore, some initial conditions are basically some non-recursive expressions for a_0 etcetera. So, a_0 and whatever is required, some few terms in the beginning will be given to you.

And this completes the recursion formula, and then we can use the recursion to compute the later ones. So, that is the idea of recursion. Now, the problem with recursive formula is that, if I want to find out the 10,000 step, or like 2549th term, then I have to compute all the previous value up to 2548. So, all the previous terms I might have to compute before I can find out what is this a_n.

Now, this may be an unnecessarily work, because often it is the case that we only need to know what is the value of let us say, a_{10000} after 10,000 steps, what has happened, that is what we want to know. But why are we finding out all the previous 9999 values, we do not need. So, to do away with this kind of work, one can try to use to put this recursion formula and then try to come up with a nice function to represent a_n.

And for that, these methods that we are going to learn from the power series, formal power series will be very useful. So, let us see some examples with some definition.

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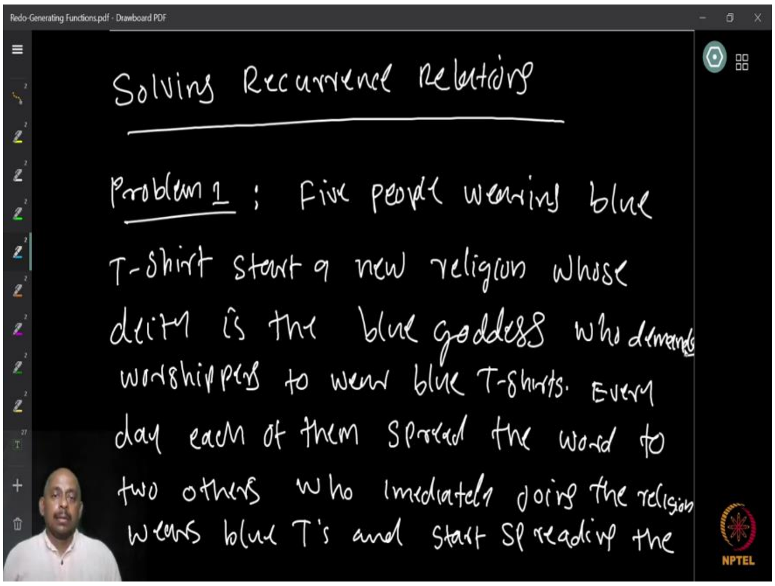

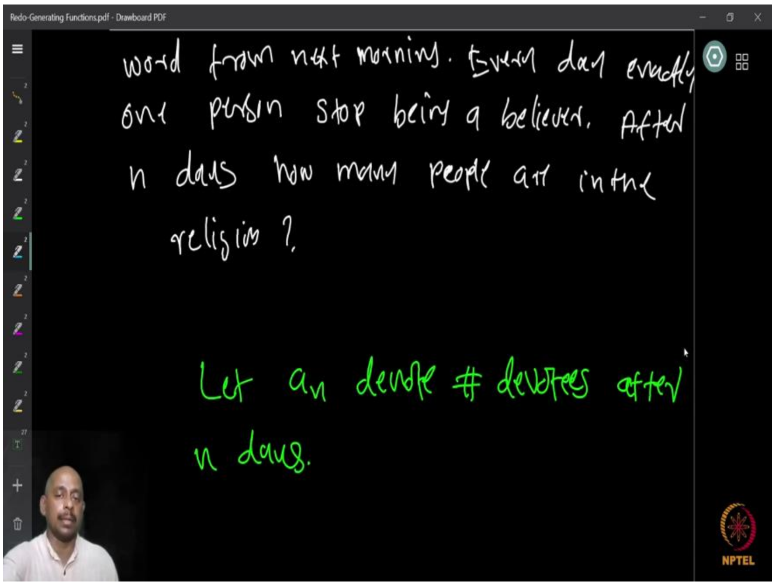

So, we want to solve recurrence relation. So, the first problem we want to look at is the following. So, let us say that there are five people start a new religion. So, what they do is that they start wearing blue T-shirts all the time. And then they said, okay, we believe in the blue goddess. Now, what does this goddess say that, so she says that all the worshipers must wear blue T-shirts, and if they do not do that, they will not get heaven, they will not get entry into heaven.

Now, every day, what the devotees do is that, they spend time to spread the word, they will try to convert other people to their religion. So, they will convince others to join them, so let us assume that they convinced two others every day, and once they are convinced, they immediately join the religion and start wearing the blue T-shirts and the next morning onwards they start spreading the religion again.

But we will also assume that every day one person will stop believing. So, after n days, we want to know how many people are there in the religion. So, we want to know only what happens after let us say n is equal to 100 or n is equal to 70. Now, after these many days, how many people are there in the religion?

Now, can you form a recursive formula for this from the given information. So, we have given all the information required. Now, a nice exercise will be to form yourself a recursive formula. So, we can start assuming that a_n denotes the number of devotees after n days, so then what is the recursive formula for a_n? So, you think about this for some time and then continue.

So, what is given to us is that five people start wearing T-shirt. So therefore, initially, there are five people. So therefore, $a_0 = 5$. So, this is very clear. Now, since there are exactly five people in the first day, they start spreading the religion to 2 others. So, by the evening, they must have converted, each of them converted 2, so therefore 10 more people. So, 10 plus 5, which is 15 people have been converted after the end of the day.

Now, we know that every day 1 person will stop believing, so therefore, 1 person will lose his belief by the end of the day again. So, how many people will be there in the next day morning, next morning there will be exactly 14 people. So, this you can see, and then it tells us something else. So, we can now form the recursion relation as follows, we know that if there are exactly, let us say a_n people or a_{n-1} people at the morning.

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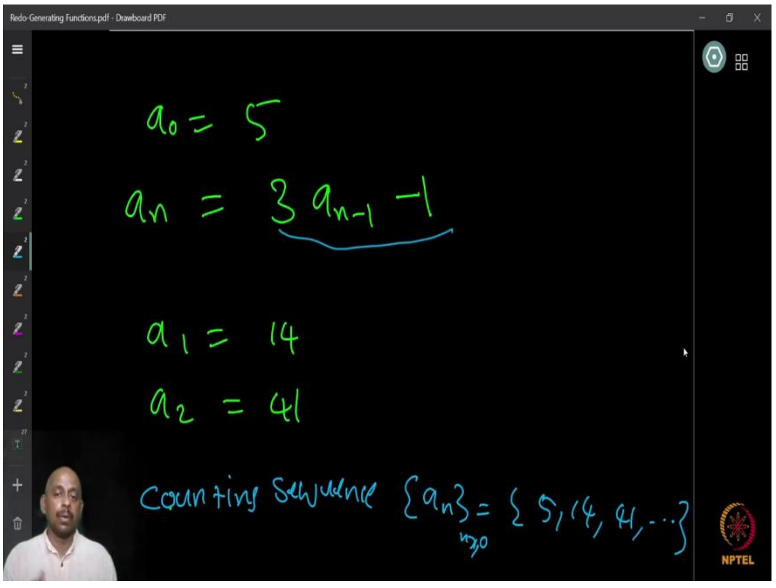

Then by the end of the day, how many people will be there, each of the a_n person will convince twice that many people, so I will get two times a_n more believers. So, this will make it $3a_{n-1} - 1$. So, end of the (n-1)th day, I have $3a_{n-1} - 1$ persons.

Now, the next day morning that is which $is\ a_n = 3a_{n-1} - 1$. So therefore, I have a recursion relation for a_n and the initial condition is that $a_0 = 5$. So, that is where we start. And as we computed $a_1 = 14$. Now, we can clearly see what is a2 because it is (3 × 14) – 1 = 41.

And then therefore, we have the counting sequence, associated counting sequence the number of devotees after n days is $\{a_n\} = \{\ 5, 14, 41, \dots\ \}$. So, we have the counting sequence. So, in this case, we can associate a power series. So, I can say now, $5\ \ +14x\ \ + 41x^2 + \dots$ is my corresponding power series.

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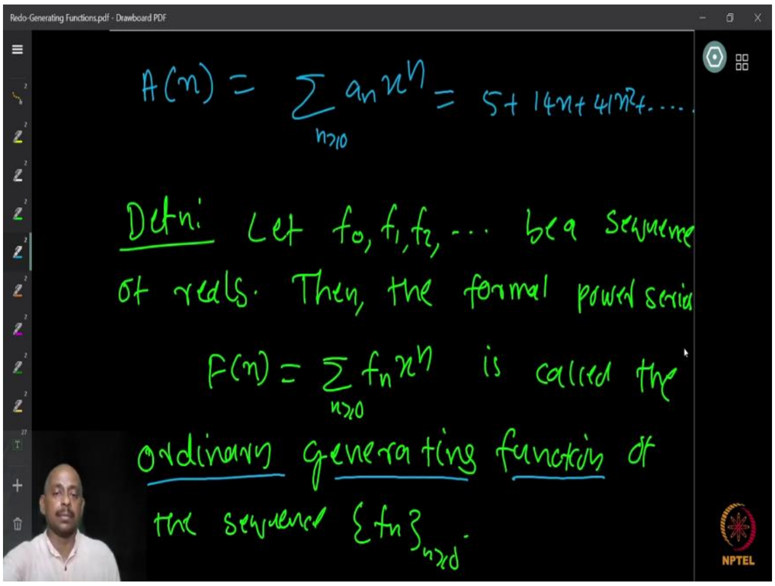

So, I have done that, I have $A(x) = \sum_{n\geq 0} a_n x^n = 5 + 14x + 41x^2 + \cdots$.Now, I want to call this with a new name. So, I am going to define that.

Suppose, we are given a sequence of real numbers f_0, f_1, f_2, etcetera, then the formal power series $F(x) = \sum_{n\geq 0} f_n x^n$ is called the ordinary generating function of the sequence $\{f_n\}$, where $n \geq 0$

Now, why this is called ordinary, it will be clear when we see that there are other types of generating functions. But for the time being, we will just say generating functions for this and most of the time, it will be clear, we are looking at ordinary generating functions. So, often we can also write OGF, as a initials of O, G, and F to denote ordinary generating function.

And so, when we say a generating function, in this part of the lecture, what we mean is the ordinary generating function. Now, so, ordinary generating function is just the power series and if you can write it as a function all the better. So, what our idea is to try to write this power series as a function and then if we can succeed, we can use that to do other operations with that.

So, let us see whether we can write our series $5 + 14x + 41x^2 + \cdots$as a nice function. And then from that, can we find a formula for a_n. For that, we want to try to use the method of generating functions.

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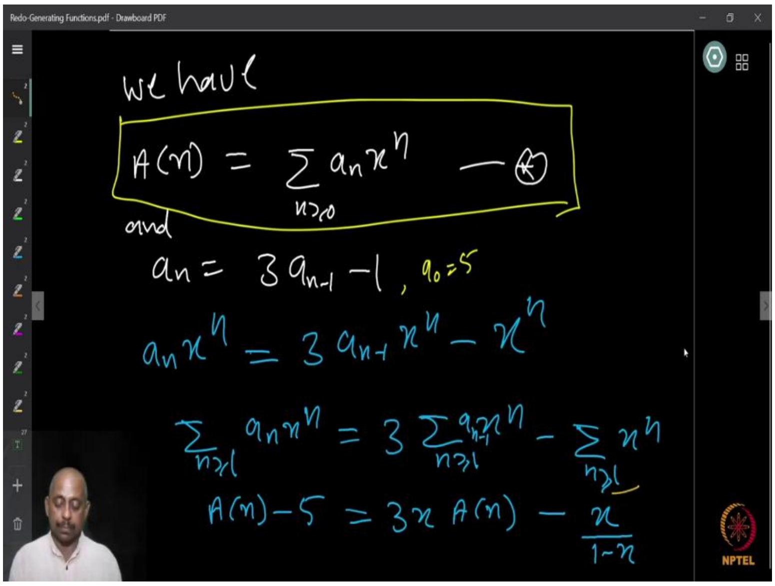

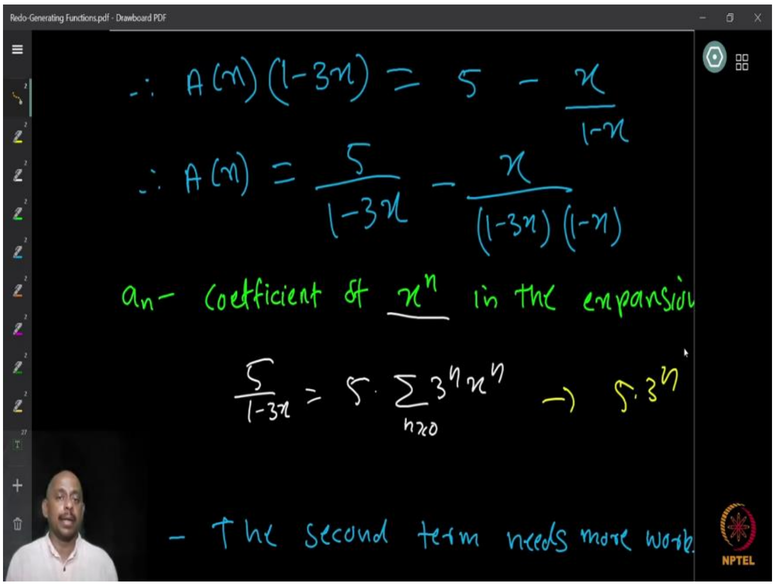

So, let us do this. So, we have $A(x) = \sum_{n \geq 0} a_n x^n$, and we have the recursion relation is $a_n = 3a_{n-1} - 1$. We also have the initial condition $a_0 = 5$. Now, I take this recursion relation, multiply with x^n on both sides of the equation and then add them for every n we get

$a_n x^n = 3a_{n-1}x^n - x^n$

Now, what does this give us? So, this gives us, when you sum over all n, I get summation n greater than or equal to 1, because I am looking at a_{n-1} here. So, because we start from a_0, I need to make sure that summation takes care of this index also.

So, therefore, I sum from n greater than or equal to 1

$$\sum_{n\geq 1} a_n x^n = 3\sum_{n\geq 1} a_{n-1}x^n - \sum_{n\geq 1} x^n$$

Now, what is on the LHS, we have summation $\sum_{n\geq 1} a_n x^n$ except the first term, which is n equals 0. So, the first term n equals 0 is $a_0 = 5$, that we know. So, therefore, using that we can write this as $A(x) - 5$. So, that is our LHS.

$$A(x) - 5 = 3xA(x) - \frac{x}{1-x}$$

So, now I have a nice equation with A(x) and x. So, I can now write A(x) in terms of x. So, how do I do that, I take all the terms with A(x) to one side and then I get $A(x)(1-3x) = 5 - \frac{x}{1-x}$. Dividing by (1-3x) we get

$$A(x) = \frac{5}{1-3x} - \frac{x}{(1-3x)(1-x)}$$

Now, what we want to find is a_n, this is what we started with. Now, we know that a_n is the coefficient of x^n in the expansion of A(x). Whatever is A(x), we expand it as a series look at the coefficient of x^n, that is going to be a_n. Because we defined A(x) as $\sum a_n x^n$. Now, we know $A(x) = \frac{5}{1-3x} - \frac{x}{(1-3x)(1-x)}$.

Now, if I know the coefficient of x^n in this each part in the RHS, then their difference is going to be the coefficient of by the rule of addition. So, from the first term, we can immediately get the coefficient of x raised to n.

So, because we know that $\frac{1}{1-3x} = \sum 3^n x^n$. So, I get $\frac{5}{1-3x} = 5\sum 3^n x^n$. So the coefficient of x^n from that is 5.3^n. But now, that is only from the first part. Now, from the second part we need to find, but the second part is not as a simple form, it is a product of two, such terms.

But now, how do you do this, we do not know how to work with this. So, therefore, we need to do some more work to simplify this. So, the second term, it needs a little more work. So, from the first time we get the coefficient 5 into 3 raised to n, we will use it later. So, now, let us work with the second term. So, there are several ways to work with the second term and what we are going to look at now is called the method of partial fractions.

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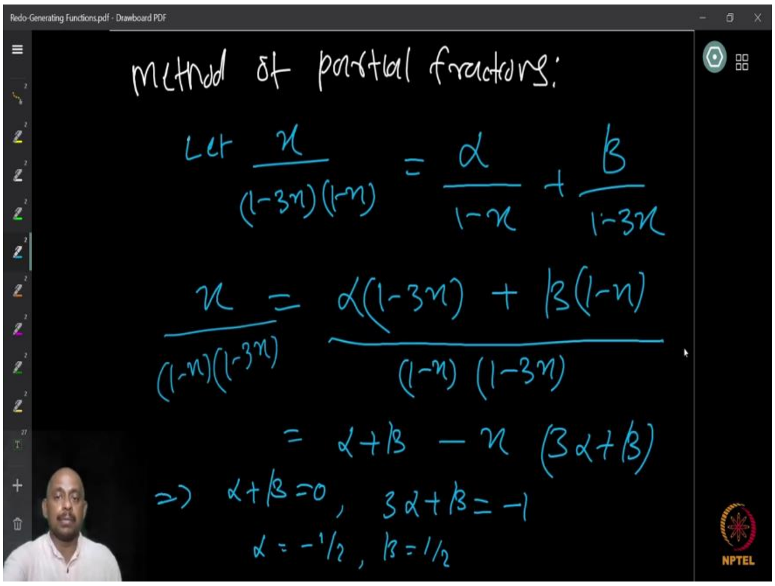

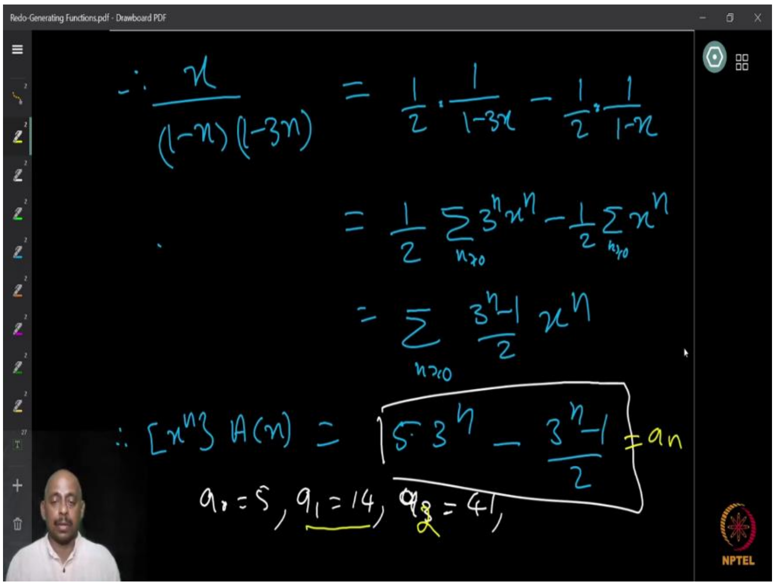

So, we write, $\frac{x}{(1-3x)(1-x)} = \frac{\alpha}{1-x} + \frac{\beta}{1-3x}$

Now, I want to solve for alpha and beta, what I do is just multiply both sides by $(1-3x)(1-x)$. So, I get

$$x = \alpha(1-3x) + \beta(1-x)$$

But now, it is a formal polynomial identity and therefore, the coefficients of the corresponding terms, the like terms of the same degree terms must be the same, but what is the coefficient of the constant term which is a constant term is 0 on the left side. So, therefore, $\alpha + \beta = 0$. Similarly, the coefficient of x is 1. So, therefore, $3\alpha + \beta = -1$.

And once we have this we can immediately solve for alpha and beta, we will get alpha equal to -1/2 and beta is equal to 1/2, by just taking these two linear equations. Now, once I have this, I can write

$$\frac{x}{(1-3x)(1-x)} = \frac{-1/2}{1-x} + \frac{1/2}{1-3x}$$

$$= \frac{1}{2}\sum_{n\geq 0} 3^n x^n - \frac{1}{2}\sum_{n\geq 0} x^n$$

$$= \sum_{n\geq 0} \frac{3^n - 1}{2} x^n$$

So, the coefficient of x raised to n in A(x) is the sum of the first term and second term, which is $5.3^n - \frac{3^n-1}{2}$.

Now, what is this? This is precisely the term a_n, the n'th term which is the number of devotees after n days. So, $a_n = 5.3^n - \frac{3^n-1}{2}$. Now, let us verify whether what we calculated, the formula for a_n is true. So, let us say that n is equal to 0, I will get 5 into 1 minus 1 minus 1 by 2 which is 5. That is $a_0 = 5$

And what is a_1, a_1 is 14, we know, but let us verify, when n is equal to 1, I get 5 into 3 which is 15 minus 3 raised to n which is 3 minus 1 which is 2 by 2 which is 1, so 15 minus 1, which is 14. Then what is a_2? So, n is equal to 2, I will get 5 into 3 square which is 45 minus 3 square which is 9 minus 1, 8 by 2 which is 4, so 45 minus 4, which is 41.

So, the first 3 terms are same, so hopefully all the terms will be the same. So, now we can see why this is such a powerful method because now if you want to find out n is equal to 100, I just need to put 3 raised to 100. And then I have the formula 5 into 3 raised to 100 minus 3 raised to 100 minus 1 by 2. On the other hand, if I wanted to find out using the recursion, I need to calculate this for each of the 99 steps before.

So, this is extremely useful when we want to find out the only the values and solve the recurrence relations. We get a nice close formula for the n’th term.