Combinatorics

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Formal Power Series, Ordinary generating functions

Hello and welcome back. In the next few lectures we are going to discuss what is called the method of generating functions in combinatorics. This is a very powerful method and it is especially useful when we are dealing with the following kind of situations. So as we have noticed we have several situations where we have a parameter like it is probably a natural number and associated to this natural number we have a number which comes as the number of objects of certain combinatorial structure. For example, you can talk about the number of permutations of an n element set which we know is $n!$ or you can talk about the number of graphs on an n element vertex set or you can talk about binary words of specific length, let us say n, binary strings of length n or let us say lattice paths of n steps or let us say triangulations of a polygon or we can talk about the number of polygons with n sides. And we can talk about the number of ways of bracketing a product of n matrices. So all these examples we have a natural number n and associated to this number we have the number of objects of a particular type.

So now this particular situation gives rise to defining what we can call sequences or counting sequences. Now generating function gives a nice way to encode the entire sequence into a function form, it is a nice form which can be immediately decoded or like decoded to get a specific value for a fixed parameter that we want or a parameter that we want. So, let us first look at what are counting sequences.

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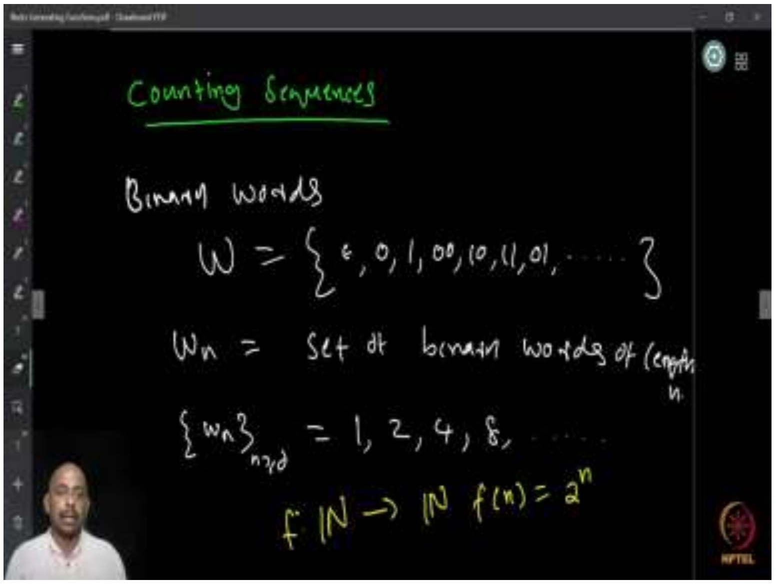

So what is a sequence? So a sequence is nothing but a function from the set of natural numbers to real numbers or natural numbers itself. I mean it could be anything like that. So basically a sequence is something which we can put it into order according to the natural numbers, say for every natural number including 0, 0, 1, 2 etcetera we can associate this number and then that defines a sequence.

For example, here is a sequence 1, 2, 4, 8 etcetera, so how this sequence came about is that like I am looking at the number of binary words of length n. So let us say that W is the set of such binary all over binary words, then I can arrange W as follows, like we put the number of strings with 0 letters which is the empty string.

Then we have 0 and 1 which are the 1 letter strings then you have 0 0, 1 0, 1 1 and 0 1 which are the 2 letter strings. This way we put this set and then we can talk about W_n which is the set of binary words of length n. So W_n. How many numbers are there? We know it is actually 2^n strings, using 0 and 1.

And therefore, we have the sequence W_n which is sequence 2^n. So basically it is a function from the natural number 0 1 2 3 et cetera to 0 going to 1, 1 going to 2, 2 going to 4, 3 going to 8, etcetera. So basically n going to 2^n. So this is basically a counting sequence because it counts the number of binary words of that length. So that is one nice example of a counting sequence.

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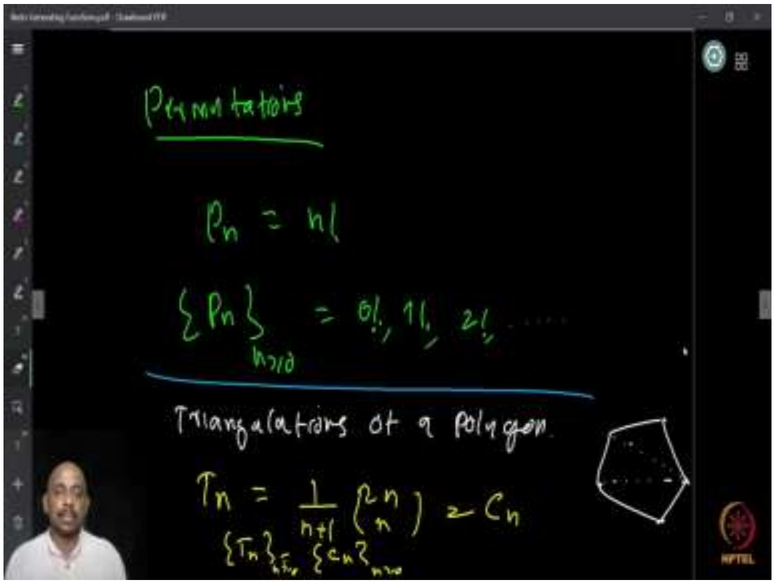

Now we can look at other examples as we said like we can look at the number of permutations we know it is n factorial for an n element set. So therefore I can talk about the sequence P_n which is basically $0!, 1!, 2!, \ldots$ and that is the counting sequence for the permutations.

Similarly, we have T_n the sequence T_n counts the triangulations of a polygon with n sides and we know that it is basically the Catalan number, we discussed this before. And therefore we have this sequence T_n to be sequence C_n where C_n is $\frac{1}{n+1}\binom{2n}{n}$. So these are examples of counting sequences.

Now, once you have a counting sequence our idea is to put it into a nice function form and then try to use this function to do several manipulations and whenever we want we can retrieve this information immediately that is the idea.

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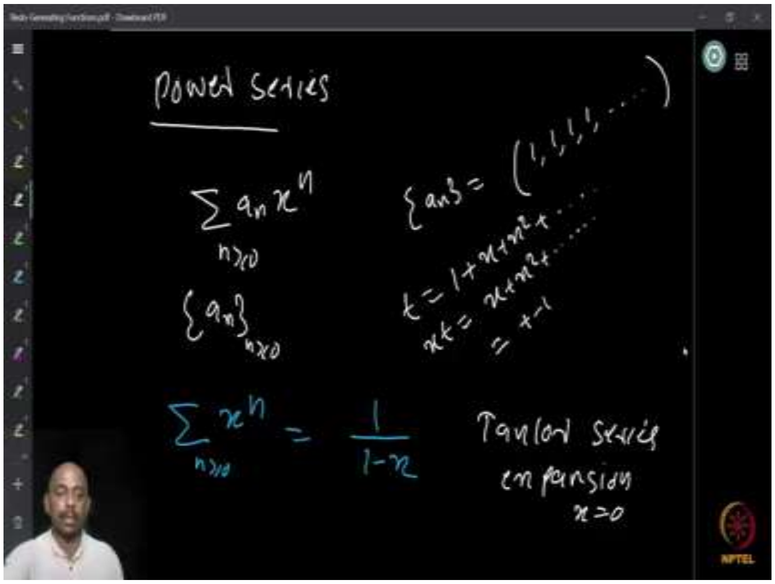

So, to do this what we are going to do is to define what is called formal power series. So before going to formal power series let us see what is a power series? So given any sequence, sequence of real numbers let us say sequence $\{a_n\}_{n\geq 0}$ we can define we can associate a power series by multiplying a_n with x^n and summing overall n.

So this does 2 things. One is that like you can now write it as a sum and when you do this sum we can still recover a n as the coefficient of x^n because when you have powers of x which is a variable or indeterminate we cannot add them together. So we can just write it as a sum, but if I have x and x^2 I cannot add them. I can write $x + x^2$.

So this helps to get back a_n by just looking at what is the coefficient of x^n . Now in some cases like in power series we know that if the power series converges, it usually converges to a function and if you have a nice function then we can use that function as a placeholder for the power series.

And then whenever we want we can retrieve back the coefficients by using the Taylor series expansion of the function and evaluating at x equal to 0 and then you can you can find out this series and its coefficients. So this is something it is standard we know can be done. So in the following example of a power series $\sum_{n\geq 0} x^n$ where you have the constant sequence 1 1 1 et cetera.

So, when a_n is sequence 1 1 1 etcetera. So now, if all the constants are 1 then we have $\sum_{n\geq 0} x^n$ and we know that this series converges to $\frac{1}{1-x}$ whenever |x|<1. So how do you do this?

How do you prove this? And this is something that we have studied in calculus? So, suppose I have $\sum_{n\geq 0} x^n$, so I have, so let us say that $t = 1 + x + x^2 + \cdots$. Then I know that $xt = x + x^2 + \cdots$, but this is the previous sequence minus 1, the first term is missing.

So, $tx = t - 1$. So from this we can find out what is t so $t = \frac{1}{1-x}$ so this we can we can find easily. So we solve for this and we get the nice function and if you take the Taylor series expansion you can recover the series.

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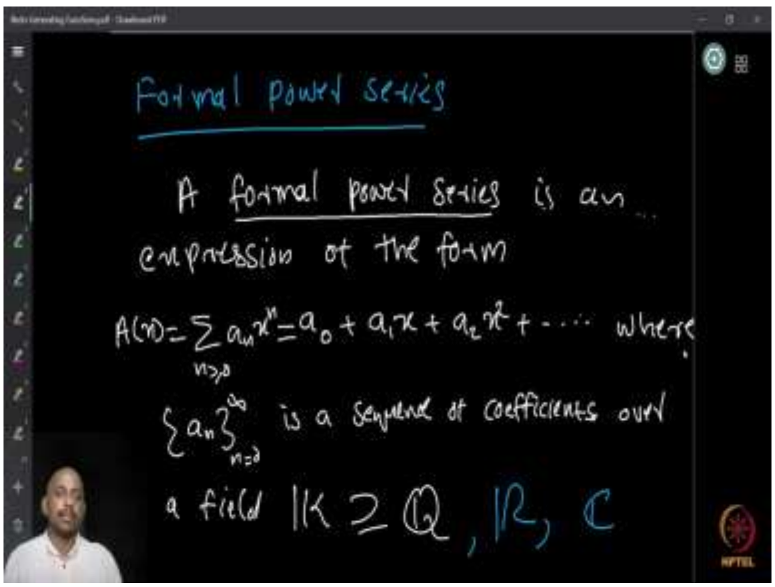

We are going to work with what is called formal power series so when we say formal it is formal in the sense that when we add or like do some operations with this series we have to keep in mind that we are not really adding we are only writing it as a formal sum and it also emphasizes the fact that, the series that we look at may not always converge and we do not really worry about the convergence because what we want is a way to put all the information about the sequence in a nice form. And we are not going to evaluate it at a point for x just to, so therefore the convergence really does not matter and we know that when x equal to 0 it is going to converge but it does not converge in an interval, still we can use this series.

But when we work with power series and if it does not converge, people might object that this is not really, we cannot add these 2 things because they are both diverging. But then we say that it does not matter because we are not really looking at the sum, we are only looking at the coefficients and whenever we are going to look at we are only going to look at finite coefficients.

And therefore we are only worried about that and it does not matter whether the entire series converges. So, therefore we define what is called formal power series where we do not really worry about the convergence issues. So the analytic properties need not always hold, but whenever it holds we can use it to get some nice form so we will write it as formal power series most of the time.

But when we can really use the analytic properties, sometimes we can use the convergence, but that we do not discuss it in this course only when we look at more advanced topics we might need to look at evaluation of the generative functions or power series at some point. So without further ado let us see what is a formal power series.

So formal power series is an expression of the form $\sum a_n x^n$ where sequence a_n is a sequence of coefficients of elements from typically a field. So, we are going to look at with the assumption that we are going to draw the coefficients from a field. And let us assume field K and most of the time we will assume that the field of coefficients contains the set of the field of rational numbers.

For example, we look at $\mathbb{Q}, \mathbb{R}\ or\ \mathbb{C}$, the rational field to the real field or the complex field. But it is not necessary it is always a field. We can also work with the cases when K is a ring also. Now we will not discuss these things for the time being we will just assume this to be the case. So we have the formal power series which is defined over this field K and it basically takes the sequence a_n and then multiplies the term a_i with x^i and then sums over all i.

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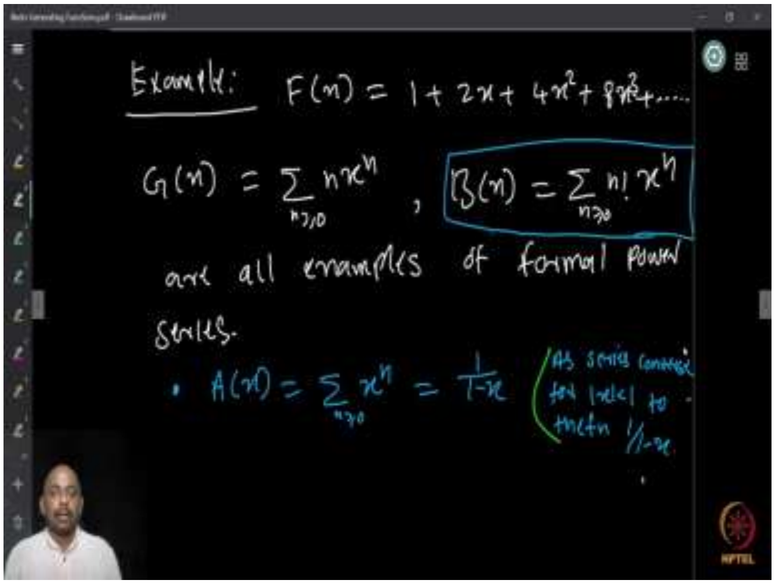

Now, let us look at some examples. Here is the first example: $F(x) = 1 + 2x + 4x^2 + 8x^3 + \cdots$. So this is basically coming from the sequence of binary words, the number of binary words of length n and that will tell you that the function is $1 + 2x + 4x^2 + 8x^3 + \cdots$ because the a length n string has exactly 2 raised to n possibilities.

Similarly, we have another sequence let us say sequence n where 1, 2, 3, etcetera is the sequence. So what is the power series which $\sum nx^n$. Then you have $\sum n!\, x^n$ which is another series, which represents the number of permutations of n element set.

Then, so these are all examples of power series and the formal power series and in this particular case when we have the constant sequence 1, we have $\sum x^n$. That sequence for example, can be written as $\frac{1}{1-x}$ as we showed before because it converges for values of x where |x| is less than 1.

And in that interval because it converges to $\frac{1}{1-x}$ we can always write it as $\frac{1}{1-x}$. So if the series converges for any interval any small interval no matter how small we can write it as that function. So we can use this to help putting the entire information in the series directly to this function and that is what we are going to use in our calculations.

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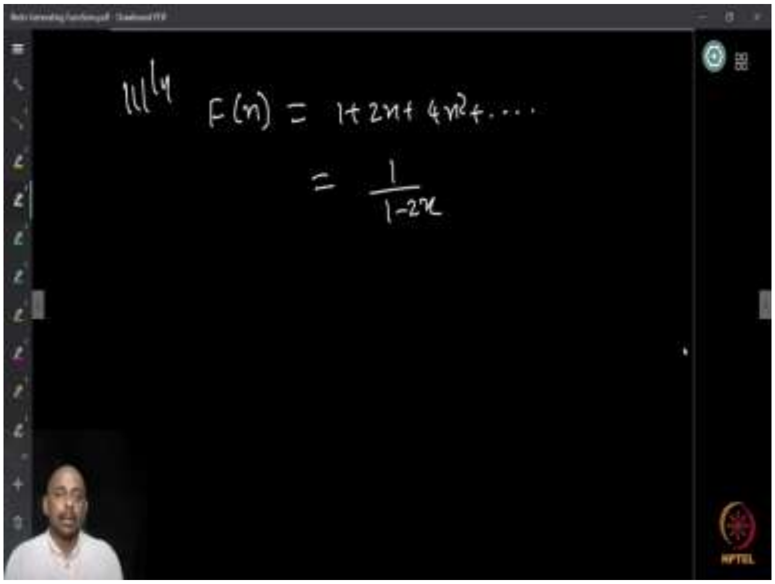

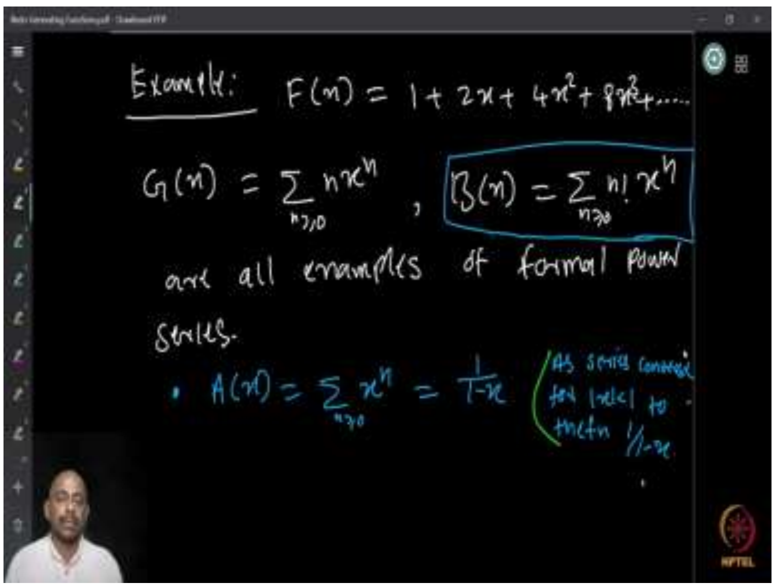

So we saw that f(x) = 1 + $2x$ + $4x^2 + \cdots$. Since we know that 1 + x + $x^2 + \cdots$ which is the $\sum x^n$ is $\frac{1}{1-x}$, let us replace x by 2x so x is then indeterminate, so therefore 2x is also indeterminate. So if I replace x with 2x then I will get the series 1 + $2x$ + $4x^2 + \cdots$..

But since this converges to $\frac{1}{1-x}$, I can say that this is a series 1 + $2x$ + $4x^2 + \cdots$. converges to $\frac{1}{1-2x}$. So because I just replace x with 2x I get this. Now of course the convergence of this is a smaller interval, so now instead of converging for |x| less than 1, it converges for 2|x| less than 1.

So for any constant k, $\sum(kx)^n$ we can we can put it into $\frac{1}{1-kx}$ kind of form.

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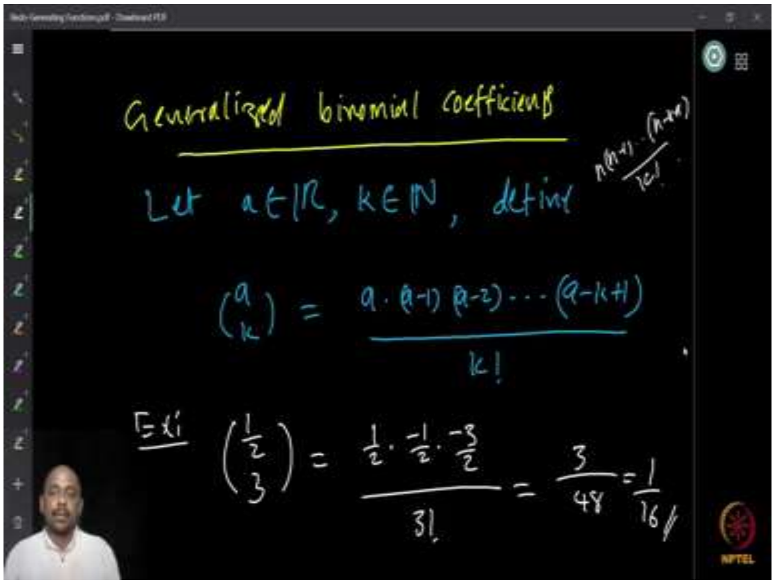

Now a slight deviation from our formal power series this is because we will need this generalization of this binomial coefficients to use sometimes in calculation, so this will help. We already saw the binomial coefficient. So given 2 natural numbers let us say n and k where n is greater than or equal to k.

We have the binomial coefficient n choose k, which we came across because it is the number of ways to choose a k-element set from an n-element set. Now, so we found out that this n choose k is actually $(n.(n-1).(n-2)\ldots(n-k+1))/k!$ this is something that we proved earlier.

And then we said that we can also write it as $\frac{n!}{k!(n-k)!}$ and this was possible because n factorial is basically1.2.3 etcetera up to n. And what we are looking at is $.(n-1).(n-2)\ldots(n-k+1)$, so therefore if I take n factorial it has all these terms. But then to remove the last n - k terms, I have (n - k) (n - k – 1) etcetera up to 1 and that goes to $(n-k)!$.

So therefore, I could have done is to replace this by n factorial by $(n-k)!$ and therefore I got this nice expression. On the other hand when we look at generalized binomial coefficient we use the previous case where by the definition what we have is instead of n factorial we have n (n -1)… / k! .

So take this definition and then let us define for a real number, now, a is a real number and k is a natural number so for this case we are going to define a choose k as the generalized binomial function. So what is a choose k? It is $a.(a-1).(a-2)\ldots(a-k+1))/k!$ Now, 1 can ask like why does this make sense, so we will see this, when we see some applications. But of course this is perfectly well defined because we have all these things, even though we cannot define for example a factorial we can define this easily. So we have the generalized binomial coefficient a choose k in this way.

Now as an example let us look at this 1/2 choose 3 it is $\frac{\frac{1}{2}.\frac{-1}{3}.\frac{-3}{2}}{3!}=\frac{3}{48}=\frac{1}{16}$. So we have this expression and the binomial coefficient for 1/2 choose 3, so this could be useful as we will see later.

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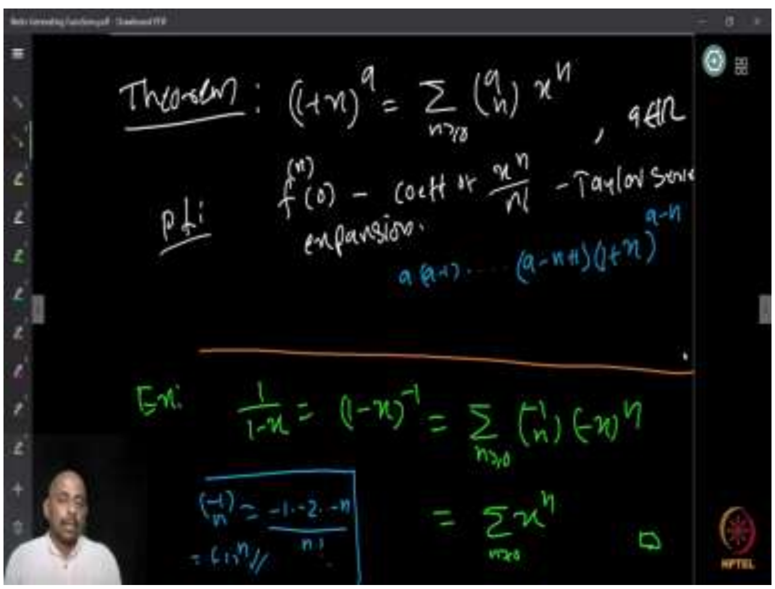

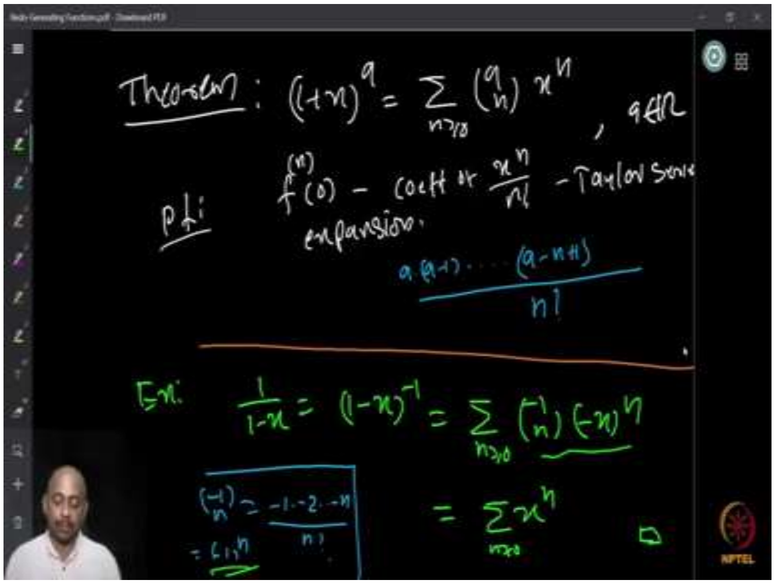

Now we can also now define the generalized binomial theorem. So binomial expansion is $(1+x)^a = \sum_{n\geq 0} \binom{a}{n} x^n$, for any real number a and n being natural numbers n greater than or equal to 0.

Now how do you prove this theorem? Well what you do is that we look at the Taylor series expansion of the function $(1+x)^a$. So, what is the Taylor series expansion? We take the n[th] derivative. So $(1+x)^a$ take the nth derivative, what do we get, we will get a(a -1)… .

If you take the n[th] derivative of $(1+x)^a$we will get a(a – 1)… (a - n + 1) $(1+x)^{a-a}$ because we have taken the derivative index. Now we want to evaluate this at x equal to 0, so then what we will get is a(a – 1)… (a - n + 1) and what Taylor theorem says is that this is precisely the coefficient of $\frac{x^n}{n!}$ when we try to expand.

So because it is the coefficient of $\frac{x^n}{n!}$ what is the coefficient of x^n? That is going to be this divided by n factorial. Now, what is this a(a – 1)… (a - n + 1) when I say this it is only the term up to here because when I substitute for x equal to 0 this term becomes 0. So this becomes 1, so therefore I just have this which is the definition of the binomial coefficient a chose n.

So therefore since I know that the coefficient of x^n is going to be now a choose n and therefore I have the theorem so that is the proof. Now once we have this we can use this to, for example expand a given function for example 1/(1 – x) I can write as $(1-x)^{-1}$, now minus 1 is a real number and I have $(1 + \ -x)^{-1}$, therefore I can use the generalized binomial expansion which will tell me that it is $\sum\binom{-1}{n}x^n$.

Now what is -1 choose n? It is $\frac{-1.-2\ldots-n}{n!} = (-1)^n$. So now, this is the -1 choose n term, so then I have $(-1)^n$ and $(-x)^n$. So this together will give me x^n, so therefore I get $\sum x^n$, so this is another way to expand using the binomial theorem generalized form.

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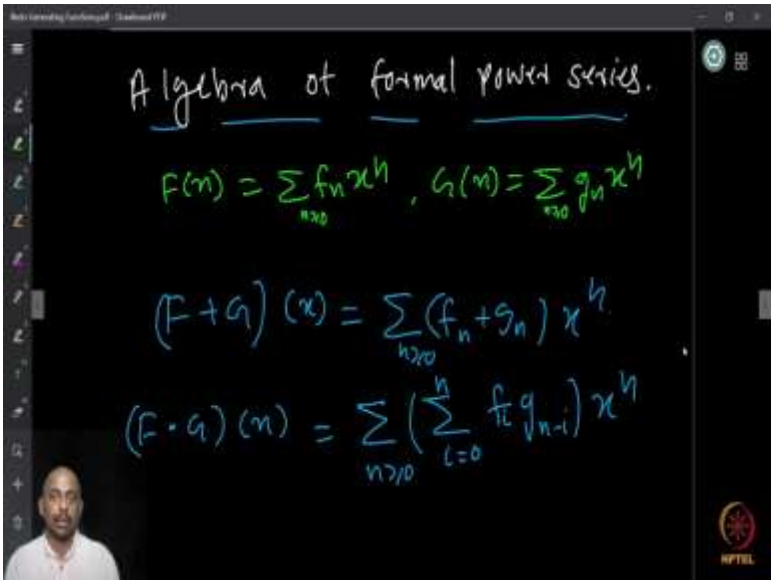

Now we can define the whole algebra of formal power series we can define addition, we can define scalar multiplication, we can define products all these things and then one can show that it actually forms a ring so we can call it the formal power series. We will not go into the details of this because that is not part of our course but we will just state a few things.

And we say how do the addition and multiplication work. So, given 2 series, let us say $F(x) = \sum_{n\geq0} f_n x^n$, which is the first series, then you have another series $G(x) = \sum_{n\geq0} g_n x^n$. Now given these two, I can define the sum of the series

$$(F+G)(x) = \sum_{n\geq0}(f_n + g_n)\,x^n$$

So this is clear because when we have 2 formal power series, when we take the sum we can only add, for example, the coefficient of the like term the corresponding times x^n I have to take the coefficient and then add with the coefficient of x^n, that is the only way we can add.

Therefore, it is a natural way to define the sum. Now, what will be interesting is when we see how this can use this kind of operations to define what happens in the combinatorial structures.