Combinatorics
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Partial Orders

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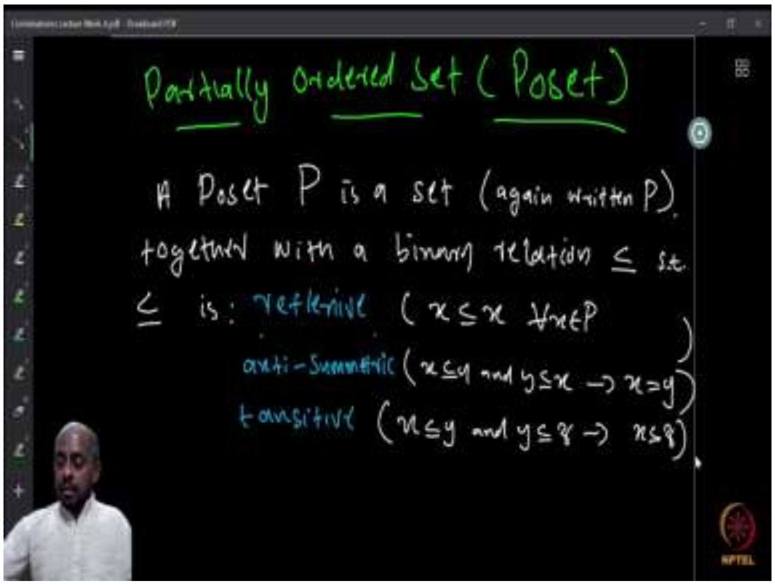

We will take a very quick review of partially ordered sets just in case some of you are not familiar with partially ordered sets or not come across these definitions and properties. So, this will be very quick, we will not spend too much time on this, because, this is not part of what we are studying in this course, but something that is required to see some of the results or understand some of the results.

So, what is a partially ordered set, a partially ordered set P is a set. Now, the set, is also denoted P, it is an abuse of notation, but, it is clear from the context. So, therefore, we do not worry about that. So, we will represent the same set which is the base set and as well as the structure Poset both with the same letter P.

So, the Poset P is a set together with a binary relation denote that less than or equal to in our most cases or sometimes you can use other symbols does not matter, but it will be explicitly mentioned. Such that this binary relation is first of all reflexive that is $x \leq x$ for every $x \in P$, for every element it is less than or equal to itself. So, it is in relation with itself that is reflexive, then it is anti-symmetric. Which means that, if $x \leq y$, and $y \leq x$, then it implies $x = y$. So, I read it as it is less than or equal to because of the symbol that we use.

But, to be more precise, you should just say that x is related to y and y$y$is related to x, but let us not worry about that for the time being. And then the third condition is that it must be a transitive, no transitive property is there. The relation $x \leq y$ and $y \leq z$ implies $x \leq z$. So, if you have these three properties, for the set with relation, then we say that this is a partially ordered set or it is a partial order.

Now, without writing it, let me just mention, when the relation is strict, now when we do not allow reflexivity, then the relation is strict, it is a strict partial order and we will not look at strict partial order in this course.

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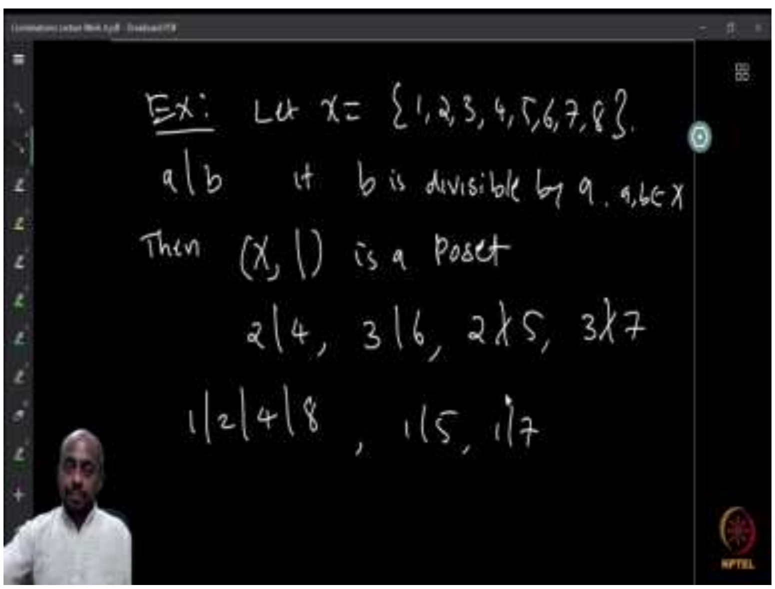

So, let us see an example. So, take the set of integers 1 to 8; 1, 2, 3, 4, 5, 6, 7, 8. And then for the relation, I look at the divisibility relation. So, $a|b,$ if b is divisible by a. So, this symbol ‘ | ‘ is used for the divisibility for a and b in X. Then, X with the relation divisibility is a poset. So, you can see that for example 2 divides 4 ($2|4$), 3|6 and 2 does not divide 5. So, 2 and 5 are not related 3 does not divide 7, 1 divides 2, 2 divides 4, and 4 divides 8 and by transitivity 1 divides 4, 2 divides 8, 1 divides 8 etcetera. Then 1 divides 5 and 1 divides 7.

So, this is a poset you can verify that reflexivity is true, that 1 is less than or equal to 5, 1 divides 5, 2 device 2 etcetera, every element divides itself and anti-symmetric. So, if 2 divides something and something divides 2, then there must be the same. So, that you can see verify that any element dividing another element and that dividing this it means that there must be the same. So, you can verify all the three properties and then see that this is actually a partially ordered set.

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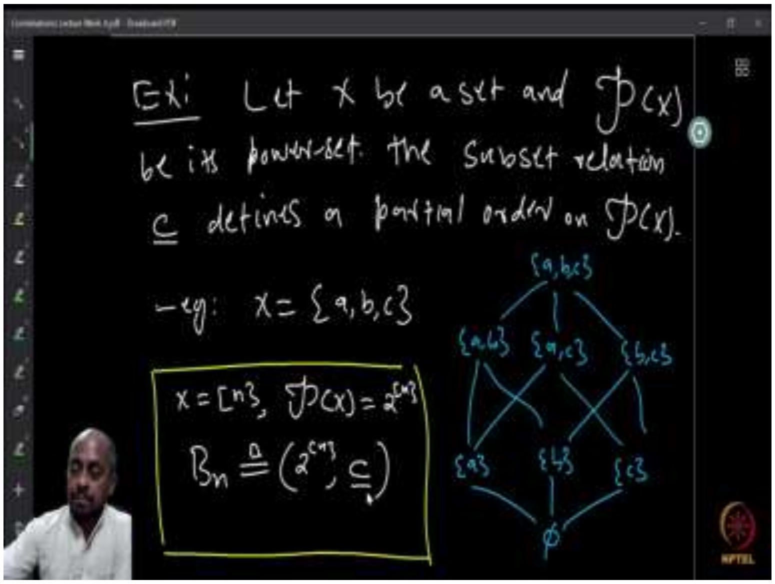

Now, another example this is a very important example that you will come to see visit many times. So, let X be some arbitrary set and $\mathcal{P}(X)$ the power set. Now, the subset relation. So, this defines a partial order on the power set. So, let us take an example X is equal to three elements set $\{ a, b, c\}$. Then we know, the empty set for example is a subset. So, empty set is as a subset of $\{ a \}$, $\{b\}$, $\{c\}$ etc.

Similarly, $a \subset \{a, b\}$, $a \subset \{a, c\}$, $b \subset \{c, b\}$, $c \subset \{a, c\}$. So, all these one can see. Now, I did not introduce formally what this picture means. So, we will come to this more formally later, but this is basically a kind of graph where you represent the elements of $\mathcal{P}$ as vertices of a graph and when you have two elements where one is contained in the other and there is nothing in between you put a line between them and if you have an element that is a subset of or less than or equal to in the relation.

Then, you make sure that the element that is the larger one and containing 1 is above in some sense, horizontally above. So, it is formerly we will define a little later, but, it is much easier to write down in all the subsets one by one. So, this defines a partially ordered set you can verify that, the reflectivity transitivity and anti-symmetric properties hold and now you can generalize to a set with n elements ($X = [n]$) and power set of X , $\mathcal{P}(X) = 2^{[n]}$ elements.

Now, the usual notation that we will use for this specific power set specific partially ordered set is B_n. So, B_n says that you are looking at the 2 raised to n subsets of n elements let us say 1 to n and then you are looking at the containment or subset relation as the order relation. So, this is a partially ordered set.

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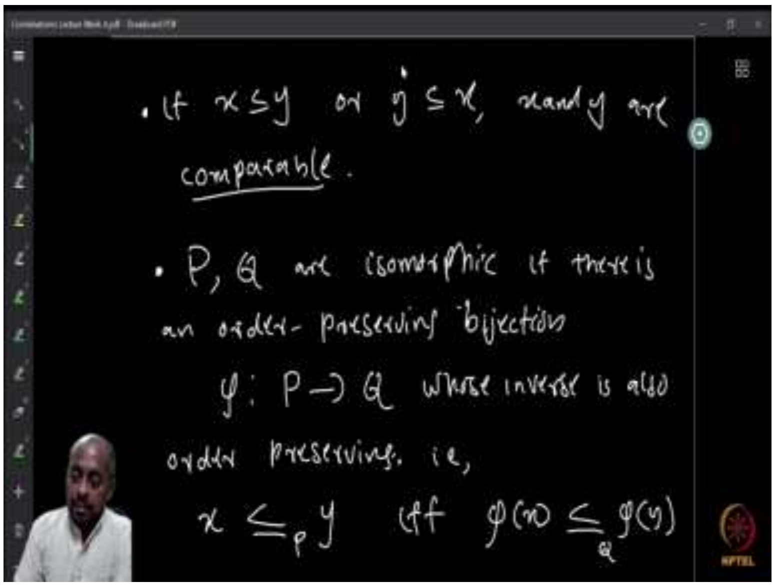

Now more definitions so, when, when I take any two elements of the poset P let us say x and y, if $x \leq y$ or $y \leq x$, then we will say x and y are comparable. So, if you look at the this example, for example, 2 and 5 if you take 2 does not divide 5 and 5 does not divide 2. So, therefore, 2 and 5 are not really comparable with respect to the division relation.

And similarly, if you take $\{a, b\}$ and $\{a, c\}$ here, they are also not comparable, you cannot say one is a subset of the other or this is sort of that, so, therefore, they are not comparable. So, when this property holds for one of the pairs then we say that they are comparable. Now, suppose you have two possess P and Q, we will say that P and Q are isomorphic if you can find an order preserving bijection, let us say $\phi: P \rightarrow Q$, whose inverse is also order preserving.

So, when I say order preserving x is less than or equal to y in P, $x \leq_P y$ So, this says that the relation is inside P. So, P and Q can have different relations. So, therefore, now, $\phi(x)$ is an element of Q and is less than or equal to $\phi(y)$ in Q, $\phi(x) \leq_Q \phi(y)$. So, if this relation is if and only if then we say this is an isomorphism. And we also want that inverse is also order preserving. Now so that is why this if and only if.

Now, suppose you have a bijection and, you know that it is the bijection is order preserving is it necessary that the inverse is also order preserving? Or can you find an example where it is not. So, think about this if you find an example, let me know or you write it as a homework.

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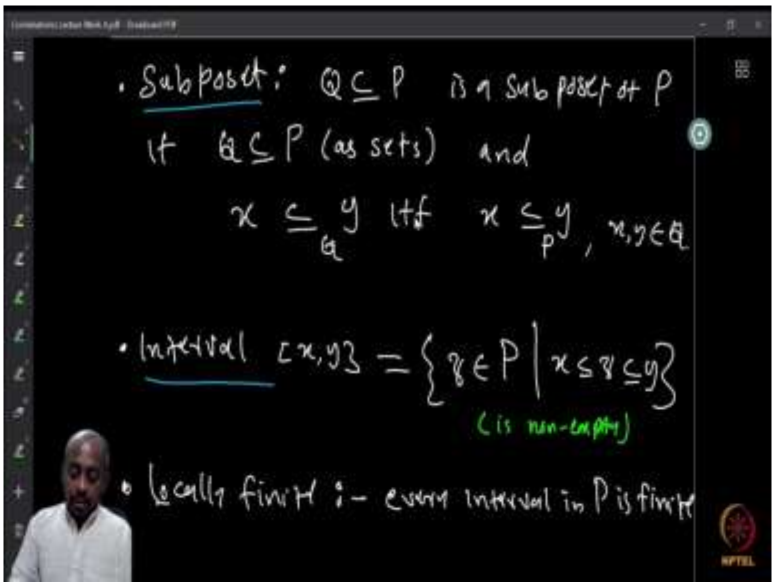

Now, given a poset we can talk about a sub-poset. The sub-poset is a subset of course, and the relation $x \leq_Q y$ if and only if $x \leq_P y,$ for $x, y \in Q$, for all the elements in Q which is the subset this should be true, then we say that it is a sub-poset. So, there is a there is a weak version of sub-poset definition where you do not insist this if and only if condition, but we will not go into that.

So, whenever we say it is a sub-poset we will say that it is a induced sub-poset or now the relation is maintained exactly as it is for all the elements within Q. Now for two elements x and y the interval $[x, y]$ is defined as the set of all elements in the poset P such that z comes between x and y, that is $[x, y] = \{z \in P: x \leq z \leq y\}$. So, this set is always non-empty, it will contain at least x and y or at least we want the, to define like that.

And so, the interval is this non-empty set. Now a poset is said to be locally finite if every interval is finite. So, the P itself may not be finite, but if you can say that every interval is finite. So, once you take fixed two elements x and y, then between x and y the number of elements that appears is finite. So, then we say it is locally finite.

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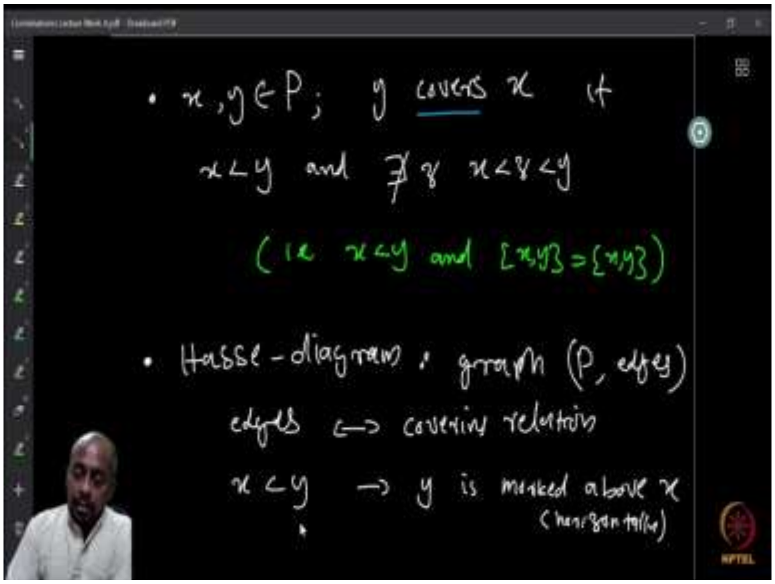

Now, when you have two elements in the poset let us say x and y, I say that y covers x if first of all x strictly less than y and there is no element z such that x is less than z less than y. So, they are comparable, because x is less than y, but there is no element in between. So, then we say that it is a covering a relation. So, comparability should be there, but there is no element in between then we say it is a covering relation.

So, which means that x is strictly less than y and the interval $[x, y]$ is precisely the set $\{x, y\}$. Now, formal definition of Hasse-diagram is that it is a graph on the vertex set as the elements of P with the vertices of the elements of P and the edges are the covering relations. So, the edges of the three other covering relations that is, when you have elements in between you do not. So, even though phi is a subset of let us say every set, you do not put an edge like this because there is already a set.

Let us say $\{a\}$ which comes in between $\{a, b\}$ and ϕ. So, therefore, I only put this edge and then the edge connecting this. So, these two together tells me that there is this what you call, the comparability but it is not a covering relation. So, I will not put the additional line. Now, we also make sure that in the Hasse-diagram when x is strictly less than y, y is marked above x horizontal. So, appears horizontally above the smaller relation.

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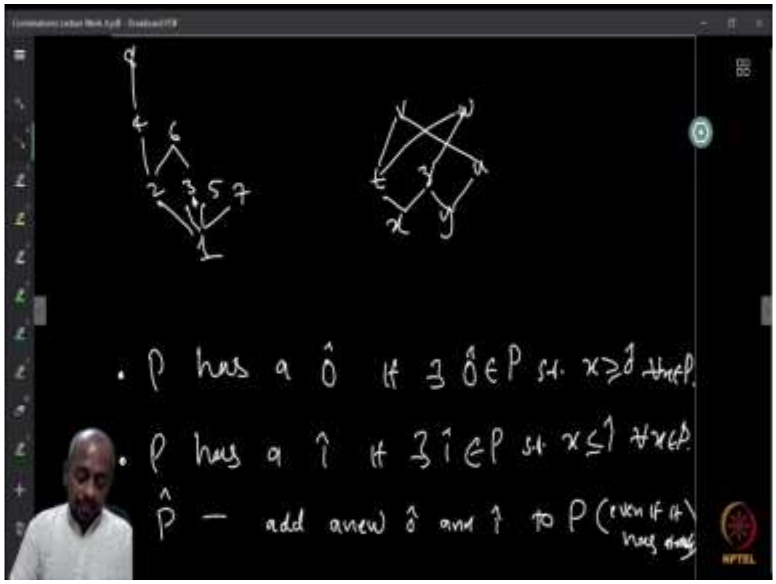

So, example here are two Hasse-diagrams. So, one for the poset that we considered earlier the poset which contains the 1 to 8 and the divisibility. So, we saw that 1 divides are 2, 3, 5, 7 they are precisely the prime numbers if you observe and then 2 divides 4, 2 covers 4, because there is nothing in between 4 divides 8. 2 divides also 6, 3 divides 6 and other elements that not divide any other elements. So, we get the complete description of the poset here and the Hasse-diagram.

So, once you have the Hasse-diagram, you know the poset. So, here is another example you have x and y and x is less than t, x is less than z, then y is less than z, and y is less than u, t is less than w, z is less and w, t and u are less than v. Now, we say that poset P has a $\hat{0}$ if you can find an element special element such that $\hat{0}$ such that x is greater than or equal to $\hat{0}$ for every x. It means that this is a kind of minimal element.

So, if every element is less than, is greater than or equal to this, then it is a 0. Similarly, P is said to have a 1, if you can find an element such that every other element is less than or equal to this element. And if you have a poset P, then you can denote by $\hat{P}$ as the new poset that you obtained by adding an additional $\hat{0}$ and $\hat{1}$ to P. So, you add two elements introduce two elements and make one as the minimum element that we know it is less than or equal to every other element.

And then one as greater than or equal to every other. If you do that, then you get a new poset P, $\hat{P}$. So, even if P originally had a 0, you can still add it, but then the original 0 may not be the 0 anymore.

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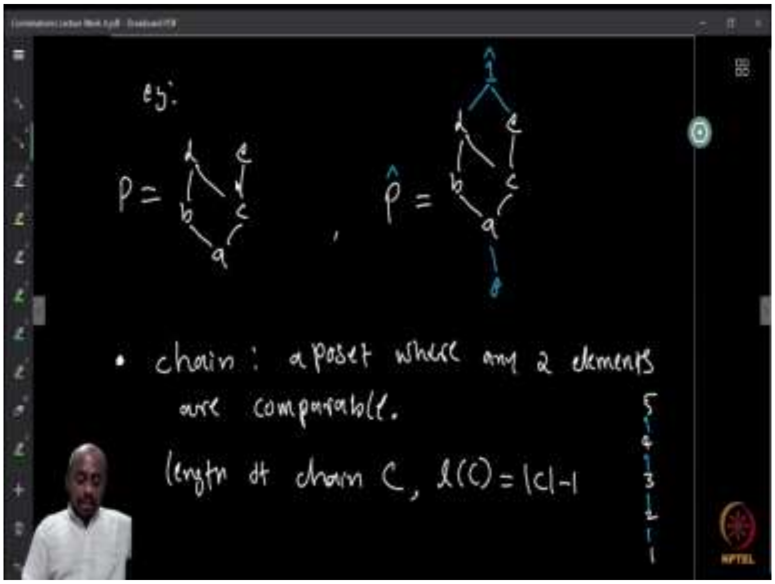

So, here is an example you start with this poset P. So, the P is the poset. And you can see that a is a minimum element because it is less than or equal to every other element, but there is no maximum element. I mean, I should not say maximum there is no 1 here because there is no element which is larger than every other element. On the other hand, if I take $\hat{P}$, I am going to introduce a 0 here and also a 1 there and now a is no more the 0. But $\hat{0}$ is and similarly $\hat{1}$ is a 1.

Now a chain is a poset where any two elements are comparable. Suppose you take a set of element and say that any two are comparable. For example, if you look at the natural numbers, natural numbers you know that every element can be less than or equal to the following numbers. So, and you know like, you can compare any two of them. So, that is a chain, you can see that so with the usual less than or equal to less the natural relation.

Here is a smaller subset 1, 2, 3, 4, 5. And 1 is less than or equal to 2, 2 less than or equal to 3, 3 less than or equal to 4, 4 less than or equal to 5, and because you can see that every element is comparable because of the structure. And the length of a chain is the number of vertices minus 1 there is the cardinality of C minus 1. So, the length of this chain is basically a 5 minus 1 which is 4, so the length is 4. So, you can think of many other chains it is very easy to see.

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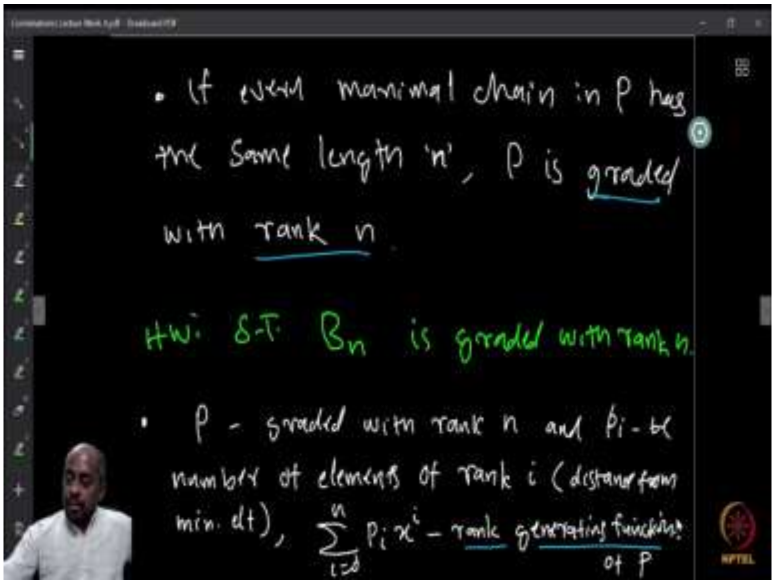
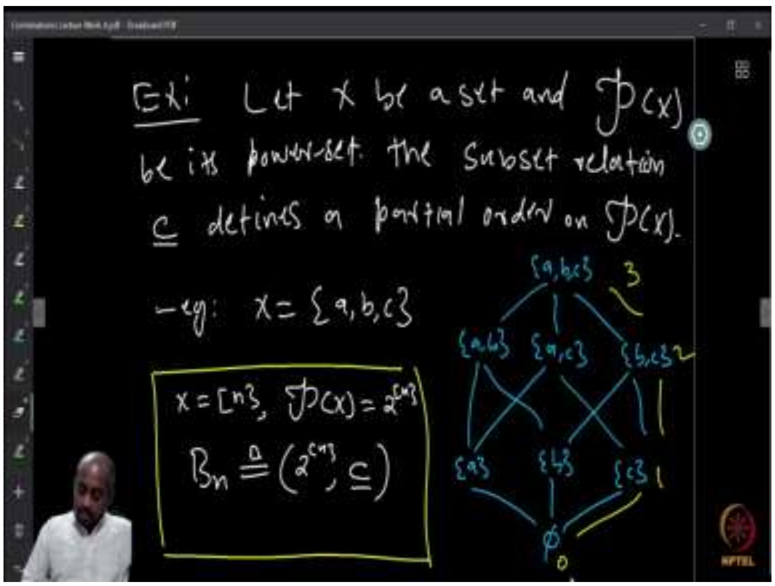
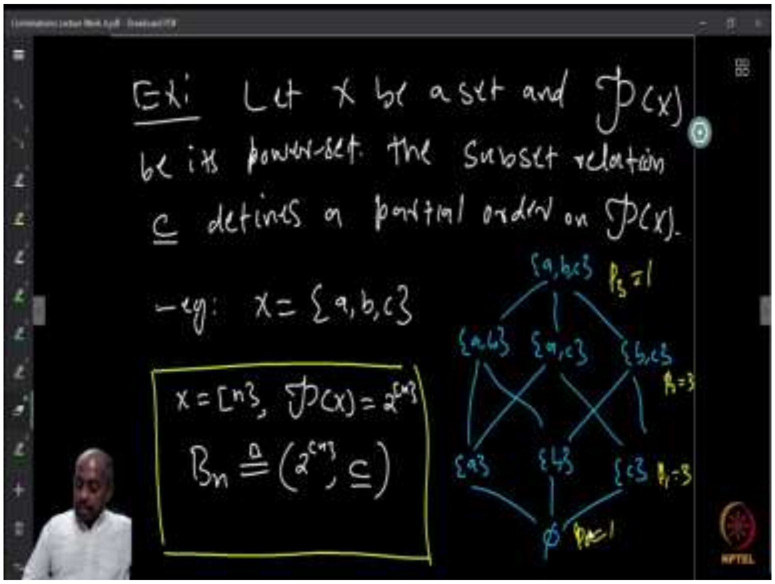

• If every maximal chain in P has the same length 'n', P is graded with rank n.
HW: S.T. B_n is graded with rank n.
• P - graded with rank n and p_i - be number of elements of rank i (distance from min. elt), $\sum_{i=0}^{n} p_i x^i$ - rank generating function of P


Ex: Let X be a set and $\wp(X)$ be its power-set. The subset relation $\subseteq$ defines a partial order on $\wp(X)$.
- eg: $X = \{a,b,c\}$
$X = [n]$, $\wp(X) = 2^{[n]}$
$B_n \triangleq (2^{[n]}, \subseteq)$
{a,b,c} 3
{a,b} {a,c} {b,c} 2
{a} {b} {c} 1
ϕ 0



Once you have defined the chain as a poset a sub-poset, isomorphic to a chain is also called a chain in the poset. So, if every maximal chain in a poset P has the same length n then we say P is graded poset with rank n. So, we looked at this example. So, if you look at the chain, maximal chains here. So, what is the chain so, basically like so, {a, b} is a chain because {a} less than or equal to {a,b}.

So, a, b is a chain, but this is not a maximal chain because I can extend the chain below and above, so, there is this maximal chain, this chain. So, the length of the chain is three here. Now, if you look through this you can verify that every maximal chain has the same length every maximal chain has length 3. So, in such a case we will say that this poset is graded and its rank is and n in this case it is 3.

So, the length of the chain is three therefore, the rank is three. Now, you can take the elements and then look at the length of the chains from the smallest minimum elements and then you can see that if the length from the minimal element is whatever I then I can say that, that is the rank of the element. So, I will say that this is 1 this is 2 and this is 3. So, elements in this level you can verify that are all having rank 2 hear all of them having rank 1 and here this is 0 and this is 3.

So, this intuitively tells you why this is graded of given rank. Now, as an exercise you show that, B_n for any arbitrary n is also graded of rank n. Now suppose you have a poset P which is graded with rank n and suppose p_i denote the number of elements of rank i which is rank i is the distance from the minimum element, then $\sum_0^n p_i x_i$ is called the rank generating function of the poset.

So, for example, in this case you have $p_0 = 1,\ p_1 = 3,\ p_2 = 3,\ p_3 = 1$. So, you have this and then this you can now write summation this values into x^i to get rank generating function. We will not look at the rank generating function at the moment. We will only look at it later, when we look at the generating functions.

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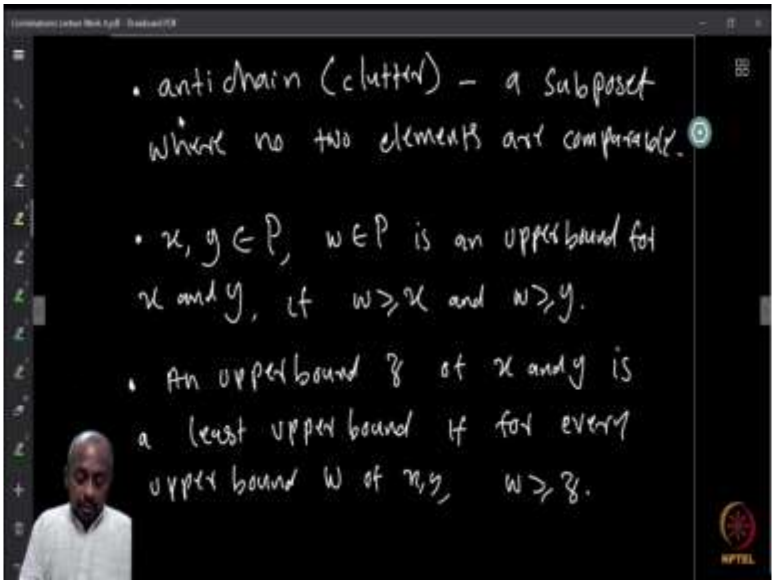

So, now, an anti-chain or a clutter is a sub-poset, where no two elements are comparable. So, if you have a subset where no two elements are comparable for example, if you take this poset and if you take b and c, b and c are not comparable. Similarly b and e are also not comparable. So, this subset {b, e} or {b, c} are clutters or anti-clutters. So, you can find several of these in these examples for example, 2, 3, 5 and 7 form an anti-chain 5 and 4, anti-chain 7 and 8 form an anti-chain.

Now, when you have two elements x and y in a poset P and say a third element w is an upper bound for elements x and y if w is greater than or equal to x and w is greater than or equal to y. So, x and y themselves may not having any relation between them, there could be also but then an upper bound is an element which is greater than or equal to both of these elements.

So, in this example, in the poset P for example, the element d is greater than or equal to b and c. So, b and c has d as an upper bound. Now an upper bound is called a least upper bound. So, x and y are elements for which a z is an upper bound. So, z is the least upper bound, if for every upper bound let us say w of x and y of course, w is greater than or equal to z, because z is the, is an upper bound, but it is there is no smaller upper bound for x and y. So, then it is a list upper bound.

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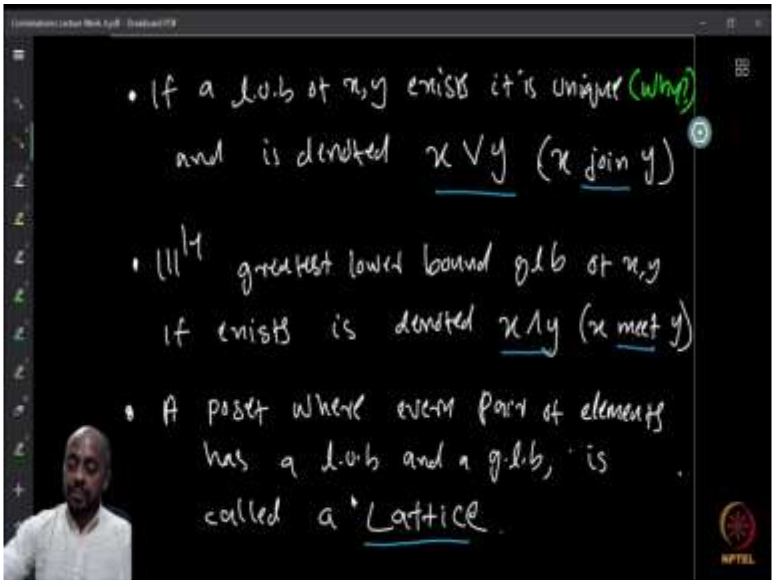

So, similarly you can define the notion of greatest lower bound and if a least upper bound of x and y exists, then it is unique. Why? you think about it and prove it, it is easy to see, but try to argue it. So, if a least upper bound of x and y exists, it is unique and this is denoted usually by $x \vee v$ where $x \vee v$ is read as x join v and similarly, the greatest lower bound again if exists is unique and it is derived by $x \wedge v$. So, we read it as x meet v..

Now a poset where every pair of elements has a least upper bound and a greatest lower bound it is called a Lattice. Again, there is a whole branch on Lattice theory, but we will not look at much of lattices just in case of some exercise or something mentions Lattice, I just want you to know that what is it? So, that is the only reason we defined Lattice and it has very nice properties, maybe in some of the exercises you can see why it is nice.

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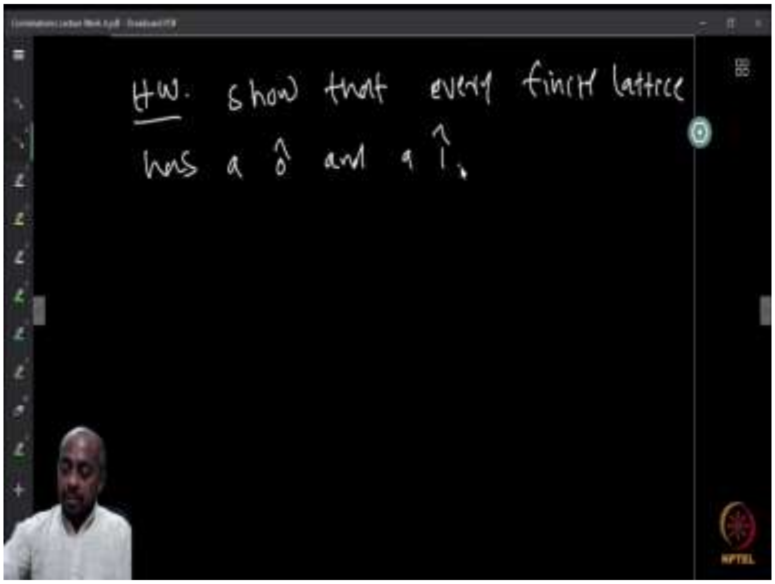

Now, show that every finite lattice has a $\hat{0}$ and a $\hat{1}$. So, every pair of elements has a least upper bound and greatest upper bound that says that it is a lattice but if it is also finite, then there is a unique minimum element and then maximum element. So, that is why it's a nice homework. I think that is all about our introduction to partial orders.

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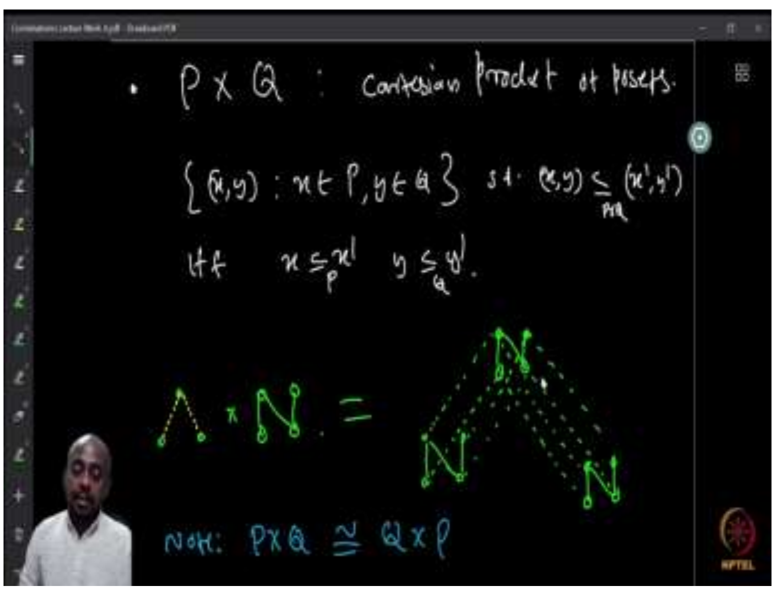

So, if you how to posets P and Q, the Cartesian product of P and Q is;

$P \times Q = \{(x, y): x \in P, y \in Q\}$ such that $(x, y) \leq_{P \times Q} (x', y')$ if and only if $x \leq_P x',\ y \leq_Q y'$
So, the Cartesian product of the posets is defined this way.

It is also called direct product in some textbooks. So the Cartesian product of the two posets P and Q is precisely defined over the Cartesian product of the sets P and Q such that, the tuple x , y is less than or equal to the tuple x dash y dash if and only if x is less than or equal to x dash in P and y is less than or equal to y dash in Q. So, here is the pictorial representation or Hasse-diagrammatic representation of the product.

So, you have this poset and you have this poset or take the product of these two is obtained by replacing for example, every copy every vertex of the first poset with a copy of the second poset. So, I replace every vertex with a copy of the second poset and whenever there is, a relation here the corresponding elements of this two posets are joint in a relation the same relation that we are having here and similarly.

So, we get the product. Now, from this definition if you look at this definition, you can say that the product it does not depend on which way you are going to do the substitution you can instead you can take these vertices and substitute copies of this. So, what happens is the Hasse-diagram looks very different from the Hasse-diagram that we got here. But they the posets are indeed isomorphic.

So, try to work out some examples and convince yourself that if you replace in a different way you will get a different structure of the Hasse-diagram, but the posets are indeed isomorphic. So, take it as homework take some small example and work out take the product and see that they are they are isomorphic if even if you take P cross Q or Q cross P.

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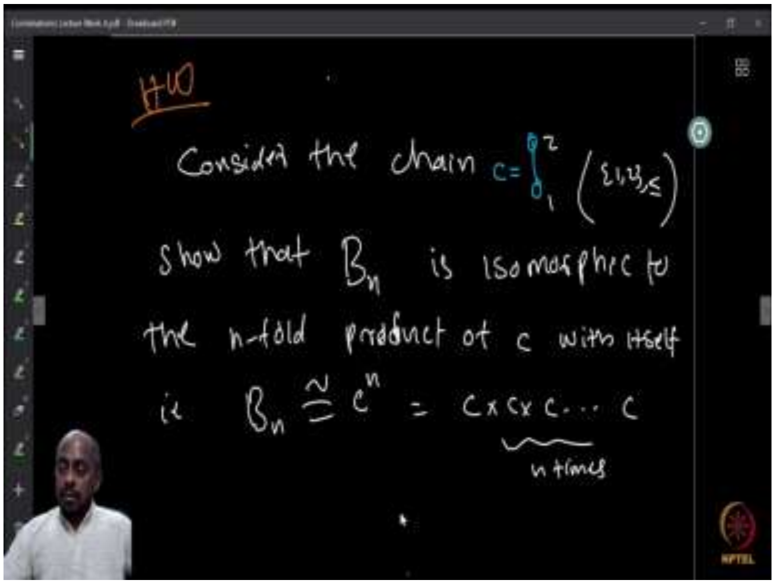

A homework: Consider the chain of length 1 which is two element chain. So, just 1 and 2. So, 1 is the smallest element 2 is the largest element, so, 1, 2 from the chain. So, the chain of length 1 or the chain of 2 elements is denoted C. Now show that the poset that we looked at B_n is isomorphic to the n-fold product of C with itself. That is you take this poset multiply it with itself n times and then you will get a poset and that was it is isomorphic to B_n.

So, this is the subset poset with the containment relation for the power set of n elements set. So, that is B_n. So, this is what so, product of C with itself n times is isomorphic to B_n. So, this is a very nice homework and with that we finished the introduction to the posets and try to look at a few more examples from the textbook and then convince yourself that you are comfortable with the notion of posets and all the related concepts that we have defined here.