Combinatorics
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Inclusion exclusion: Linear algebra view

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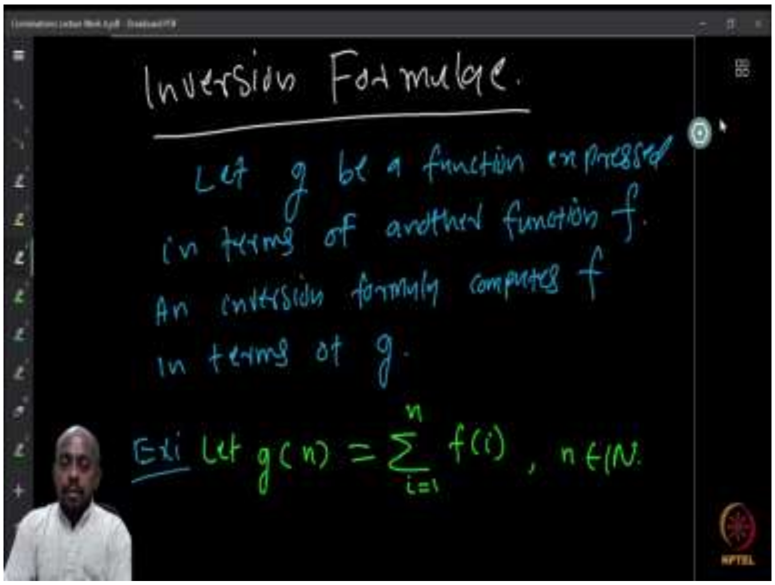

Welcome back, in the previous week we looked at a technique of counting called the principle of inclusion and exclusion. We saw several applications of this method. Now, we want to look at the same technique with a slightly different viewpoint and the idea is to generalize this even further. So, before going further let us look at what is called inversion formulae. So, what is the inversion formula? So, suppose that you are given a function g that is expressed in terms of some other function f.

Now, an inversion formula is a formula that computes the function f in terms of g. So, you are given a formula g in terms of f. Now, you want to find out what is f in terms of g. So, this is an inversion formula. Let us look at very basic example to begin with. Suppose, you look, let us say

$$g(x) = \sum_{i=1}^{n} f(i), n \in \mathbb{N}$$

So, g is defined for every natural number n and f is some function.

Now, the question is that can you recover f given g. If you know g can you recover f? So, this is something you must be pretty convinced that you can find it very easily. So, think about it for a minute.

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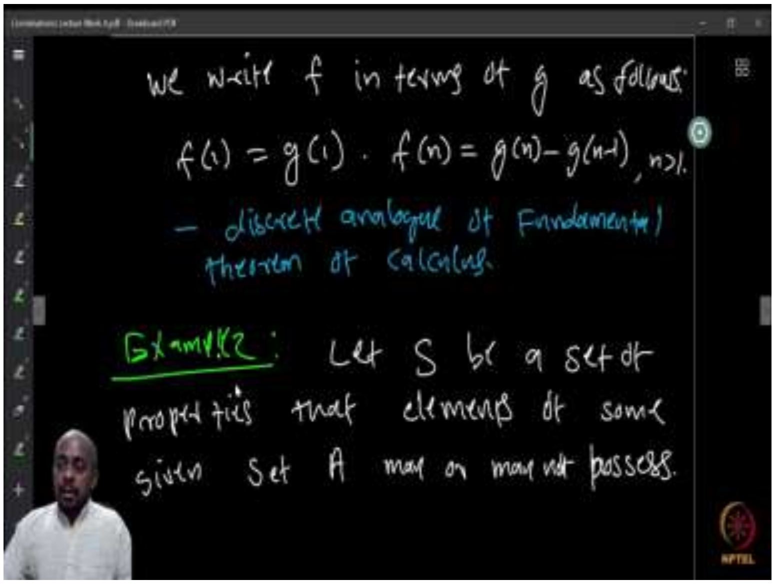

Now, we can write f in terms of g as follows:

$f(1) = g(1)$ from the above summation it is very clear. Then $f(n) = g(n) - g(n-1), n > 1$ this is also something that you can verify immediately from the previous definition of g in terms of f. But this is slightly more than that, if you remember your calculus you can see that this is a discrete analog of the fundamental theorem of calculus. That is what we say in calculus also, the fundamental theorem of calculus expresses the one function in terms of the difference of the other at the endpoint.

So, one can try to prove in fact the, the non-discrete version also using this kind of an approach, but it takes a little bit of work, but it is possible. Now let us look at one more example. Suppose, S is a set of properties that, elements of some given set A may or may not possess.

So, you have a set, big set A of several sets maybe, or some other elements whatever. But now, you collect some properties that some of the elements of this set might have or might not have. You take all these properties and say that, these elements have this property, these elements have this property, now, these elements have this property and that property like whatever. So, you have this collection of properties that each element may or may not have.

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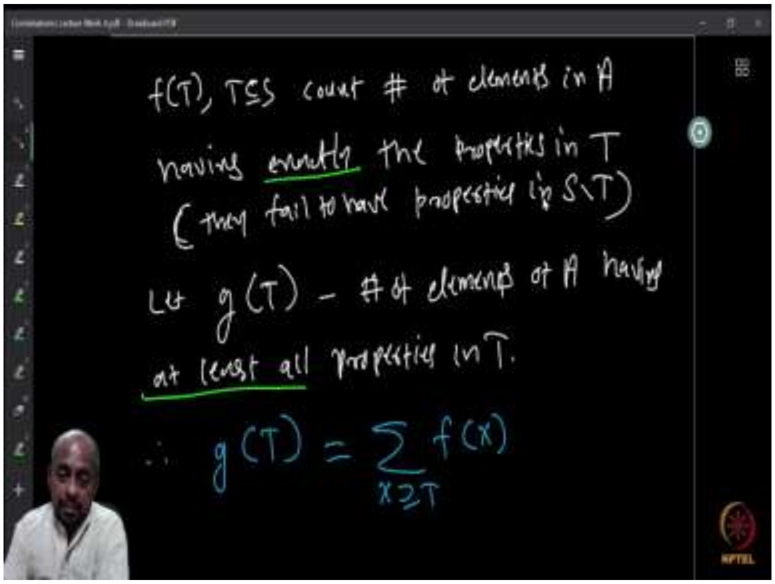

Then suppose, f is some function. So, suppose you are given a subset of S, let us say T, that is $T \subseteq S,$ then $f(T)$ counts the number of elements in A that has exactly the properties that is described in T.

Now, so, when I say exactly, accounting exactly, I mean is that they fail to have properties in $S \backslash T$. Because, S is the set of all properties that we are considering and $f(T)$ count only those precisely in T. Now, let $g(T)$ counts the number of elements in A that has at least all the properties in T. So the g basically counts those properties where, you take all those elements, where these elements have all the properties in T that is given, but it can have other properties, we do not care about that, but it should have all the properties in T.

So, now one can immediately see that $g(T)$ can be expressed in terms of f very easily. So, you take all super sets of T, and then say that sum over all this f and that will give you g. That is,

$$g(T) = \sum_{X \supseteq T} f(X)$$

Because $f(X)$ counts all those having precisely the properties X and now $g(T) = \sum_{X \supseteq T} f(X)$. Which means that it will have all the elements having properties in T and more. So, that is what $g(T)$ is.

Now, the principle of inclusion exclusion says that you can find the inverse. That is, you can express now f in terms of g also. If you recall the principle of inclusion, exclusion, what the way we used it and think about, exactly what we are doing that we were expressing.

So, we wanted to find some the number of elements having precisely some property and then we said that, we count those which has at least these many properties, and then we use that to get a formula for the properties that is precisely in this given set. So, if you are not convinced go back and think about it, look at some example and see whether it is precisely the same thing.

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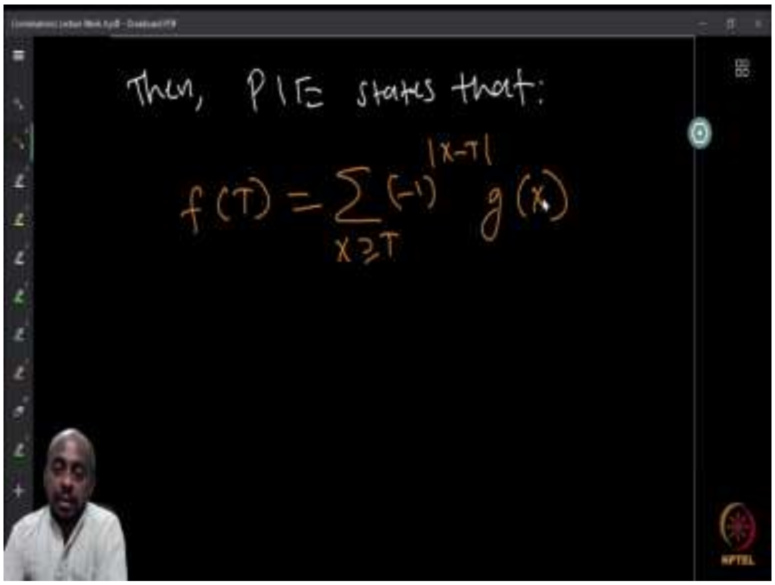

So, what principle of inclusion exclusion states is the following:

$$f(T) = \sum_{X \supseteq T} (-1)^{|X-T|} g(X)$$

Now, we did not state it precisely in this form, but you can see some similarity and if you work out the details, you can verify that, this indeed expresses the principle of inclusion and exclusion.

So, we will come back to it again, but, for the time being, I want you to go through this point of view and then see whether this expression matches with the way we have defined the principle of inclusion and exclusion. So, think about this and also think about this way of thinking that as two functions f and g instead of the way we presented earlier. What we are doing is actually a kind of inversion. So, once you think about it, it will be very clear.

Now, I want to look at this also in a different viewpoint. So, the classical way or not classical, not just the classical, the standard way or the more popular way of introducing principle of inclusion and exclusion is by looking at the small examples, like, you have 2 sets or 3 sets, then what you are doing is that, you count the union, you take the union and then you subtract, the intersection of pairs, and then you again, go back and then, add the things that we have subtracted too many times.

So, this way, and then try to generalize that into a formula, this is what we did and when we tried to prove this from the, the basics. And the, I mean then of course, it is the way probably, the technique evolved also. But then this technique, once you see it as an inversion, you see something more to it.

And then you can realize that it is actually a very minor theorem from linear algebra. So, inclusion exclusion principle is just a small theorem in linear algebra. And, it became so important because of its wide variety of applications, wide applicability. And so, let us see how this is a theorem in linear algebra.

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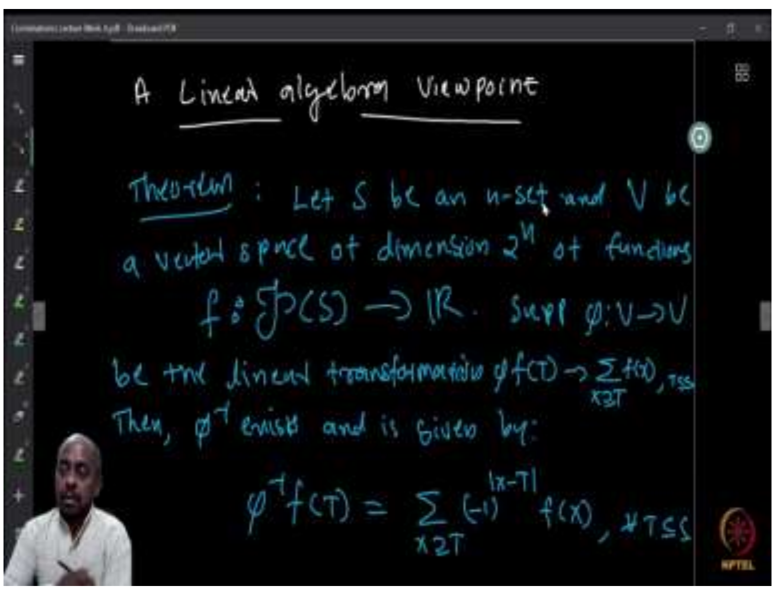

So, here is the theorem: let S be an n-element set, and V be a vector space of dimension 2^n of functions that maps the elements of the power set of S to $\mathbb{R}$, so basically it is a real valued vector space over the power set. Now, look at a linear transformation ϕ that maps V to V the following way that, $\phi f(T) = \sum_{X \supseteq T} f(X), T \subseteq S$ So, this is the way the linear transformation works for any given $T \subseteq S$.

Because the domain of f is the subsets of S. So, therefore, $,T \subseteq S$ means that the function is defined for that and therefore, this is a well-defined linear transformation. Now, if you define the linear transformation this way, the theorem states that, its inverse exists, its ϕ^{-1} and the inverse can be given by the following.

$$\phi^{-1}f(T) = \sum_{X \supseteq T} (-1)^{|X-T|} f(X), \forall T \subseteq S.$$

So, this theorem is precisely what the inclusion exclusion principle is saying if you just compare it with the previous statement, you can see immediately why this is the same statement.

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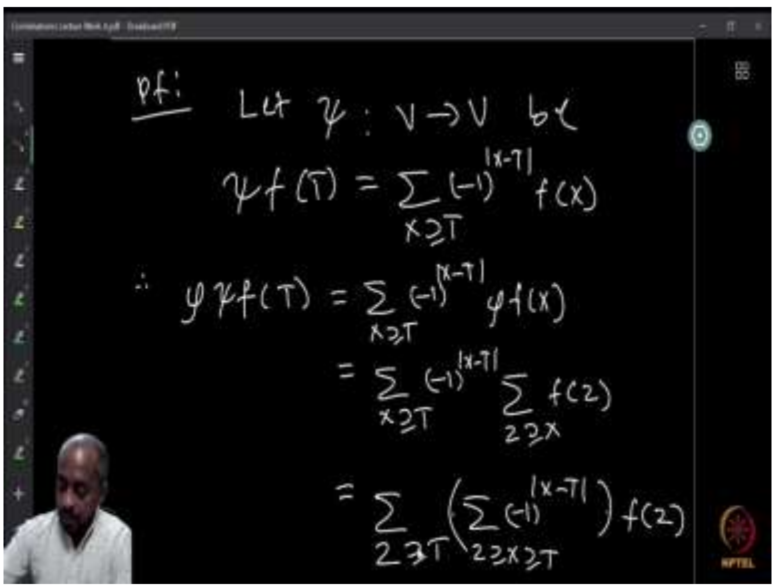

And let us prove this, the proof is by assuming that $\sum_{X \supseteq T}(-1)^{|X-T|}f(X)$ is the inverse So, let us take this as some function let us say $\psi: V \to V$ given by

$$\psi f(T) = \sum_{X \supseteq T} (-1)^{|X-T|} f(X)$$

So, this is the definition of ψ. Now, we show that ψ is the inverse of ϕ by multiplying them together and seeing that it returns identity. So,

$\phi\psi f(T) = \sum_{X \supseteq T}(-1)^{|X-T|}\, \phi f(X)$, by definition

$$=\sum_{X \supseteq T}(-1)^{|X-T|} \sum_{z \supseteq X} f(z) = \sum_{z \supseteq T}\left(\sum_{z \supseteq X \supseteq T}(-1)^{|X-T|}\right) f(z).$$

Now, $\left(\sum_{Z\supseteq X\supseteq T}(-1)^{|X-T|}\right)$ is independent of the choice of the Z.

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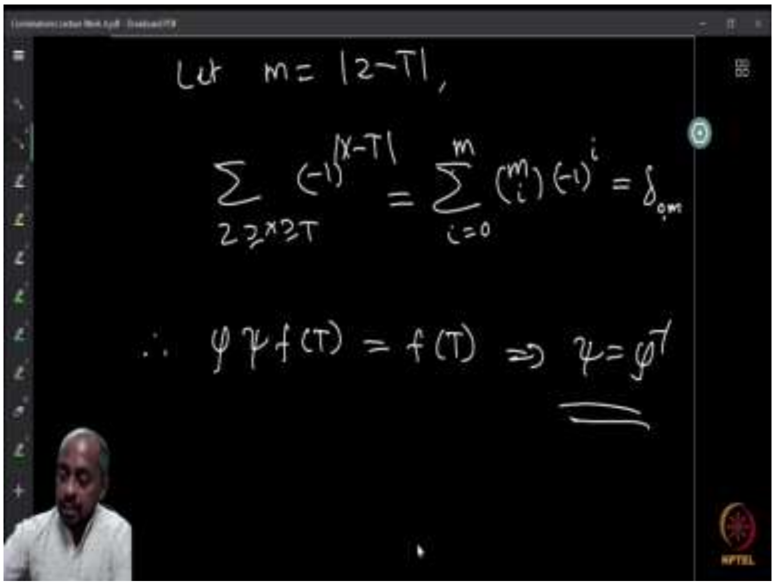

So, therefore, I can compute it. For computing that, let us put $m = |Z - T|$. Now consider the term $\left(\sum_{Z\supseteq X\supseteq T}(-1)^{|X-T|}\right)$. Here, X can vary from T to Z. So, when $X = T$ you will get 0 and when $X = Z$ we will get m. So, now we will count according to the cardinality of the set that we are considering. So, now, we can say that

$$\left(\sum_{Z\supseteq X\supseteq T}(-1)^{|X-T|}\right) = \sum_{i=0}^{m}\binom{m}{i}(-1)^i$$, because precisely $\binom{m}{i}$ ways to select an i element set

$$= \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases} = \delta_{0,m}.$$

Now, so the internal part is Kronecker delta $\delta_{0,m}$. So, now, look at what happens here? So, when it is precisely a nonzero that is when m is equal to 0. So, when m is equal to 0 this is nonzero.

So, therefore, the only term that survives is m is equal to 0. So, what is m is equal to 0 that is Z equal to T. So, therefore in the summation the only term that remains is or survives is $f(T)$ and everything else disappears. And in that case, the summation here is precisely 1. So, therefore, we get $f(T)$ on the right hand side. So, therefore, $\phi\psi f(T) = f(T)$, which means that $\phi\psi$ is the identity or $\psi = \phi^{-1}$. So, that proves the theorem.

So, it is a very simple proof and this is the result from linear algebra and you see that this statement is precisely our statement of the principle of inclusion and exclusion. So, you can express f in terms of g. So, given a function g which is in terms of the function f which counts exactly the properties in subsets. Then you can express f in terms of g also.