The twelvefold way
Professor. Doctor Narayan N
Department of Mathematics
Indian Institute of Technology, Madras
The twelvefold way

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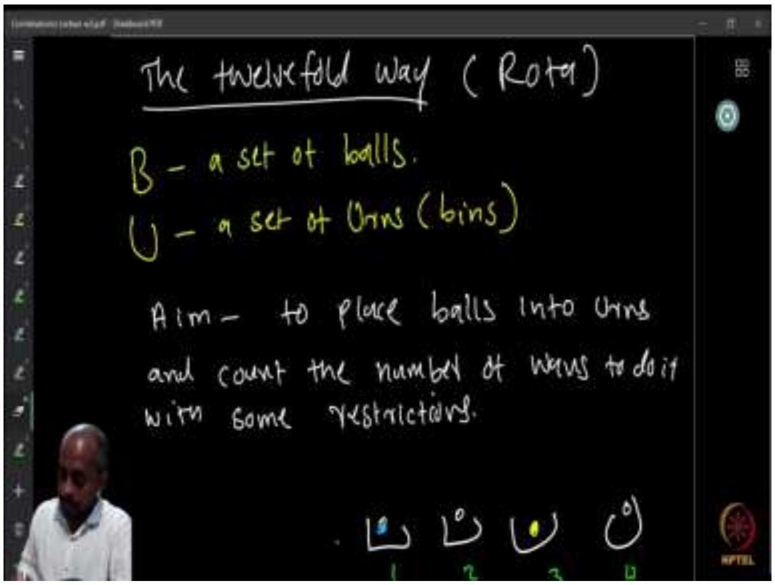

So, what we want to look at is a way to collectively, visualize the several of the questions that we already solved in the same framework. So, this particular way of looking at several counting problems in the same framework was done by or at least to us that is what is known is introduced by Gian-Carlo Rota. So, who was the advisor of Stanley and then in his work, when Stanley has written about this in detail and we are going to just look at that.

So, what we have is the following setting we have a set B of balls. Then you have a set U have urns or bins or boxes. So, you have a set of boxes and you have a set of balls. Now, what you want to do is to distribute the balls to boxes or place the balls into the urns and then count the number of ways to do it. So, that is all we are going to do.

So, we are going to take the boxes and balls and then put each ball into some of the boxes and then we can say like , there are some restrictions that we want to look and this is the general setting. What are the possible things that you can do in this, what kind of restrictions you can do? Let us look at these restrictions one by one. So, one of the things that you can think is that let us say if you have your boxes and then you have balls, so you can say that there is no restriction at all.

We will say that, you have to just distribute the balls to boxes, you can do whatever you want, whichever way you want you can do it. Then you can also say that, I do not want you to do anything like that I want you to put at most one ball into a box. So, a box cannot contain more than one ball, that is one restriction that you can put. Another restriction, that you can put is that, every box should have at least one ball.

In other case you can say that every box must have at most one ball. So, the first case was the at least one ball then you will say that you can have 0 but you must not have more than 1. So, these are some kind of restrictions that you can put. Then what other restrictions that you can put I mean other conditions that you can put ? For example, you can say that I have labelled the boxes, the boxes are different now.

So, boxes are labelled as box 1, box 2, box 3, box 4 etc. Then I can ask the same questions with the labelled boxes. That is, if the balls are going to the boxes, then how many different ways are there if I allow at most one, at least one, or arbitrary. Then you can say that, my balls are also coloured and they are not identical. So, then if they are colored then, how many different ways I can do?

Then you can say that, the balls are colored but the boxes are not labelled, they are identical. So, if that is the case, then what you can do? So, if the boxes are not labelled and then the balls are labelled, then what you get? So, these kind of different questions you can ask and you can see that there are altogether 12 such possibilities.

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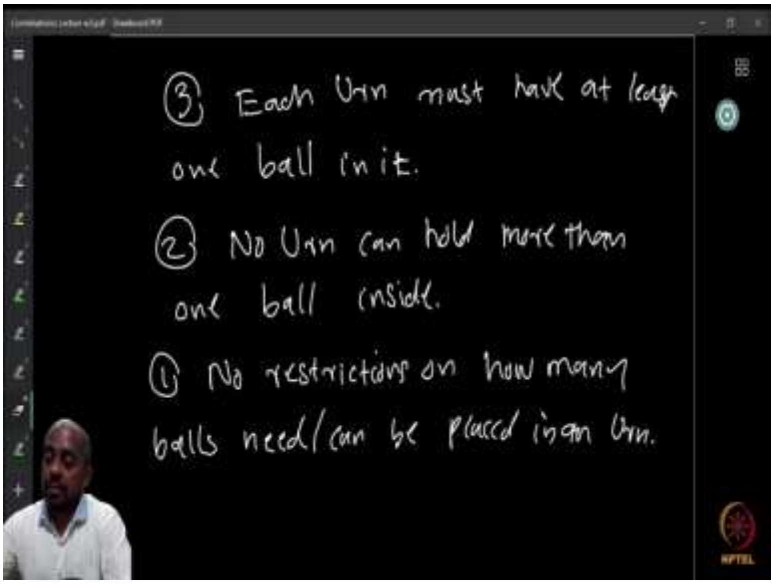

So, let us list them. One is that like each urn or boxes must have at least one ball in it, second is that no urn can hold more than one ball inside and there is no restriction on how many balls can go to a box or an urn.

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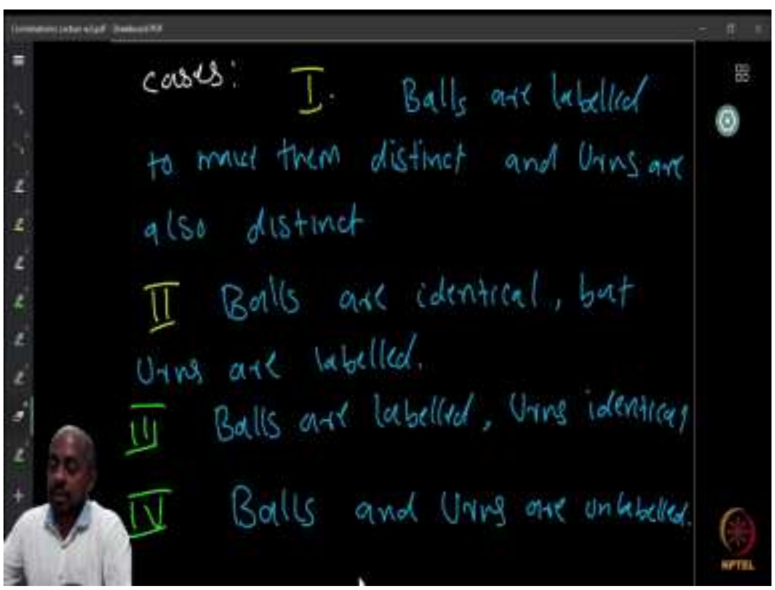

So, then what we said is that, the different cases that we can have is that the balls are labelled to make them distinct and the urns are also distinct. So, both the boxes and the balls are labeled. Then second case is that the balls are all identical, but the urns are labelled and third case is that ball are labelled and urns are identical. Then balls and urns both are unlabelled, that is, all the balls are identical and all the boxes are also identical.

So, in this case, what you can see is that, for each of the four cases, you can ask the previous three situations? So, there are $4 \times 3 = 12$ possible ways. So, this is called 12-fold way.

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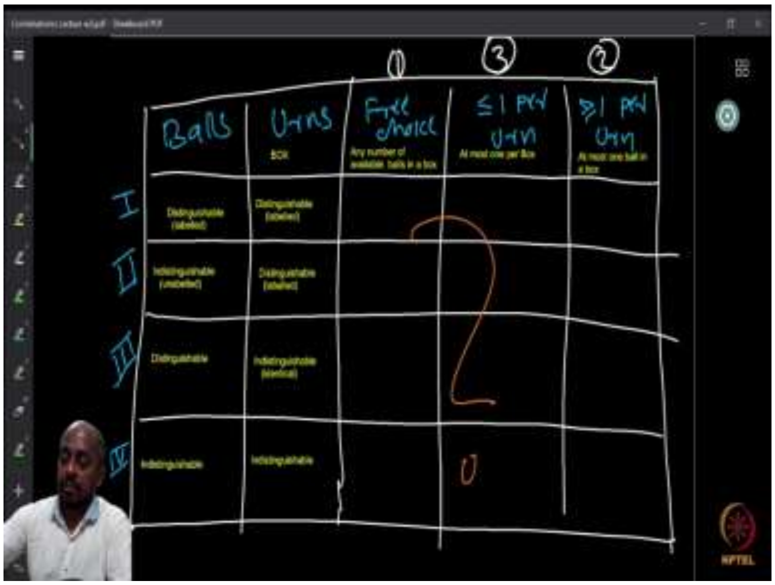

Now, let us try to put it into a nice table. So, you have the balls, the urns, then the cases that are distinguishable or labelled and the indistinguishable or unlabelled. Then these are the 4×3 choices, where the 3 choice are, (1) we can have any number of balls in a box (2) at most 1 per each box is allowed and, (3) at least 1 per each box is allowed. So, then you want to find out what are the entries in each of these boxes?

These are the 12 cells you need to fill and that will give you the answer to that 12 fold way question. So, what are these numbers? So, I want you to stop this and then think about this and fill up the number in yourself, because we are already solved all these questions. So, each of these questions we have we have solved, at least enough material we have covered it using which we can solve all these questions.

So, think about this and try to work out the entries and then once you fill all the entries see whether they match with the observation that we are going to find.

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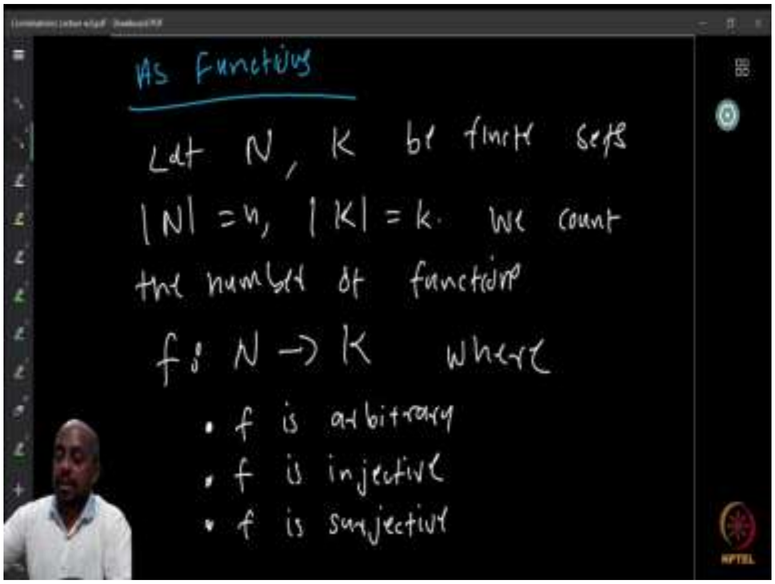

So, we can also look at this question in terms of functions, as many of you might have already immediately noticed. So, let N and K be finite sets, where $|N| = n$ and $|K| = k$. Now, we want to count the number of functions let us say $f: N \rightarrow K,$ where first case is that f is arbitrary, or f can be any function.

Then we say that we insist that f must be an injective function, then you say that f must be a subjective function. So, these are three possible restrictions that you can put.

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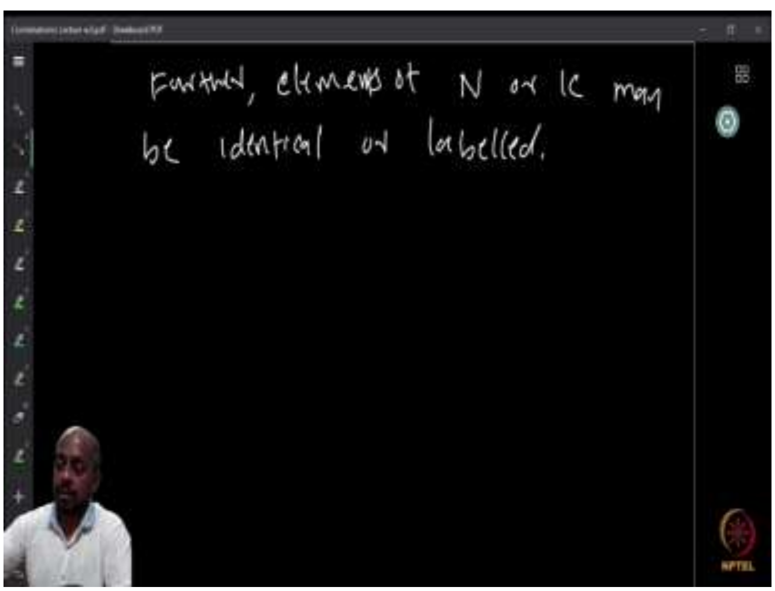

And then similarly, we can say that the elements of N or K may be identical or they are labelled to make it distinct. So, these are the four different scenarios. So $4 \times 3 = 12$ again.

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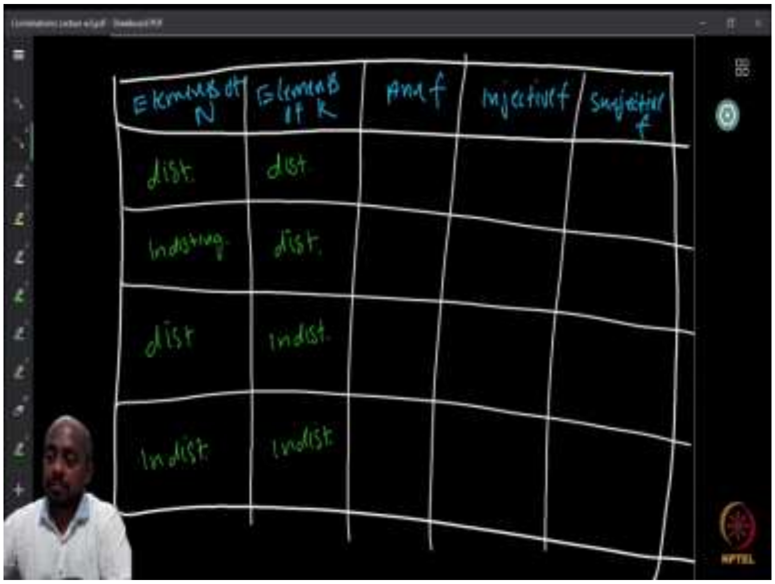

So, what are the cases here? Elements of N elements of K, they are distinguishable or indistinguishable. Indistinguishable means that there is no label they are all identical distinguishable means that they are labelled they are all distinct. And then you can say that any function is allowed or injective functions are only allowed or surjective functions are only allowed. So, these are the 12 case and these precisely correspond to the previous cases of balls and bins.

So, there is no difference it is just another way of looking at it. So, we are going to count this or find out this cells and by looking in one of these cases, we will either look at it as a function or as a balls and boxes question and this will give you some different kinds of intuition, at least I hope so. So, let us begin.

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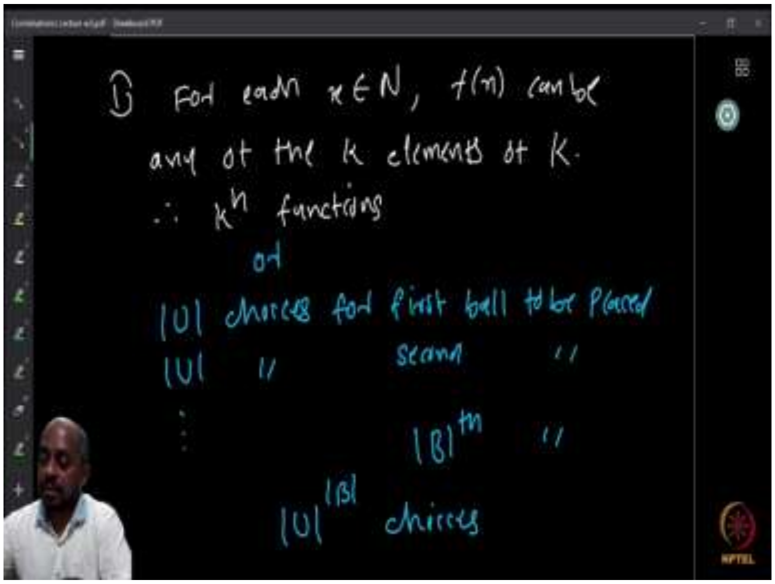

The first case. So, what is the first entry here that we want to fill? So, that is, we have distinguishable elements, the distinct elements in N and labelled elements in K also and then we want to say that any function. If the function is any function then obviously, we have already done this. It is one of the first questions that we did. So, this is the total number of functions, any function is allowed.

So, how many functions are there from an n- element set to k- element set which is k^n. Because $f(x)$ for any element x can be any of the k possible element because any element can be mapped to any of the elements in K. So, each one has k choices. Since there are n elements in the domain we get k^n. And in the terminology of urns and balls, we have, $|U|$ choices for the first ball to be placed and $|U|$ choices for the second ball to be placed and etc. Because, any possibility is allowed. You can put as many as you want in any of the urns.

And so, therefore, the ball has that many choices. If you are going to put the first ball you can say that it can be any of the U, second ball again it can be any of the U similarly, for all the balls. So, therefore, $|U|^{|B|}$ choices are there, which is what we were doing. So, the first entry we have filled, which is k^n.

Now, let us look at the next entry in the same row, the second entry which says that, f must be injective. So, then what can you say?

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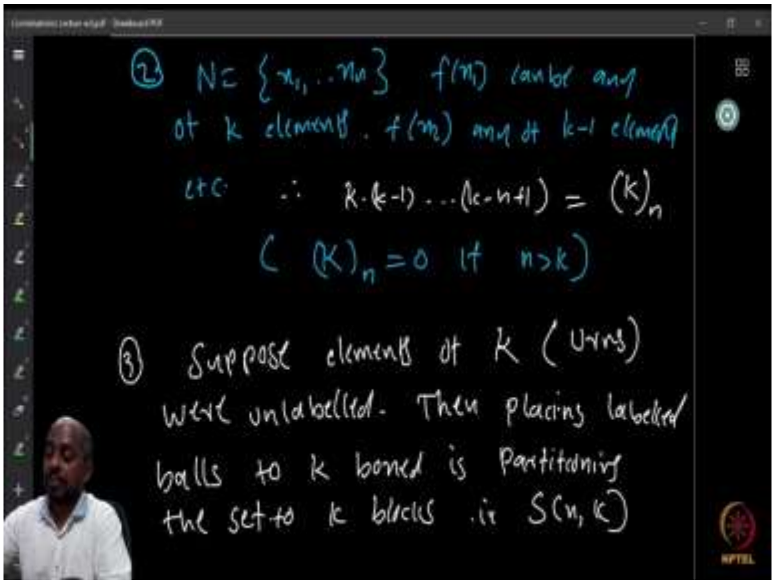

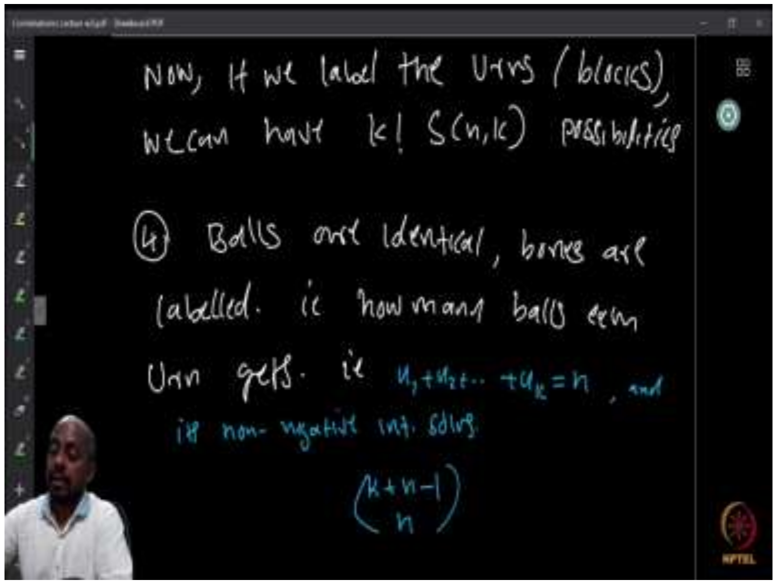

So, let us say that $N = \{x_1, x_2, \dots, x_n\}$. Then if you take the first element x_1, $f(x_1)$ can be any of the k elements. But if you take x_2 once you fix an image for x_1, then x_2 cannot have that image, because we want it to be injective. So, therefore, $f(x_2)$ must be any of the remaining $k-1$ element. So, similarly, you continue you will get $k.(k-1)\dots(k-n+1) = (k)_n$, which is the n permutation of the k element.

Now, when $n > k$, we will see that this quantity is going to be 0 because you will have $k-k$ coming there. So, therefore, this will cancel out and therefore, it will be 0. So, if $n > k$ that will be 0, but otherwise it will be $(k)_n$. Now, let us also look at the third case where you want the functions to be surjective.

Now, to count these surjective functions, let us assume that initially the elements of K or the urns are unlabeled. So, we were looking at the labeled case but let us look at the unlabeled case. Suppose they were unlabeled. Now, if the boxes were unlabeled, then placing the labelled balls to k boxes is basically partitioning the set N to k blocks because we are just saying that they are unlabeled so, which box contains what is not important.

But, each box how many elements or which elements are there, that is the only thing that makes the difference. So, therefore, is basically partitioning the set N into exactly k blocks, if the boxes were unlabeled. Now, if the boxes were labelled then it also matters that which box this has gone. So, basically the order of the partitions are also important, which means that order of the blocks are also important. So, once you have the blocks you can permute them in any way. So, there are

k blocks and $k!$ ways to permute them. Therefore, we have $k!\,S(n,k)$ possibilities to do, where $S(n,k)$ is defined to be the number of ways to partition the n-element set into k blocks.

So, we have filled up that entry. We can fill that entry now with $k!\,S(n,k)$ and the fourth entry which is the entry below this is going to be the case where the balls are identical and the boxes are labelled. So, the elements of N are identical, the balls are identical and the elements of K which correspond to the boxes are or urns are distinguishable or labelled.

Now, in this case what matters is that how many balls each urn gets, because the balls are all identical. So, which ball goes where it is not important, but only how many balls are going to each of the labelled urns. So, therefore, you can think of the urns as the variables which take the values like which is the number of balls. So, this is basically like $u_1 + u_2 + \cdots + u_k = n$, where u_i is the number of balls that the i^{th} urn gets.

The sum come to n because we are distributing n ball to the k urns and we already know how to solve this we have done this that is a non negative integer solutions to the equation which is $\binom{k+n-1}{n}$. So, we have a solution for that.

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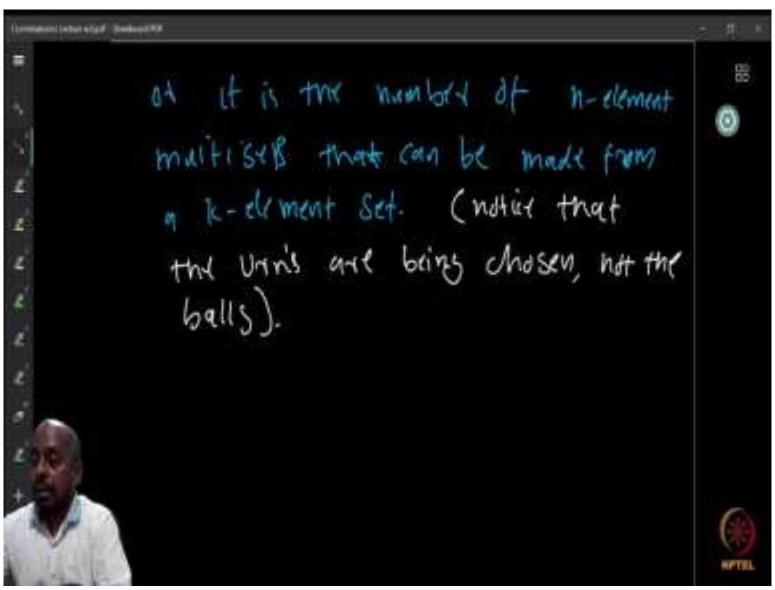

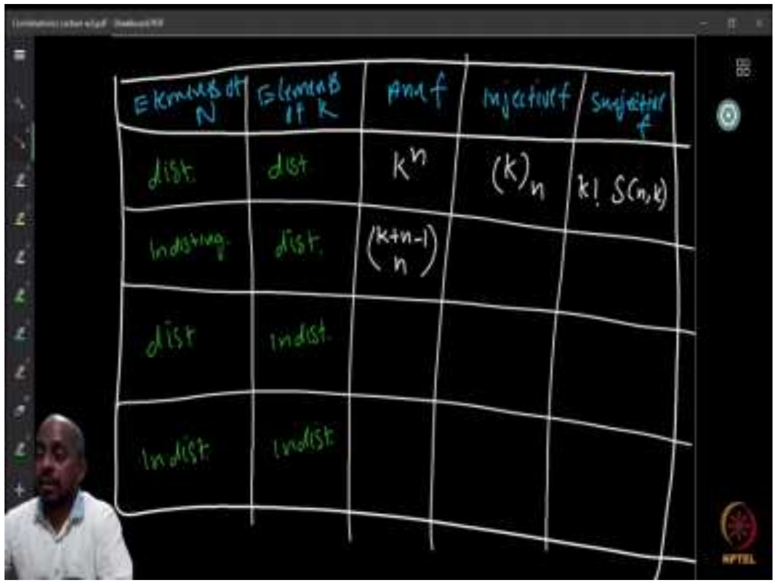

Now, this is also the number of n-element multisets that can be made from a k- element set. Because instead of looking at, choosing the balls to be placed, you choose the urns and then decide how many balls go to each of the urns. So, basically, using the urns can be looked at as the number of ways of forming a multiset from a k- element set. The k-element set is the set of urns, from this you want to select some copies of each of the urns so that you get an n-element multiset which is the total number of balls.

So, therefore, you can think of this as the number of ways of selecting an n element multiset from a k element set that is also $\binom{k+n-1}{n}$. So, we already saw this both notions for this parameter and this matches. So, therefore, we have filled up the first four entries.

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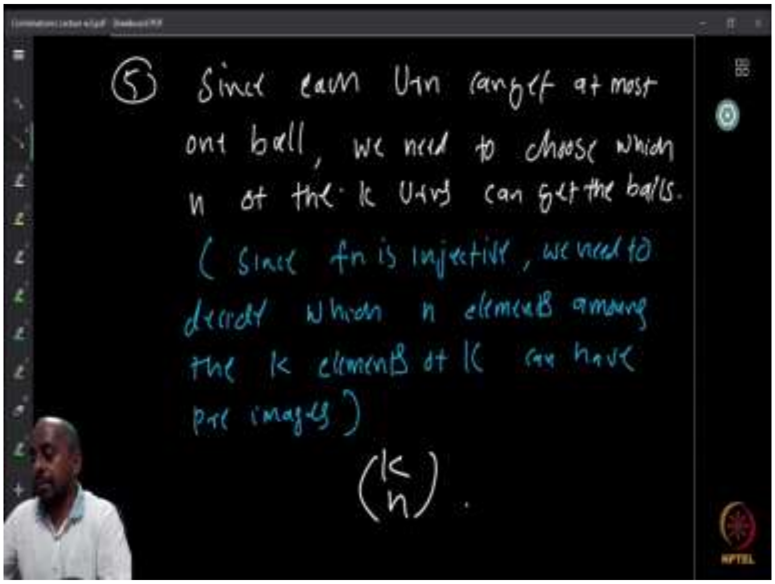

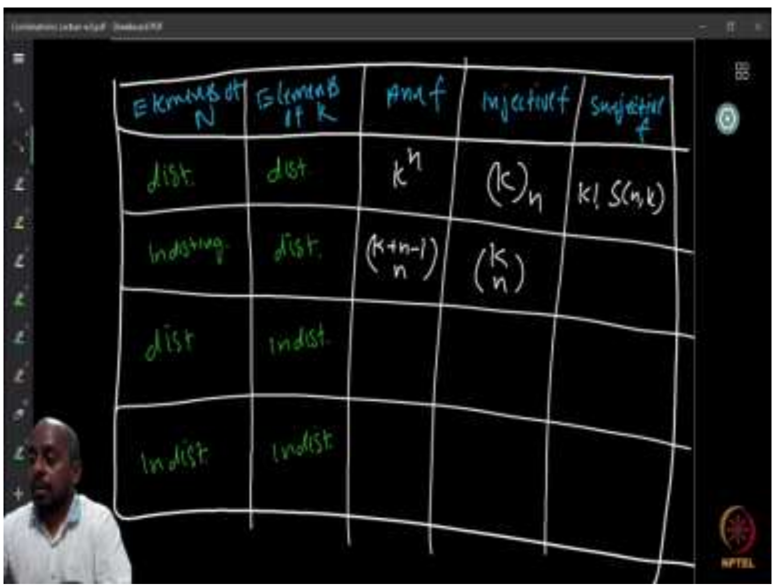

Fifth case which means that we have injective f. Injective f means that, you are saying that you have at most 1. Now, this means that each urn can get at most 1 ball. So, we need to choose which n of the urns can get the balls. You are saying that you have to distribute the unlabeled balls to distinguishing bins, which mean that each bin or urn. So, each urn now can get at most one ball.

So, this means that you have to select which of these which of the n urns are going to get the balls, because, each one can only get one and the balls are identical, so, it does not matter. So, you just choose an n-element subset of the total number of k-element subset which are the urns that are available. So, this function is injective you need to decide which n elements among the k elements of K, can have preimages.

So, that is what choosing n of these urns from the total available k urns. So, that is the solution. So, we have $\binom{k}{n}$. Now, what is the next entry? So, if the function is surjective instead of injective, then what can you say? This means that each box or each urn can get at least one ball inside.

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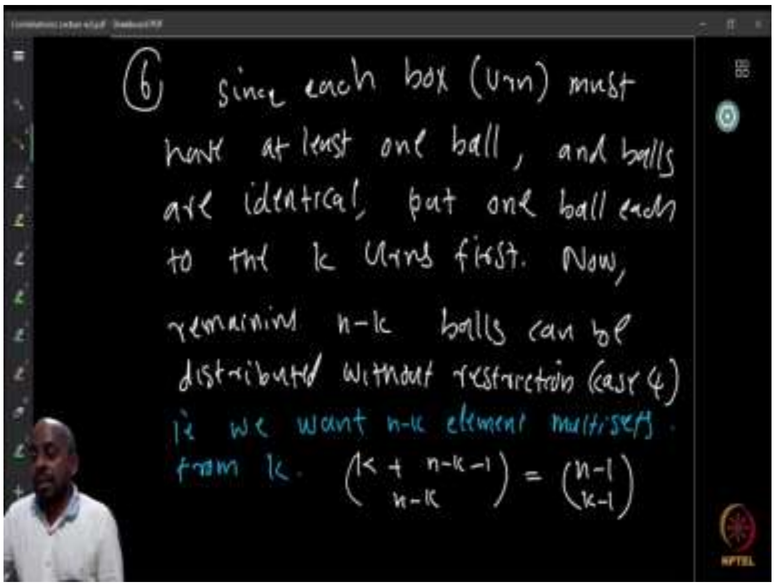

So, you should have at least one ball, because it is surjective. So, since each box must have at least one ball and you know which is identical to any other ball. So, what we can say is that, let us put one ball each to the k urns first. So, that, we are, and they are identical. So, I can just distribute them. And then now, once I do this, then my question is now reduced to a previously solved question.

I have made sure that it is going to be surjective by selecting one ball for each of the urns. So, every box must have at least 1. So, I put 1 in each of them. Now the remaining whichever way I am going to distribute the function has already become surjective. So, how many ways I can do this? The is the different ways I can do this is going to be the number.

Because this one is precondition that I need to fill up each of the urns with at least one ball. So therefore, once you take away, so k balls are now used to fill up each of the k bins. So, you have $n - k$ remaining balls, and that can be distributed, without any restriction and distributing without restriction that is basically something that we already did in case for that we want $n-k$ element multi sets from k-element set.

So, we have $n-k$ balls are there and then you are going to put it into $n-k$ of these things. And then that is basically choosing the urns. So, basically choosing an $n-k$ element multiset from the available k of the urns. And this is precisely what we did in case 4. Case 4 was precisely doing this, that you had identical balls and boxes are labelled, and then how many ways you can do, precisely that is the way to do it.

And in this case, it happens to be distribution of $n-k$. Because we already use k to fill up the boxes with one ball each. So, $\binom{k+n-k-1}{n-k} = \binom{n-1}{k-1}$. Because, we start with the k element set and then we want to make $n-k$ element multisets and the value on the top becomes $n-1$ and then choose $n-k$ will become also $k-1$ when you take the compliment. Therefore, you get $\binom{n-1}{k-1}$.

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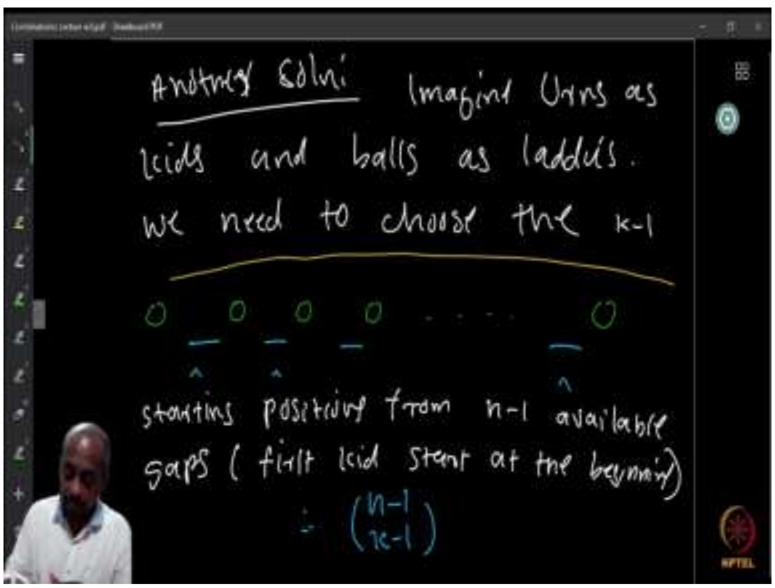

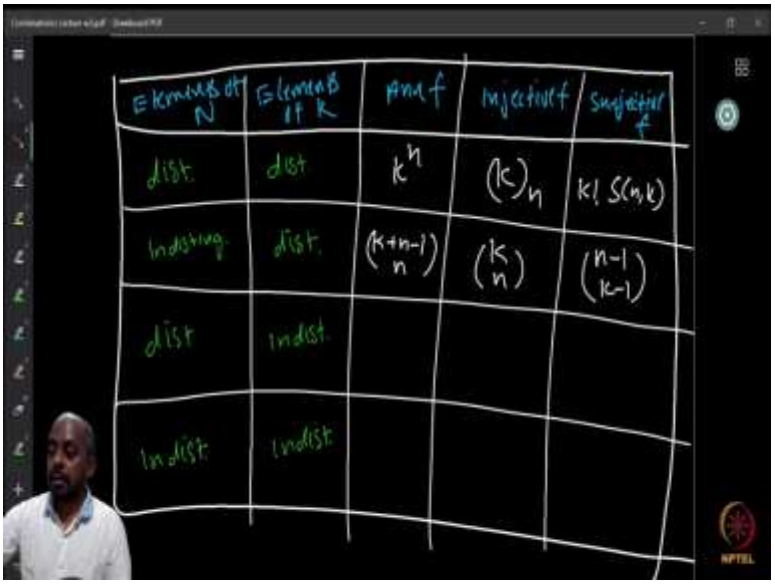

So, another way to look at the same is once you see that it is $\binom{n-1}{k-1}$, we get this idea. So imagine you have this urns or the boxes the urns are as kids and the balls as the laddus we already solved this question laddus and kids. So, we want to distribute the laddus to the kids so, that everybody is happy, everybody gets at least one and, because that is what we want when we want surjections every.

So, basically the laddus has to be distributed to the kids who are the urns. So, how many ways we can do this, we did this already, but let us review that. So, we have placed the laddus in a line we have this laddus in a line and then we let the kids go from the beginning and then pick some laddus and then stop and then continue with the next person. So, the first child always starts at the beginning position.

So, there were n laddus and $n-1$ gaps. We have to decide where each of the remaining, the remaining $k-1$ kids are going to start. That is we will choose $k-1$ positions from the $k-1$ available gaps. So, that is $\binom{n-1}{k-1}$. So, this is another way to look at the same question. So, we have filled the two first two rows now, the remaining two. So, there are 6 is done and 6 more to go.

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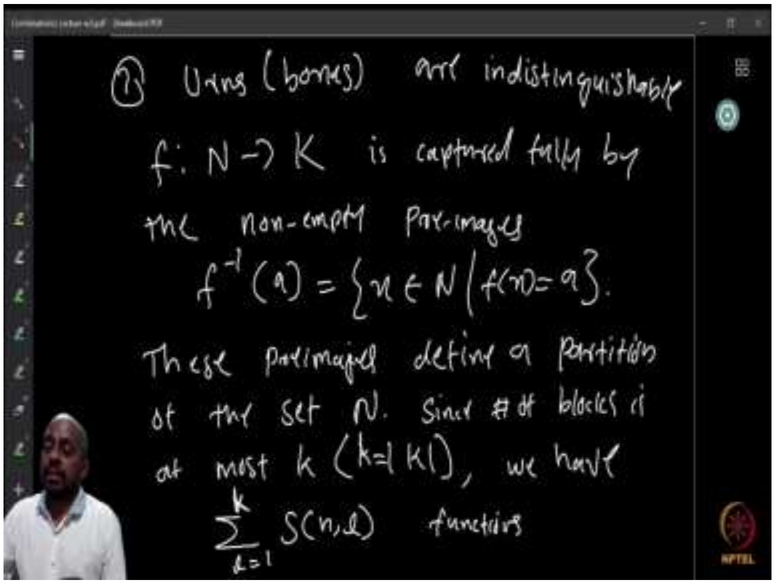

So, what is the next entry? So, the seventh entry is that you have labelled balls colored balls and you have identical boxes. So, since the boxes are identical, so if you look at the function going from N to K, so this function is captured fully by the non empty preimages of the elements of K. So, you take the elements of K and then look at the non empty preimages, like some of them may not be mapped to anything because the function is arbitrary.

But, if you take all the preimages of the elements you are going to get the set N. But the set N is now, if you collect them with respect to the its image what we are going to get is basically a partition of N. So, this basically defines a partition of the set N and the number of blocks is at most

k because we are only having at most k boxes. What we are going to see is that, you are looking at different ways of partitioning the set N into at most K blocks.

So, that is $\sum_{l=1}^{k} S(n, l)$. So, we get these seventh entry. So, what is the next entry which is f is injective. You have distinguishable balls and the identical boxes and the injective function.

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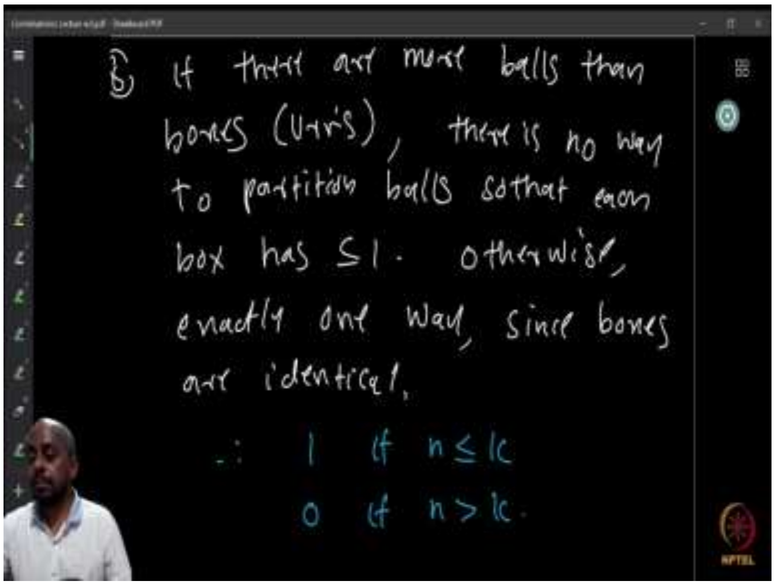

Now, if the number of balls are more than the number of boxes, there is no way to find an injective function, because the cardinality of N is greater. So, therefore, we have this injective functions only if the cardinality of N is less than or equal to the cardinality of K. And how many are there?

So, if there is only one way because the boxes are all identical. You are going to partition the balls so that each box has less than or equal to 1. So, if you are going to put less than or equal to 1 to the boxes then there is only 1 way to do it, you just put the balls to the boxes that is it. So, you just distribute them and then that is it because the balls are unlabeled there is no other way to distinguish. So, the answer is that it is 1 if $n \leq k$ and is 0 if $n > k$.

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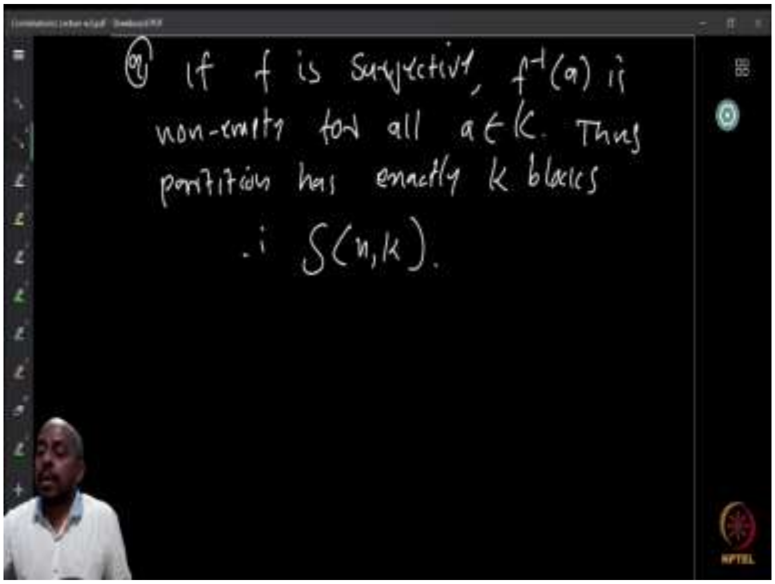

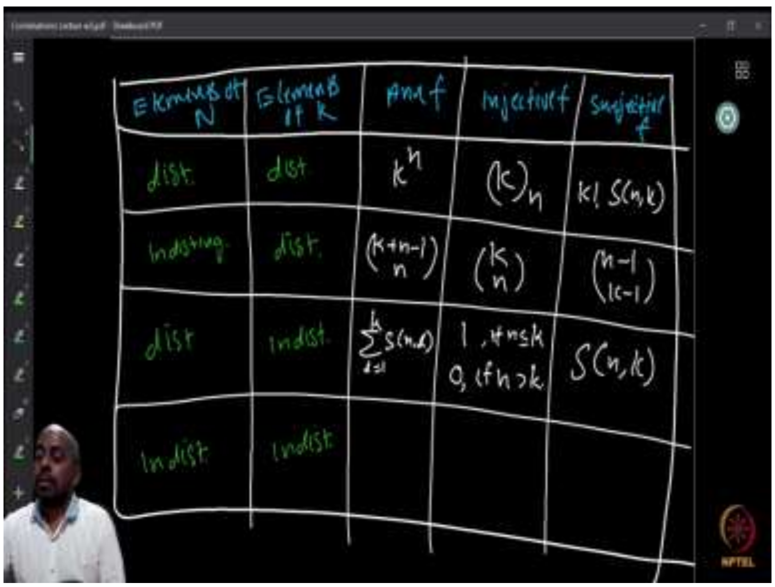

The ninth entry is that if f is surjective. So, if f is surjective, then we observe that $f^{-1}(a)$ is a nonempty for all $a \in K$. So, thus the partition has exactly K blocks. Because each of the element has a preimage and there are k elements in K. So, therefore, there is exactly k blocks each preimage defines a block that we saw earlier. So, therefore, there is exactly $S(n, k)$. That is, by definition we have done the number of surjective function from an n- element set to a k- element set.

So, this count the number of surjections when the when the boxes are identical. Now, what happens if the elements of N are indistinguishable and the elements of K are also indistinguishable this means that all the ball are identical, all the boxes are identical. So, how do you count in this case?

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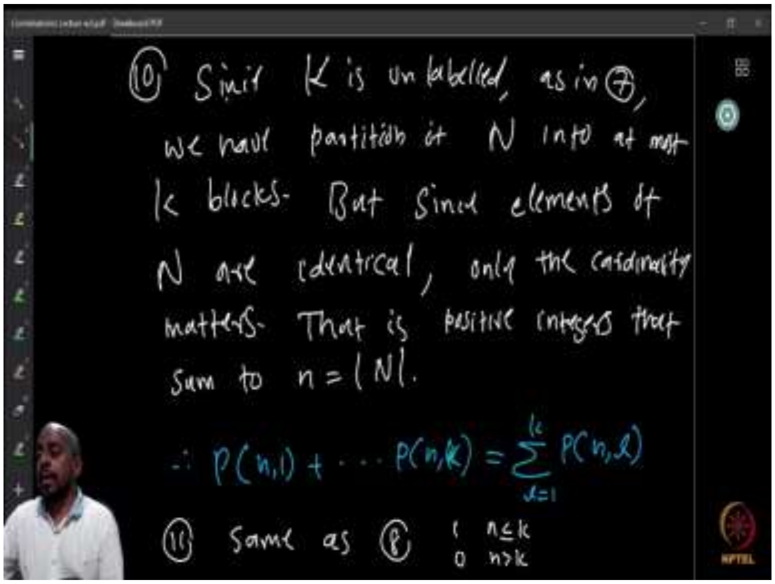

Since the urns are unlabeled we can partition N into at most k blocks. But now, the elements of N are basically identical. So, it does not matter what are the elements because they are all identical, but it matters how many are there. So, therefore, basically it is partitioning the cardinality the number of elements to block. So, that is basically integer partition, number of ways of partitioning it, but then you can have at most k blocks because the function is arbitrary.

And therefore, as in the previous case we have $\sum_{l=1}^{k} P(n,l)$ where $P(n,l)$ is the number of ways of partitioning n to exactly l blocks. So, integer partition and what is the eleventh question well eleven is as we said in the case of eight when the boxes are unlabeled, there is precisely one way to do it when $n \leq k$ and 0 is if $n > k$.

The same thing holds here because, again it does not matter whether the balls are labelled or unlabeled even they are unlabeled, there is precisely one way to do it, if $n \leq k$ and otherwise no way to find an injection.

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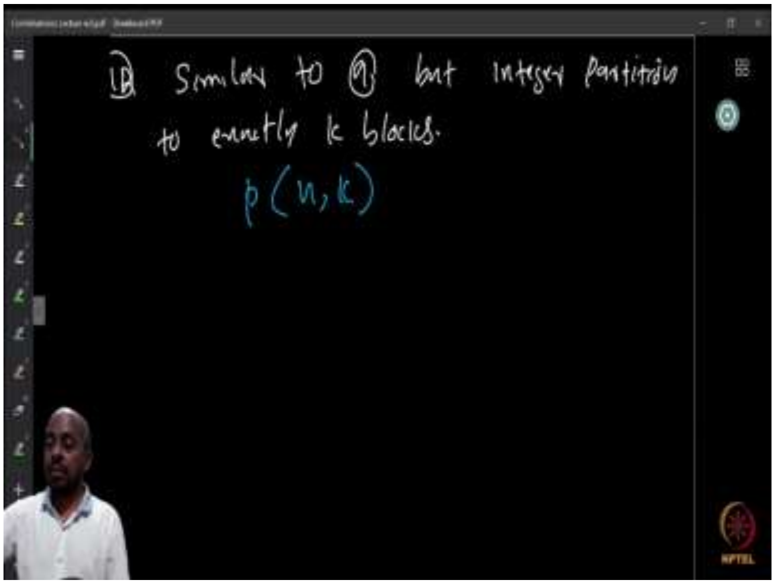

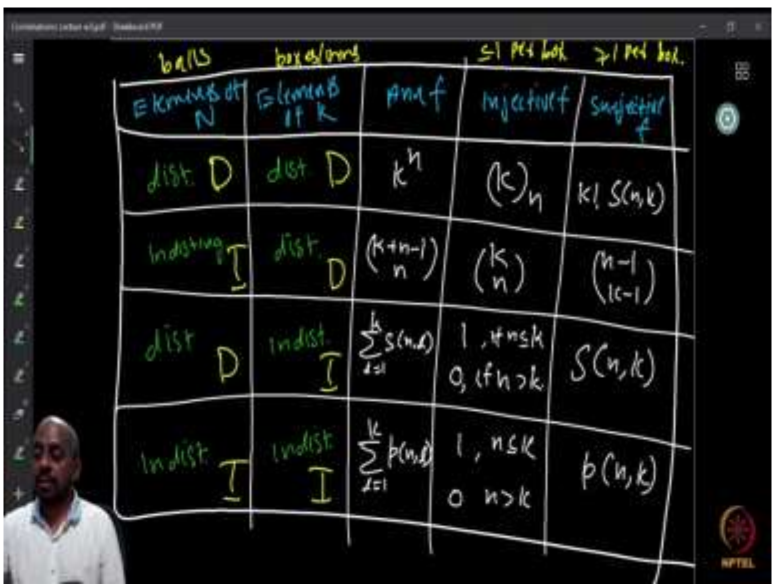

And the final question, question number 12 is again similar to 9, but now integer partitioning to exactly k blocks because the function is a surjection. So, we have $P(n, k)$. So, we have the full table here now. So, we have the balls which are distinguishable or indistinguishable, the boxes which are indistinguishable or labeled, then you have any function it means that unrestricted distribution you have at most one per box which is injective functions, you have at least one per box which is surjective function.

So, these are the twelve cases and we have filled up all the entries and each of these we have done in our previous classes. So, I think that is basically like collecting so many different counting questions into the same framework, I am distributing balls to bins, which is a nice way to look at the it. That is the reason we look at it.

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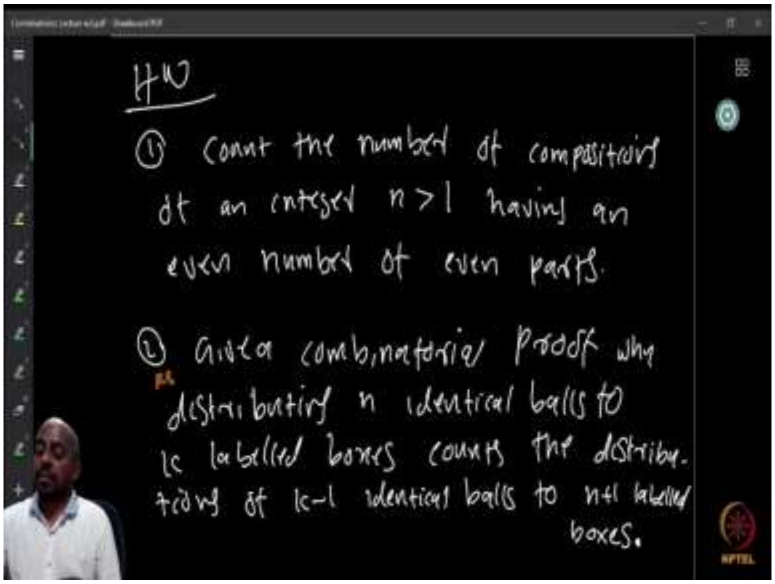

And a couple of homework question. So, one question is to count the number of compositions of an integer let us say $n > 1$ having an even number of even parts. Then the second question is give a combinatorial proof why distributing n identical balls to k labelled boxes is the same as, basically the number of distributions of $k - 1$ identical balls to $n + 1$ labelled boxes.

So, distributing n identical balls to k labelled boxes and distributing $k - 1$ identical balls to $n + 1$ labeled boxes apparently is the same, same in the sense that, they have there is a bijection between, they account the same number. So, the cardinalities are the same which is kind of interesting to see. Why should they be the same?

So, can you find a combinatorial proof ? Of course, if you want to give an algebraic proof, I think it is very easy there is nothing there. But we need a combinatorial proof why they are identical. So, that is it. We have completed the homework parts for this question and then let us see in the next class.