Combinatorics
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Lecture 02
Dirichlet theorem and Erdos-Szekeres Theorem

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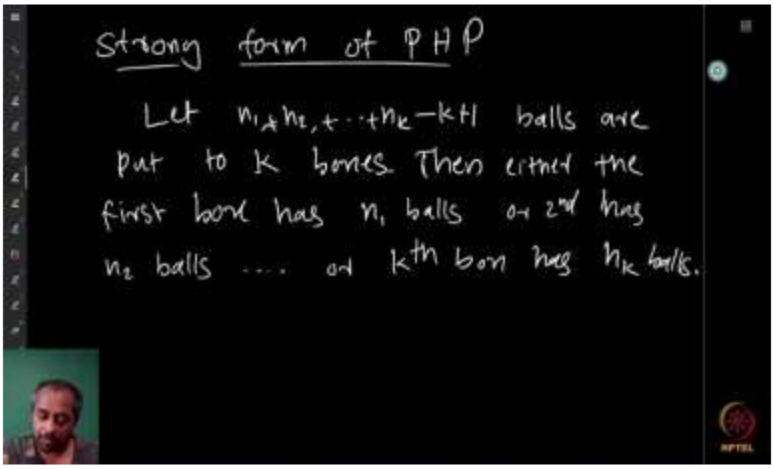

Even stronger form of pigeonhole principle. Suppose $n_1 + n_2 + \cdots + n_k - k + 1$ (This is basically $n_1 - 1 + n_2 - 1 + \cdots + n_k - 1 + 1$), balls are put into k boxes. That is, I have k boxes, and I have some number of balls say $n_1 - 1 + n_2 - 1 + \cdots + n_k - 1$, and then I am just putting 1 extra. Whatever that number is, that many balls I am going to throw into each of the different k boxes. Then either the first box has n_1 balls or maybe that is not the case; then the second has n_2 balls, if that is not the case, then third box has n_3 balls or something up to k^{th} box has n_k balls. One of these must be true, all of them cannot be false altogether.

Because if the first one had only $n_1 - 1$, second had only $n_2 - 1$ etc last one had $n_k - 1$, then in total, we have only $n_1 + n_2 + \cdots + n_k - k$, and the plus 1 will be missing. So, therefore, one of them must satisfy this. So, now this gives more structure. Now I can quantify and say that, this has n_1 or this has n_2 or this has n_3 or etc. I mean, I can change the order also. We can say that, first one has n_2, second one has n_1 etc. and that order does not matter. This is something we can prove and we can use it to prove more interesting results.

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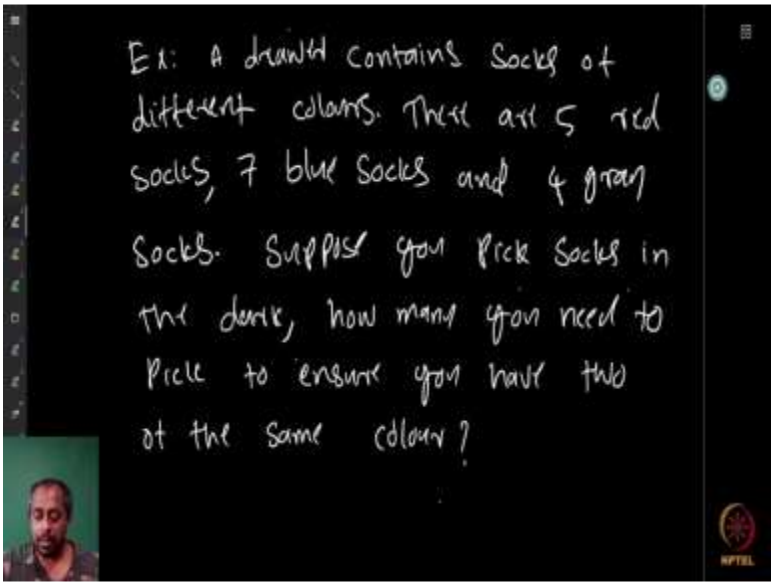

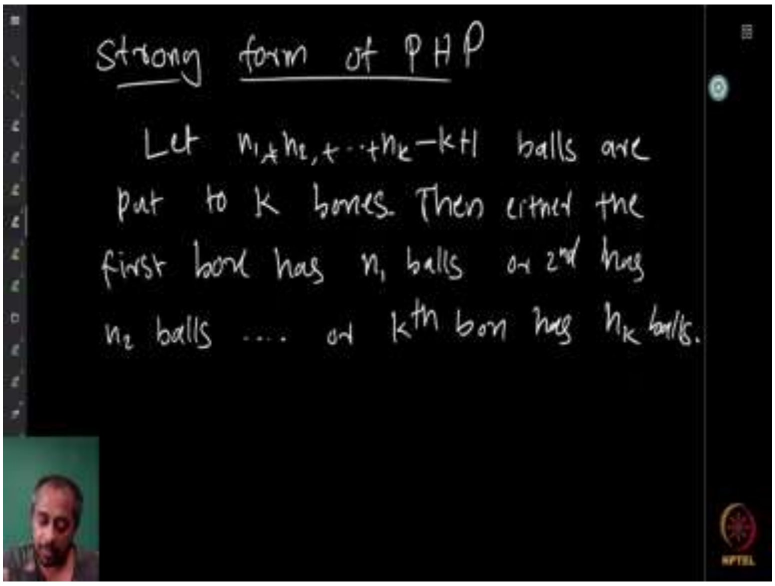

So, here is a simple, very simple example. So, we have a drawer and this drawer contains socks of different colors. Every day I go out I want to let us say put a matching color for my dress, the socks must have the matching color, I do not know. So now let us say that I have 5 red socks, 7 blue socks and 4 gray socks. Now, one day you go to pick up the socks for the next day morning, you want to make it already and then there is no power or something, then you are picking up the socks from the drawers in the dark, so you cannot see the color.

So, the question is that how many socks you need to pick to make sure that you have at least 2 of the same color, because I do not want to put a red socks on one leg and the blue socks on the other one. So, therefore, how many you need to pick to make sure that you have at least two of the same color. So, this is a problem very well suited for the generalized version that we were just presenting, the strong form. You have n_1, n_2, etc, $n_k - k + 1$. So, what are these numbers that we need to figure out?

Now we know that the socks are going to be the, socks are going to be the pigeons and the colors are going to be the pigeonholes because we need to pick 2 of the same color. So therefore, we can already see that, now 2 of the same color must fall into one box. That must be the blue box or red box or gray box.

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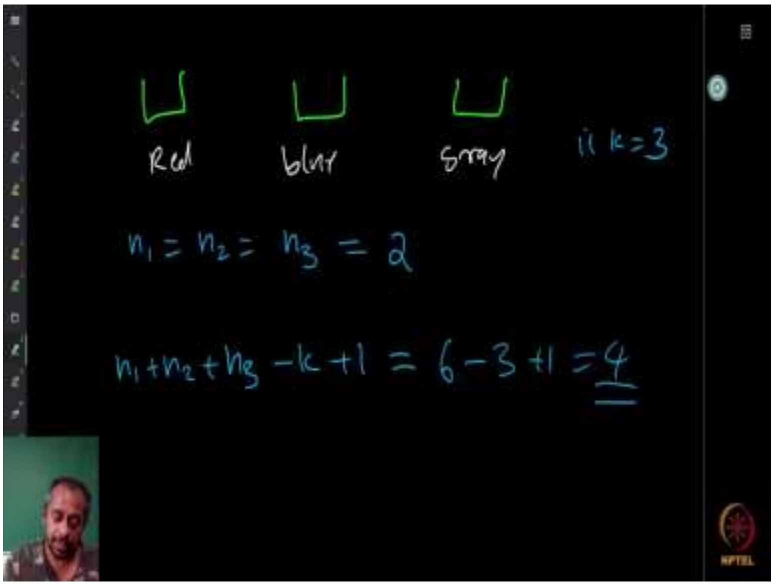

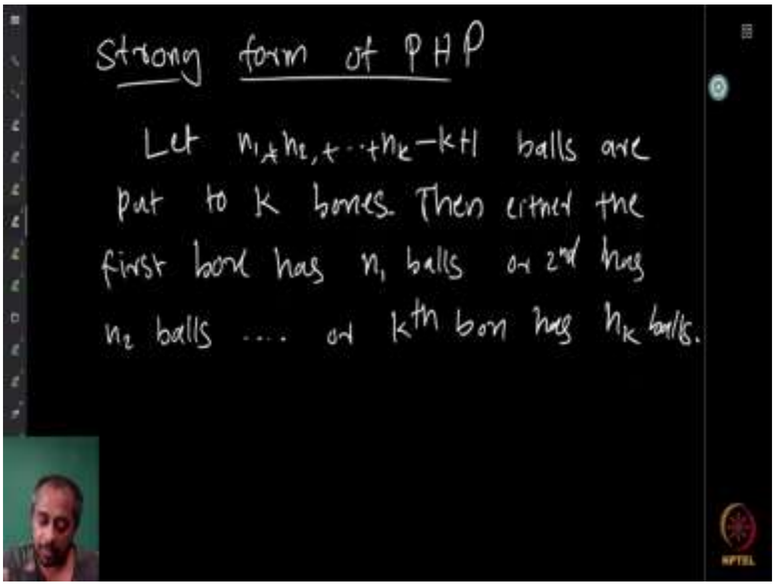

Now, what are these numbers that we want to talk about? So, we have a red box, we have blue box and a gray box. So, we have k is equal to 3. Then we have n_1, n_2, and n_3. We say that, you need to guarantee there will be either n_1, n_2, or n_3. We need either 2 red or 2 blue or 2 gray, one of them. So $n_1, = n_2 = n_3 = 2$. And therefore, by applying the generalized form, the minimum number of socks you need to pick will be $n_1 + n_2 + n_3 - k = (2 \times 3) - 3 + 1 = 4$. So, is you pick 4 socks, then definitely one of them, one of the colors, there will be 2 of them. There will be either 2 red, if it is not that, there will be 2 blue, that also is not the case then there will be 2 gray. So that is it. So, we are now masters of pigeonhole principle.

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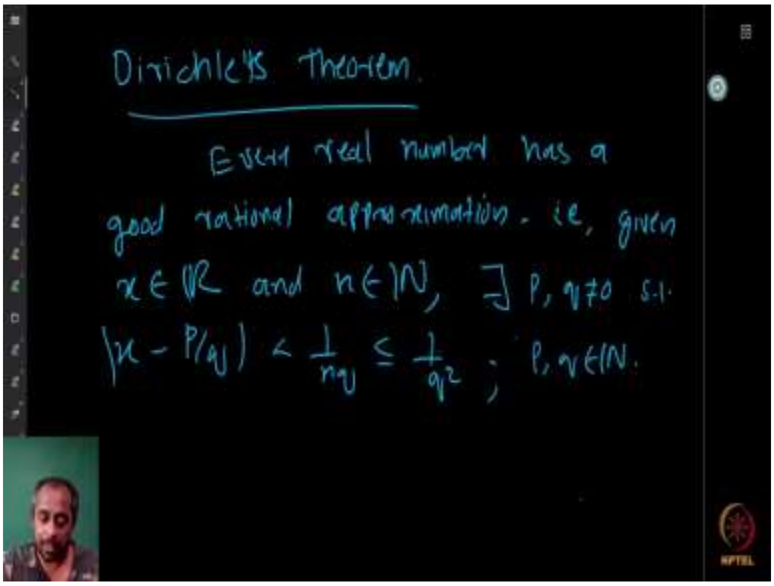

Now we want to see some really amazing applications of this theorem. We are going to prove a very famous result called Dirichlet theorem. Dirichlet is the name of the mathematician who proved this for the first time at least, that is what people believe. And this is a result from analysis, you can say if you want, that says the following. You are already familiar with the numbers, you have integers, you have rational numbers which can be written of the form p/q, where $q \neq 0$ is an integer.

And then you have real numbers. The real numbers are the extension of rational numbers. And you have also numbers which are not rational, called irrational numbers. Irrational numbers does not have $\frac{p}{q}$ form, there is no rational form. But what Dirichlet theorem says is that every real number has a very good rational approximation. That is, given any real number and given any epsilon, any very, very small number that you give me, like 0.0000000… hundred 0s and 1 or something. I can find a rational number, which is close to this number.

Now the difference between them will be less than the number that you gave me. So, no matter what is the epsilon you give me, I can always find a number even closer. So, this is called Dirichlet theorem. This says that there is a very good rational approximation as good as you want, you can make it as small as you want, the difference can be as small as you want. So let us state it in the formal way.

So, you have real number x is given and positive natural number n is given, then you can find 2 natural numbers p and q where $q \neq 0$, such that $\left|x - \left(\frac{p}{q}\right)\right| < \left(\frac{1}{nq}\right) \leq \left(\frac{1}{q^2}\right)$. So, $\frac{1}{nq}$ means that n can be arbitrarily large, which means that $\frac{1}{nq}$ will be a arbitrarily small, you can make it going to 0 like as close to 0 as you want.

So, the difference can be as close to 0 as you want, that is what it says. Now, how do you prove something like this using pigeonhole principle? That is an interesting question. Can you prove this? I want you to think about this, stop and think about it before we proceed.

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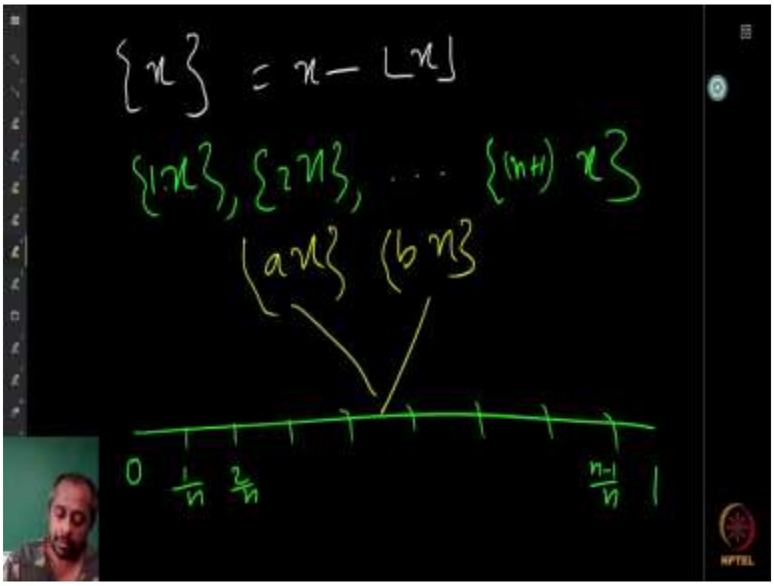

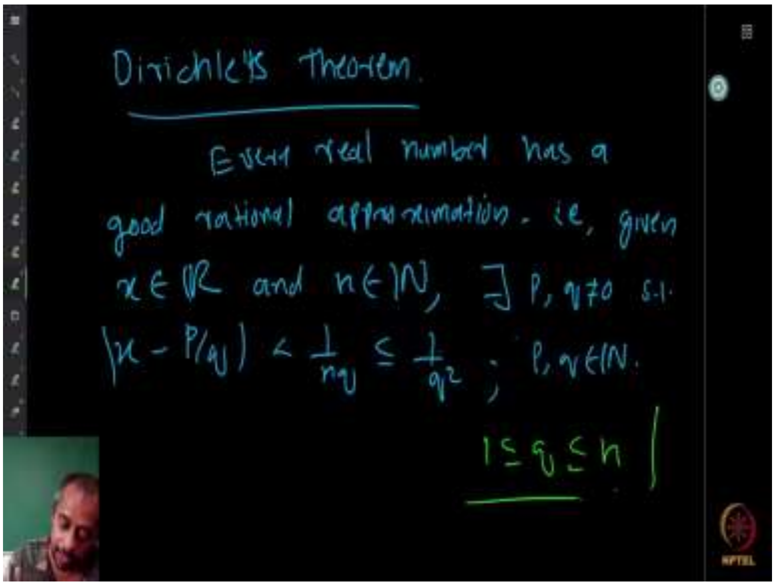

So here is the idea. When we are talking about real numbers, the real number always have some integer part, which is not interesting really, because the integer part is there maybe it is the larger part. But the fractional part is what is making it interesting. And we will start by assuming that since the rational number does not need a rational approximation because that number is itself a rational number. So, therefore, the difference between the best approximation which is itself and that is going to be 0.

So, here $\left|x-\left(\frac{p}{q}\right)\right|<\left(\frac{1}{nq}\right)\leq\left(\frac{1}{q^2}\right)$ is trivial, because the difference is 0. So, therefore, we can assume that number is irrational. So, for the irrational number the interesting part is the fractional part. So let us look at the number x given and then look at the fractional part. I will denote the fractional part of x as {x}. So, this says that, I remove the integer part from this. So, $\{x\} = x - \lfloor x \rfloor$.

So, I take the fractional part. Now fractional part is, something we know about fractional part is that it is going to be less than 1 and is going to be between 0 and 1 we are talking about positive for the time, where we can just change the sign. It is not a big deal. So, we have a property that it is going to be between 0 and 1. Now, this inequality $\left|x-\left(\frac{p}{q}\right)\right|<\left(\frac{1}{nq}\right)$ gives us a clue. So, $\frac{1}{n}$ is something that we can understand.

Because you give me n, then $\frac{1}{n}$ is a very, very small number epsilon (ε). Now, if I want to show that the difference is going to be less than $\frac{1}{n}$, then, basically what I am saying is that these 2 numbers happen to be in an interval of size less than $\frac{1}{n}$. So, this is what we are basically saying.

In this interval, we have these 2 numbers, the number that we are talking about, and it says that, they must be as close. So, the difference is going to be very small.

Now, how do you go about showing this, something like this. So, since we already have a fractional part, and I know that it is going to be between 0 and 1, I am going to subdivide the, this interval $[0,1]$ into n subintervals like $\left[0,\frac{1}{n}\right]$, $[\frac{1}{n},\frac{2}{n}], \ldots [\frac{n-1}{n},1]$. So now, I want to somehow bring the settings so that 2 things are going to fall in the same interval. Now what are these 2 things?

So, I need to somehow generate enough things to say that, if I want to apply pigeonhole principle, I need to generate at least $n+1$ numbers to be able to put it into, say that 2 of them are in the same interval. I am going to have n intervals as my pigeonholes, then $n+1$ numbers must be there. So, what are these $n+1$ numbers? That gives us a clue. So, for that we are going to generate the numbers by using the fact that irrational numbers even if you multiply with an integer, it is still going to be irrational.

So, what I am going to do is that I am going to take our irrational numbers x and then multiply it with numbers let us say, $\{1.x\},\{2.x\},\ldots.,\{(n+1)x\}$. So, I have generated now $n+1$ rational numbers, I am looking at again the fractional part of each of these, so I get $n+1$ different rational numbers, these are all related to x also.

So, we get a relation and the interesting part is that, if you look at the coefficients, what you have multiplied with, they are in the range 1 to $n+1$. So that the difference is going to be at most n. That is what this $\frac{1}{nq}$ and where our q is going to be, in fact between $1 \leq q \leq n$, this is an extra condition that you can give if you want.

So, that gives us a clue to what is going to be our q etc. So, what we know is that by pigeonholes principle, the fractional parts $\{ax\}$ and $\{bx\}$ must be belonging to the same interval, whatever some interval.

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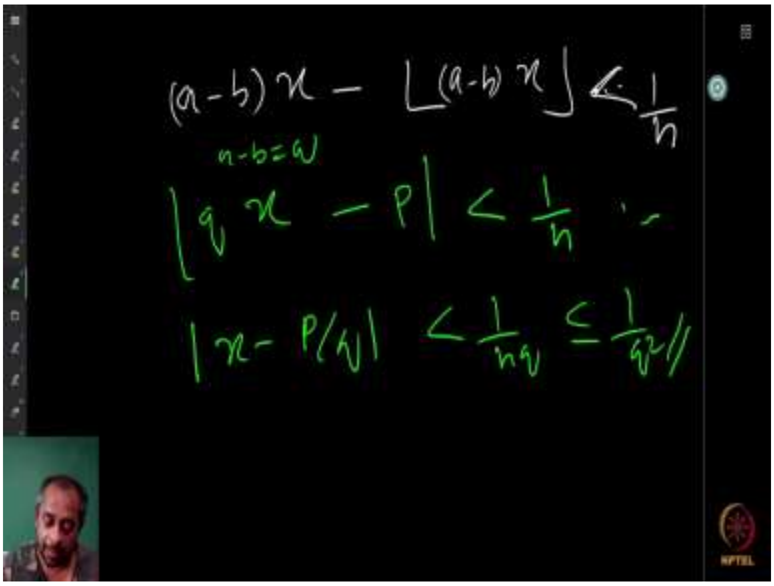

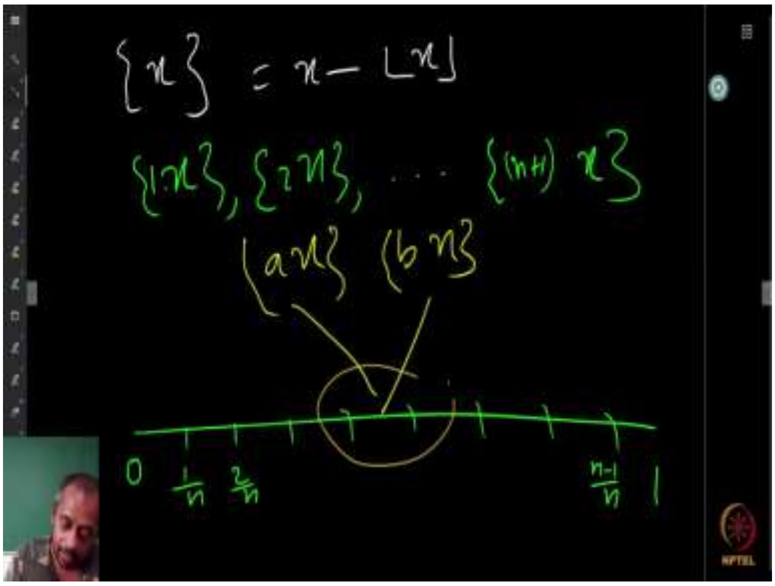

So, we know that $\{ax\}$ and $\{bx\}$ belong to the same intervals. So, that says that $(a-b)x - \lfloor (a-b)x \rfloor < 1/n$, this is in the same interval which means that it is the difference between these two $(a-b)x$ and integer part of $(a-b)x$, that difference is less than $\frac{1}{n}$, because they belong to the same interval.

Now, why it is strictly less, because these numbers can never be in the boundary because they are irrational numbers and $\frac{1}{n}$, $\frac{2}{n}$, etc are rational numbers. So, the fractional part will never be in the boundary. So, it will be strictly inside. If it is in one interval like this, then we know that they are going to be strictly inside. So, the difference is to be less than $\frac{1}{n}$,.

Now, things are easy. So, I am going to put $a - b = q$, so q into x because a minus b is going to be between 1 and n now, because the numbers are from 1 to n plus 1. So, $(a - b)$=q and this integer part of $\lfloor (a - b)x \rfloor = p$. That is, $|qx - p| < \frac{1}{n}$. So, dividing throughout by q, because q is non-zero, I have taken distinct numbers a and b, the difference is going to be non-zero and it is going to be in the range 1 to n.

So, therefore, this is dividing throughout by q, I get, $\left|x - \frac{p}{q}\right| < \frac{1}{nq} \leq \frac{1}{q^2}$, because $1 \leq q \leq n$. And that is what they wanted to prove. So, we have proved the Dirichlet theorem by applying pigeonhole principle, by considering the intervals $[0, \frac{1}{n}]$ etc, as the pigeonholes and the numbers that we generated ($\{1.x\}, \{2.x\}, etc$), as the pigeons.

And this is what requires some ingenuity, this is what that makes the pigeonhole principle difficult to apply because we need to figure out which are the pigeons and which are the pigeonholes and that is not always the easy job. So, I hope that you have cleared up the question. If there is anything just think about it, and let me know.

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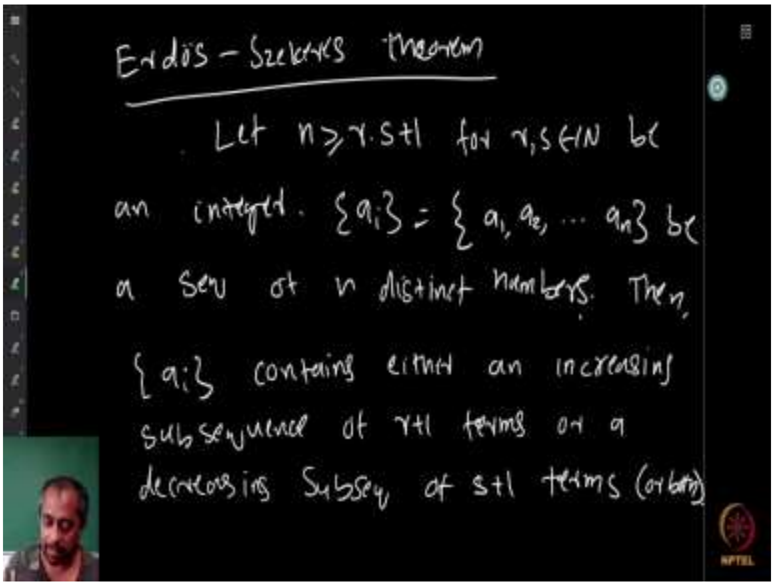

Now, another very important result and very interesting application of the pigeonhole principle, this is Erdos-Szekeres theorem. So, we can prove this theorem using the generalized form or the strong form, but I want to use the other form for the time being, because it gives a different flavor to it. It gives a different way in approach, it is a very beautiful approach. So, the Erdos-Szekeres theorem says the following.

Suppose you are given n numbers in a sequence like $a_1 a_2, \dots, a_n$, in some order. Now, this n happens to be a number which is greater than or equal to $r.s + 1$, for some positive integers r and s. Now, no matter what the numbers that you are given me. So, these are distinct numbers, no matter what order that you have given me these numbers you can show that the sequence contains either an increasing subsequence of $r + 1$ terms or a decreasing subsequence of $s + 1$ terms.

What is an increasing subsequence, you take the sequence as it is and just remove some of the elements from there, what remains is a subsequence. And if in this subsequence, every number in the sequence in the order whenever you go to the right, it increases, that is an increasing subsequence. Decreasing subsequence is exactly you take a subsequence, but it decreases each time.

Now, what the Erdos-Szekeres theorem says is that, you can find either an increasing subsequence of at least $r + 1$ numbers or a decreasing subsequence of $s + 1$ numbers, this is something that you can always prove. Now, how do you go about doing something like this? In the earlier one, suppose we want to use the first form that we studied, rather than the general form. Here we have, several things that you want to show.

You want to show either $r + 1$ or $s + 1$. General form allows this, but let us try to use the other form. How do you, can you use the previous form to do this? That requires some thinking. So, why do not you pause and think about it. Now, to do this, I am going to, look at the sequence and then do some analysis. So, $a_1, \dots, a\_n$ is given.

Now, I am going to show you that I can produce either a decreasing subsequence or an increasing subsequence of these many terms. So, you have given me this thing, I am going to look at the sequence and study it.

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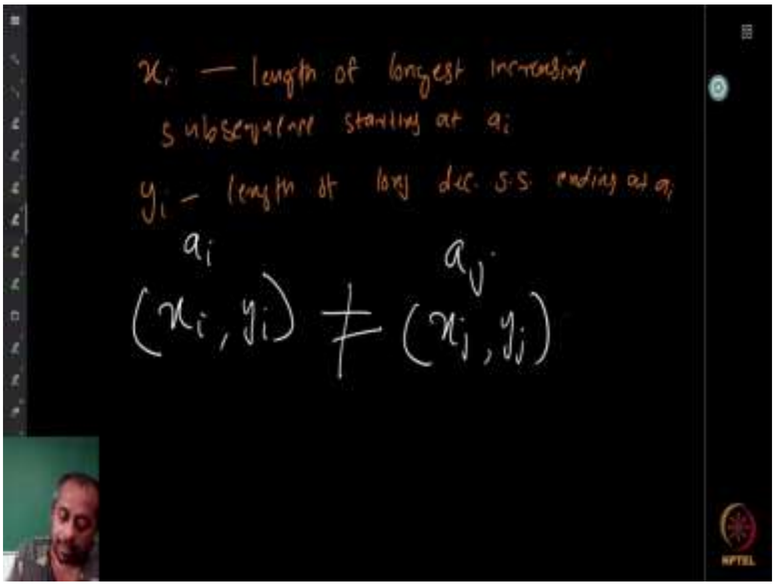

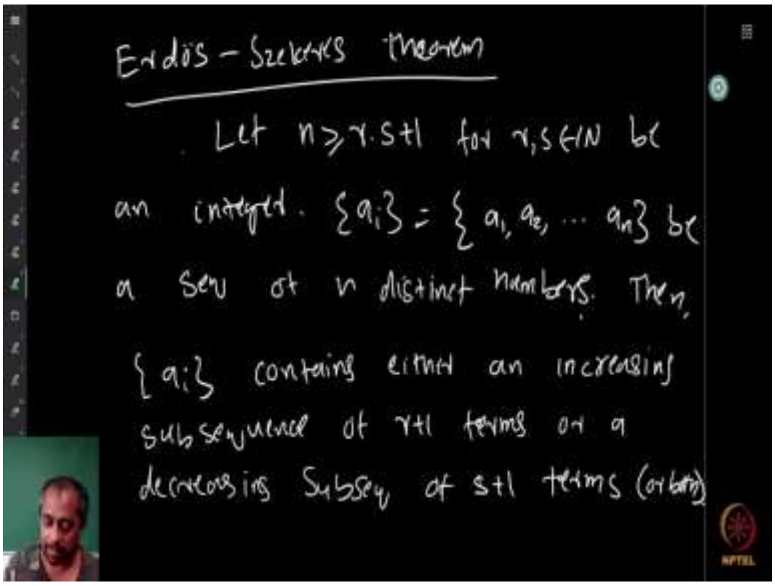

Now, what I am going to do is that I count the number of, or the length of the longest increasing subsequence that starts from a number. Suppose, if I look at the position, let us say, a_2. Now, I start from a_2 and look at what is the maximum the subsequence that you can create which increases. So, after a_2, I can only select numbers which are larger than it, then again, select only numbers larger than that, etc. So, I look at this and see how many I can select, this I can do, whatever I am going to make an argument. So, I am going to say that whatever is that number, that I will call as x_i.

So, x_i is the length of the longest increasing subsequence starting at a_i. So from a_1 this is x_1 , from a_2 this is x_2, from a_3 it is x_3 etc., from a_n it is x_n. So, you know I have this x_i. Then I also look at the longest decreasing subsequence that ends at a_i, now that is the clever part. So look at the decreasing subsequence, but not starting from a_2 but that ends at a_2, not starting at a_n but ending at a_n.

So, what is the longest decreasing subsequence, ending at a_i, so this is y_i. Now, what is the property of or the advantage of selecting something like this is that if I look at the number that we are deciding now. For corresponding to a_i I have this numbers (x_i, y_i). Now, suppose I select any other number let us say a_j and j different from i, I have(x_j , y_j). Now, if I look at x_i and x_j, maybe x_i and x_j are the same. That is possible.

Because x_i is the length of the increasing subsequence starting from a_i and x_j is the one starting from a_j, maybe from, whatever is the longest sequence starting from a_i is the same as the one starting from a_j, that is possible. So, x_i can be equal to x_j.

Similarly, y_i can be equal to y_j that is also possible. For any number the decreasing subsequence ending at a_i and a_j could be the same. Because after that everything is larger, then I cannot select anything. So, therefore, it can be the same. But what I know is that (x_i, y_i) and (x_j, y_j) as ordered pair can never be the same. Why is that?

See, if I select (x_i, y_i) that says that x_i is the length of the largest increasing subsequence starting from a_i and y_i is the length of the decreasing subsequence that ends at a_i. When I go to a_j what happens, suppose x_i does not increase, then y_j will increase, because the numbers are distinct. So, if x_i does not increase, that means that the number is going to be smaller, so therefore, y_j will increase.

Otherwise, if y_j does not change, then the x_i will change, one of them will change. Now, because of this, as ordered pairs they will never be the same. That is $(x_i, y_i) \neq (x_j, y_j)$ So, this helps to design a pigeonhole application.

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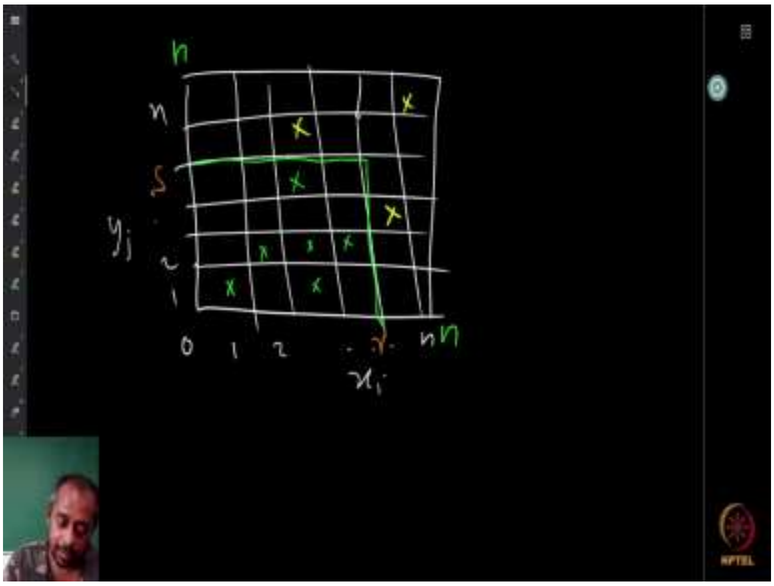
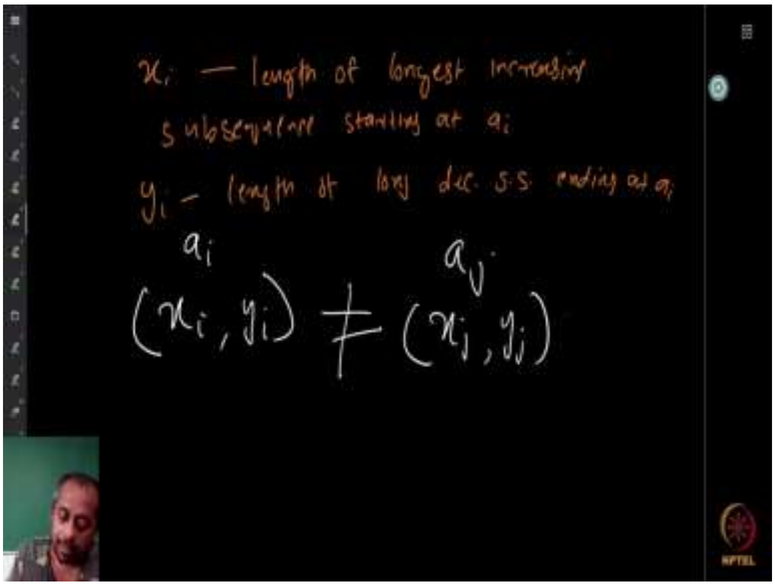
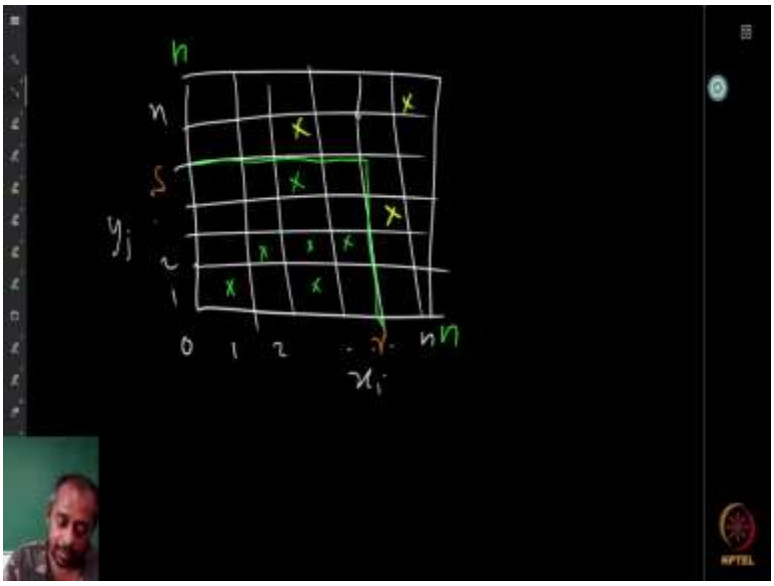




So, what you do is that, we know that these numbers x_i and y_j can never be more than n, because maximum number of terms is n. So longest increasing subsequence can only be at most n, decreasing also can be at most n. Finally, plot in the graph, like, I take an $n \times n$ square, then I am going to put the numbers here x_i, so this is the position, that is going to with i so that is like, 1, 2, etc up to n and 1, 2, etc up to n.



So this says that, this is where I am going to plot x_i's and this is where I am going to plot y_j's. So, how I am going to do that, well, I take x_i, y_j, for any number, and whatever is the number I am going to mark it. So this guy, this guy, this guy, whatever I am going to, so the corresponding numbers I am going to put here. The ordered pairs (x_i, y_j), if that pair is appearing, I am going to put a cross mark wherever it is.

Now, what I know is that $n \geq r.s + 1$. Let me look at the rectangle of say, s and r. So, If am going to look at the $s \times r$ sub rectangle here, this rectangle will contain at most $s.r$ boxes.

But $n \geq r.s + 1$. So, therefore, even if you fill up all these boxes with cross you will still have some cross that must appear outside this box, it must can be either here or maybe here or maybe here, wherever. Now, if the cross appearing here means that for some pair (x_i, y_j), the x_i had crossed r, which means that the longest increasing subsequence is greater than or equal to $r + 1$. If it was here, y_j had crossed s so therefore, the longest decreasing subsequence is at least $s + 1$.

And similarly, if it is here, both might have happened, the longest decreasing subsequence as well as increasing subsequence is larger than $r + 1$ and $s + 1$. So, these are the possibilities, and using the pigeonhole principle, the boxes as the pigeonhole and the ordered pairs as pigeons, we can show that Erdos-Szekeres Theorem holds. And this is a very beautiful and very ingenious application of the pigeonhole principle.

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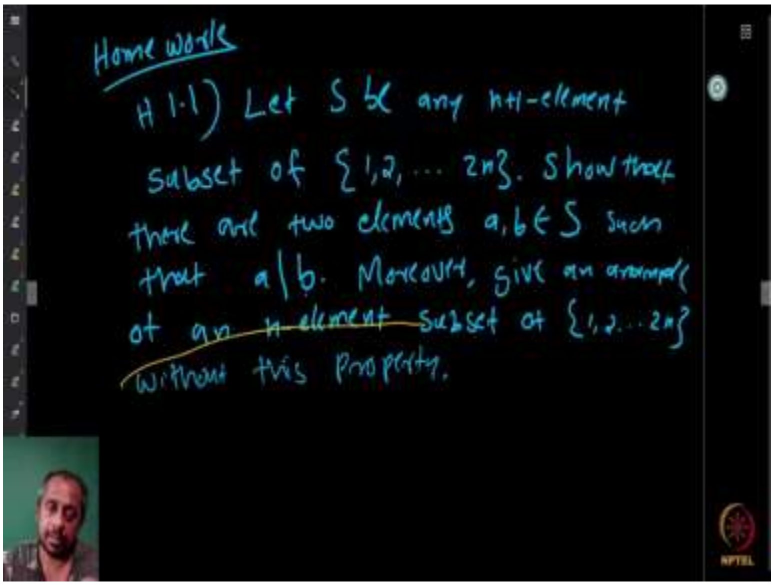

Now time for homework. I will give you some questions. As I mentioned earlier, you need to look at more questions from the textbook, I would recommend to go through all the questions and try to solve as many as you can. But at least try to do half of them, something like that. But here are the homework questions.

First question is let S be any $n+1$- element subset of the set $\{1, 2, \dots, 2n\}$. So we have an n plus 1 elements subset of 1 to 2 n. So, there is a 2 n element set exact precise set 1 to 2n.

Now show that there are two numbers a, b in S such that a divides b. This is a very classic result of Erdos. I want you to think about proving this rather than trying to look it online, you will find it very easily online. You do not bother with that. Try to try to find out a solution yourself. And it is fun, believe me it is going to be challenging, but it will be fun. Now, so, this says that any $n+1$- element subset of the set will contain 2 element such that, one divides the other.

But what is even stronger one can show is that if you take some n -element subset, that need not be the case. So, give me an example of an n -element subset of $\{1, 2, \dots, 2n\}$, where you cannot find 2 numbers with this property. That is the first question.

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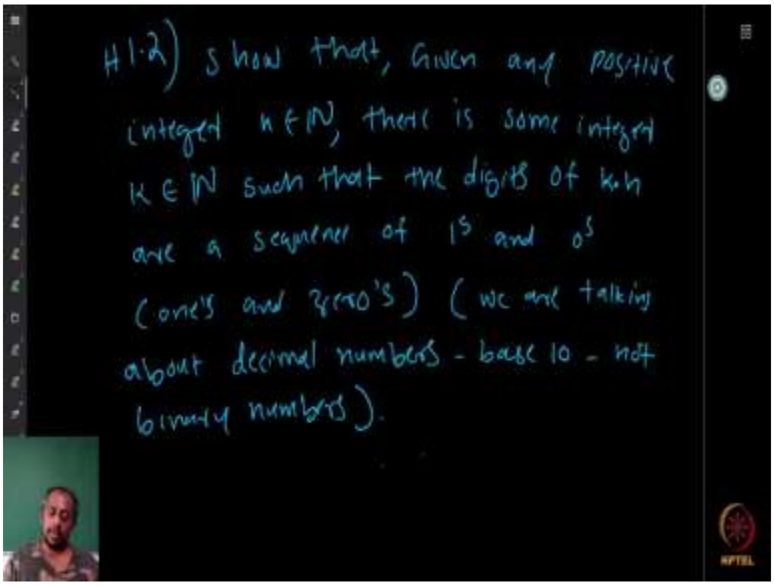

Then show that given any positive integer n, there is some integer k such that the digits of $k.n$ are a sequence of 1's and 0's. We are not talking about binary representation because any number can be written as asequence of 1s and 0s, but not that, in the decimal system itself I can find a multiple. Let us say that you take 37, you know that 111 is a multiple of 37. So, this way, but maybe like 117 1110. If you want both 1s and 0s you can add 0s of course.

So, I want you to show that there is a, there is always a multiple. No matter what number you give me, you can always find a multiple with all the digits are just 1s and 0s.

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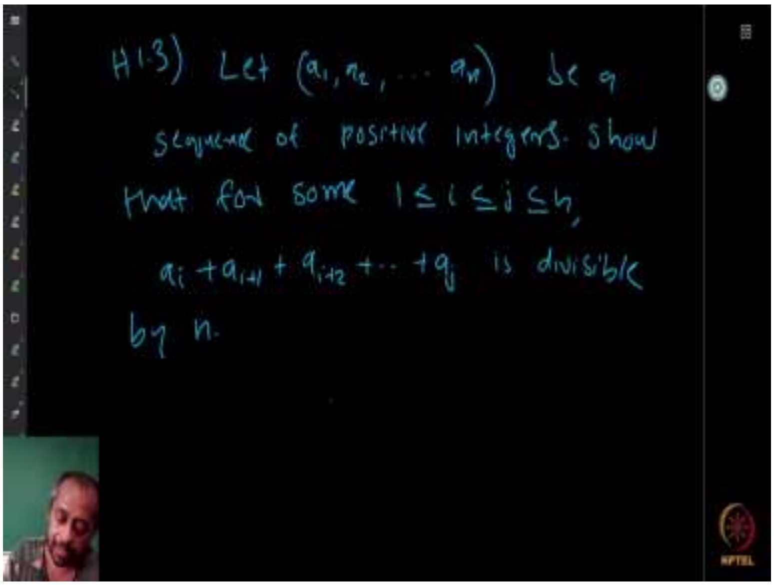

Another question, that is, given a sequence $(a_1, a_2, \ldots, a\_n)$, all positive integers. Show that for some $1 \le i \le j \le n$, $a_i + a_{\{i+1\}} + a_{\{i+2\}} + \cdots + a_j$ is divisible by n. So, there is a sub sequence of consecutive terms whose sum is divisible by n. So, that is the third question.

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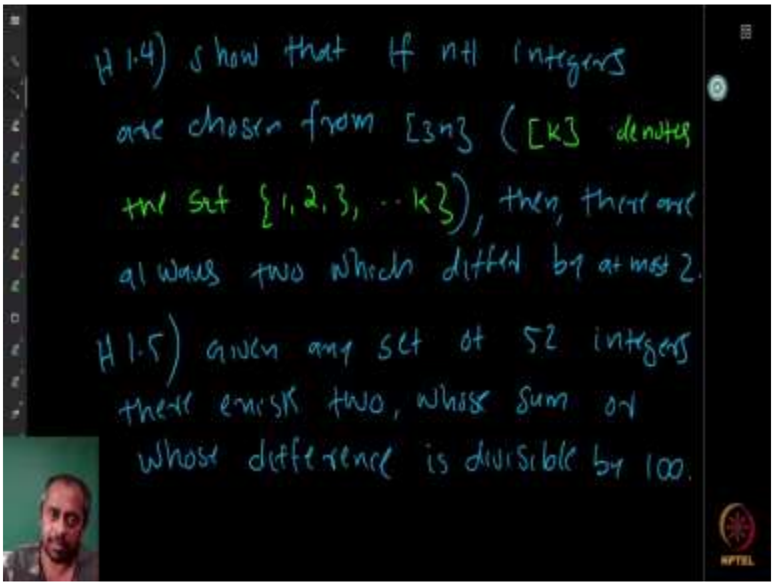

And the 4th question is that show that if $n + 1$ integers are chosen from the set $\{1, 2, \ldots, 3n\}$, then there are always 2 which differ by at most 2.

And the next question is that given any set of 52 integers there exists 2 of them whose sum or whose difference is divisible by 100. So, now its slightly different, either the sum or the difference is divisible by 100. So can you show this? It needs a little more thinking than the previous one, it is not very difficult.

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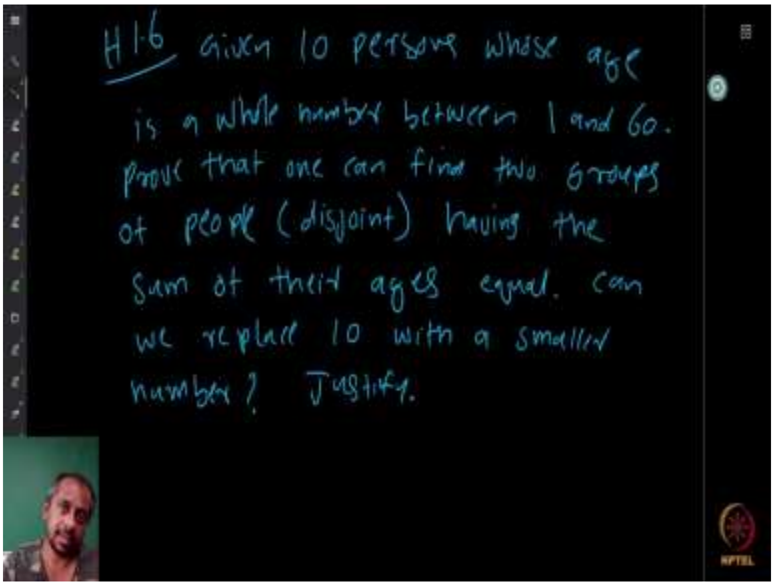

And sixth question, given 10 persons whose age is a whole number between 1 and 60. Prove that one can find 2 groups of people disjoint having the sum of their ages equal. Can we replace 10 with a smaller number? Yes or no and justify whatever you say. It is yes, you need to give justification, it is no, you have to again give justification.