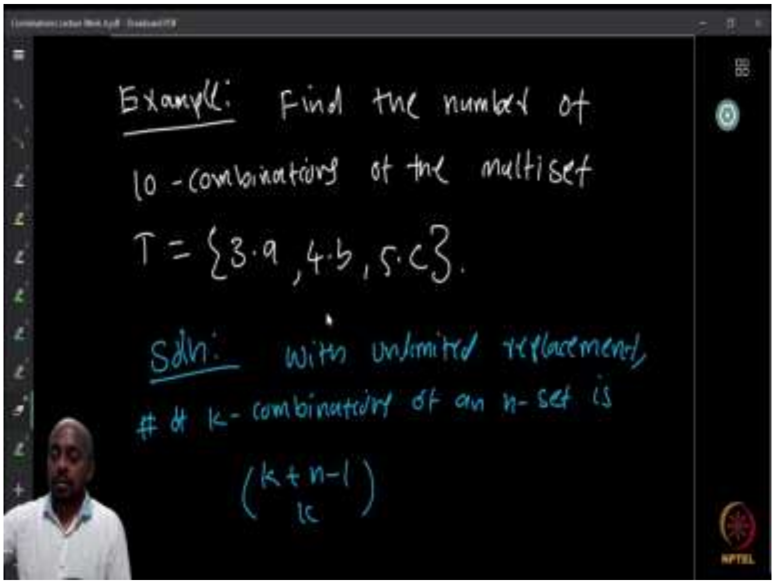
Combinatorics
Professor. Narayanan N
Department of Mathematics
Indian Institute of Technology, Madras
Applications of PIE

(Refer Slide Time: 0:14)

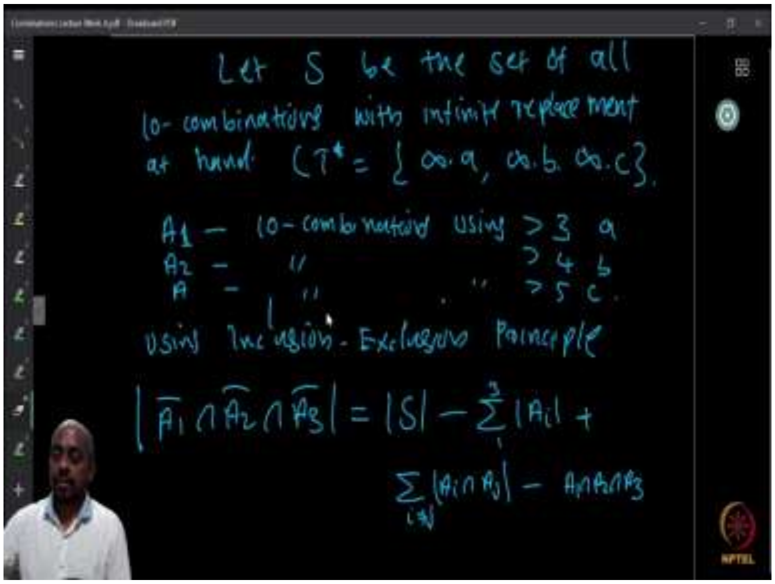


In the previous lecture, we were looking at the principle of inclusion and exclusion. We proved that and then we looked at a couple of examples, in particular, we looked at the number of derangements using inclusion exclusion, and then also using principle of inclusion and exclusion, we calculated the number of surjections from set to another set. So, these are the 2 examples that we looked at.

And so, now, let us look at a couple of more examples. So, what we look today is the first question is that, given a multiset $T = \{3.a, 4.b, 5.c\}$. We want to find the number of 10 combinations of this set this means that we want to select n element multisets, but of course which are as multi subsets of the set. Which means that we cannot have more than 3 copies of a in the subset, 4 copies of b in the subset or 5 copies of c in the subject.

So, we know how to count using the one of the results that we studied earlier is to count the number of multi sets that we can make if there is an unlimited supply of each element. So, if the repetition or replacement is allowed, then we know how to do that. So, given an unlimited replacement, the number of k combinations of n element set is $\binom{k+n-1}{k}$. Now, what can we do in this case? So, that is the question. So, as you might have guessed, we can try to use the principle of inclusion and exclusion to deal with the question this time. So how do you do that?

(Refer Slide Time: 2:52)



So, first we look at S as the set of all 10 combinations where infinite replacement is available. So, let us denote by T^*, the multiset where each copies of $a, b, and\ c$ are available as many as we want. Then, we will define as in a typical example, application of inclusion and exclusion, we define A_1 to be those combinations in S which uses strictly more than 3 copies of a we are only allowed to use at most 3. So, the combinations that use more than 3 are going to be the bad ones, we want to avoid.

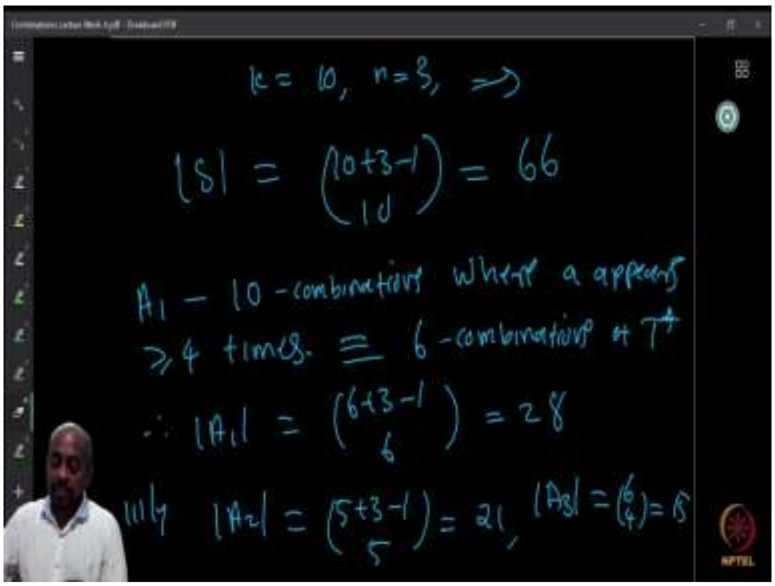
Similarly, we can define A_2 to be the 10 combinations in S that has strictly more than 4 copies of b and similarly, A_3 to be the one with more than 5 copies of c. So, now, once we have this, we know how to calculate this using principle of inclusion and exclusion, we can say that, well you take the you know the total number of combinations or the universe which is S, and then you subtract the bad ones and then you get the ones that we want.

So, what we want is that $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$, which is to say that we are looking at those 10 combinations, which does not have more than 3 a and a more than 4 b or more than 5 c. So, none of this must be there. So therefore, we are looking at $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$.

And by inclusion exclusion $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - \sum_1^3 |A_i| + \sum_{i \neq j} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3|$

So, now all you have to do is to compute each of these quantities what is cardinality of S, what is cardinality of each of these A_i's and their intersections. Once we have this, we can use and then get this. So, how do we do that here?

(Refer Slide Time: 5:47)



Well, in the case, what we have $k = 10$ and $n = 3$ for the 10 combinations of the infinite replacement situation, we want to count S. So that is

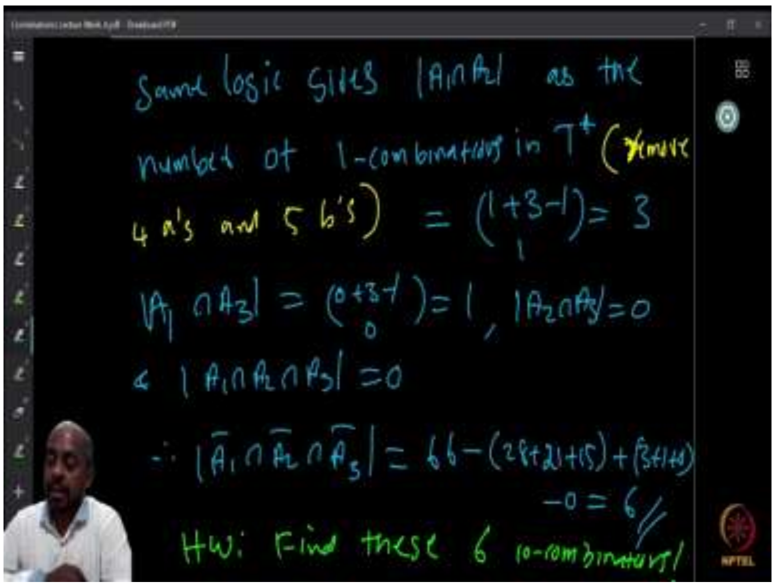
$$|S| = \binom{k+n-1}{k} = \binom{10+3-1}{10} = 66.$$

Then what is A_1? So A_1 is the 10 combinations, where a appears at least 4 times strictly more than 3. Now, if a is appearing 4 times, those are the bad guys. So, from S how can we compute this, well you look at that 10 combinations where there is at least 4 then you know, what you can do is that you can remove these 4 guys. So, remove the 4 and then what you get is going to be a 6 combination. And then it can of course, contain you know, some, some a's. But that is okay.

So, these are in one to one correspondence with each other, because you can also reverse this operation, so you have a combination of T^* where you are looking at the multi sets using this infinite possibility of elements and then you add, add the four a's that back, then you are going to get 10 combinations where there is at least 4 a's. So therefore, there is a correspondence and then we can instead count the 6 combinations of T^*.

But again, counting combinations in T^* is very easy like we already have a formula. So, here it is a $k = 6$. So, therefore, we have, $\binom{6+3-1}{6} = 28$. So similarly, you can calculate $|A_2| = \binom{5+3-1}{5} = 21$ and $\binom{6}{4} = 15$. So, these numbers we can calculate. Then what we need, now, we need to find out $|A_1 \cap A_2|$, $|A_2 \cap A_3|$ etc. So, these computations we need to do.

(Refer Slide Time: 8:36)



If you take $|A_1 \cap A_2|$, what we are saying is that we are looking for the combinations, where you have at least 4 a's and at least 5 b's. So, those are the bad guys. So, if both of them are happening together, then we mean that, 4+5 = 9 elements are basically a's and b's. So, those let us remove as in the previous case, so you will get 1 combinations in T^* that we can count and there are in one to one bijection again in the previous, so therefore we get $|A_1 \cap A_2| = \binom{1+3-1}{1} = 3$.

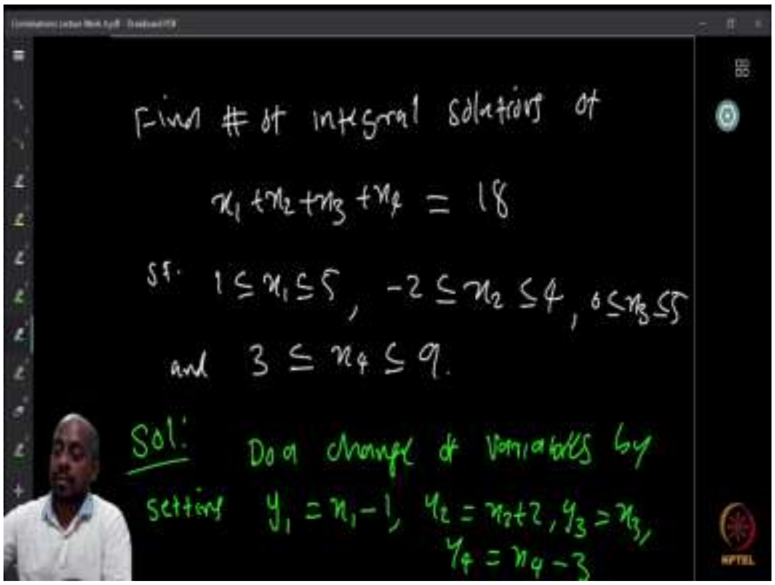
Similarly we can find $|A_1 \cap A_3| = \binom{0+3-1}{0} = 1$, and $|A_2 \cap A_3| = 0$ and $|A_1 \cap A_2 \cap A_3| = 0$. So, we have calculated all these things and therefore, we can directly apply the formula we get:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - \sum_1^3 |A_i| + \sum_{i \neq j} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3|$$

$$= 66 - (28 + 21 + 15) + (3 + 1 + 0) - 0 = 6$$

So, now there are exactly 6, 10 combinations with the property that we are looking at in this set, 3.a, 4.b and 5.c. Now, can you actually find the 6, 10 combinations and list them? So, that is a nice question that you can think about. And I will write it as a homework. Now, one more question and then we will look at something else.

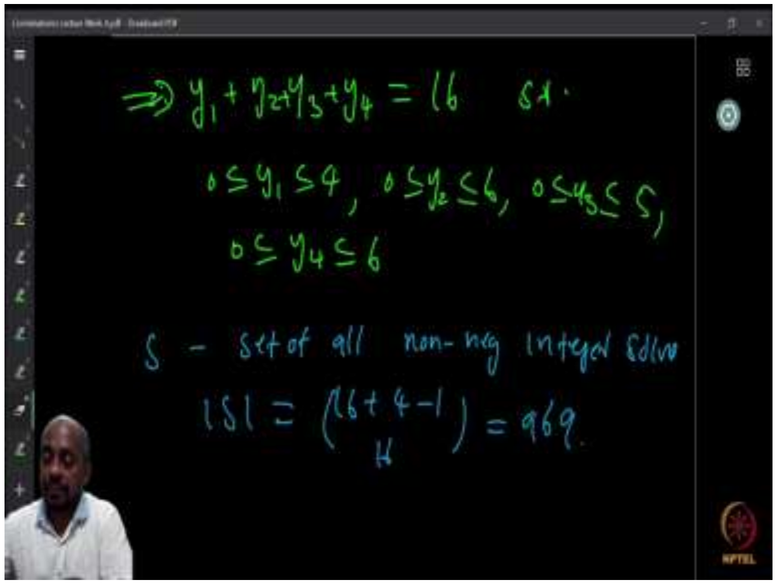
(Refer Slide Time: 11:07)



So, here what we want to do is to find the number of integral solutions of the equation $x_1 + x_2 + x_3 + x_4 = 18$. But then instead of the earlier conditions, where x_i's greater than or equal to 0 or x_i's are greater than equal to 1 both we solved, here what we have is that $1 \leq x_1 \leq 5$, $-2 \leq x_2 \leq 4$, $0 \leq x_3 \leq 5$ and $3 \leq x_4 \leq 9$. So, these are the boundary of you know x_i's, so x_i's cannot take values outside these boundaries. Now, how do you look at something like this? You think for a few minutes, you can see that like very simple change of variables will help you to change the format to something that we already know, at least partially. So, let us first do the change of variables as follows.

Let $y_1 = x_1 - 1$, $y_2 = x_2 + 2, y_3 = x_3$ and $y_4 = x_4 - 3,$ which means that I am going to get 0 on the left hand side of that variable y_i.

(Refer Slide Time: 13:30)



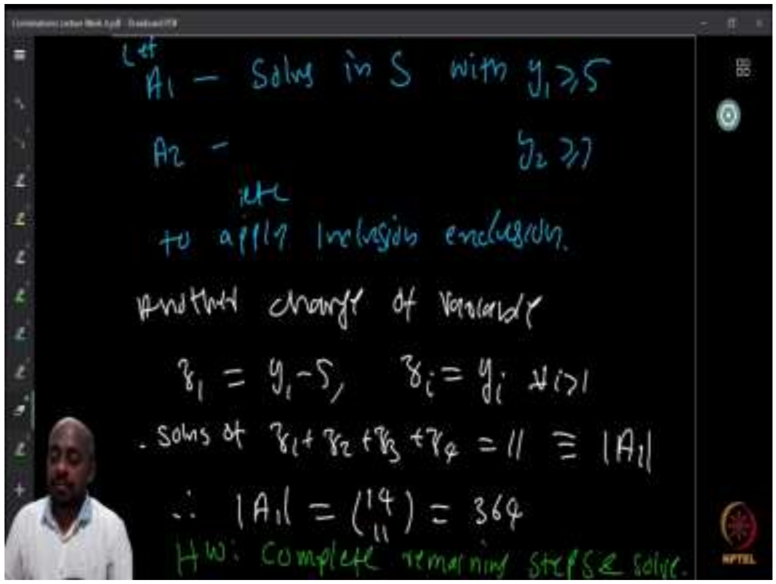
So, then what you will get is that $y_1 + y_2 + y + y_4 = 16$ such that $0 \leq y_1 \leq 4$, $-0 \leq y_2 \leq 6$, $0 \leq y \leq 5$ and $0 \leq y_4 \leq 6$

Now this again, it is very similar to the previous question, let us just look at the set of all non negative integer solutions and then subtract using the inclusion exclusion.

Let S is the set of all non-negative integer solutions to this equation above. So, since there is a one to one correspondence between this equation in y and the equation in x, we do not worry about number the numbers remains the same.

So, the total number of non-negative integer solutions $|S| = \binom{(16+4-1)}{16} = 969$.But now then we have to put the other conditions on the right hand So, how do you do that?

(Refer Slide Time: 15:05)

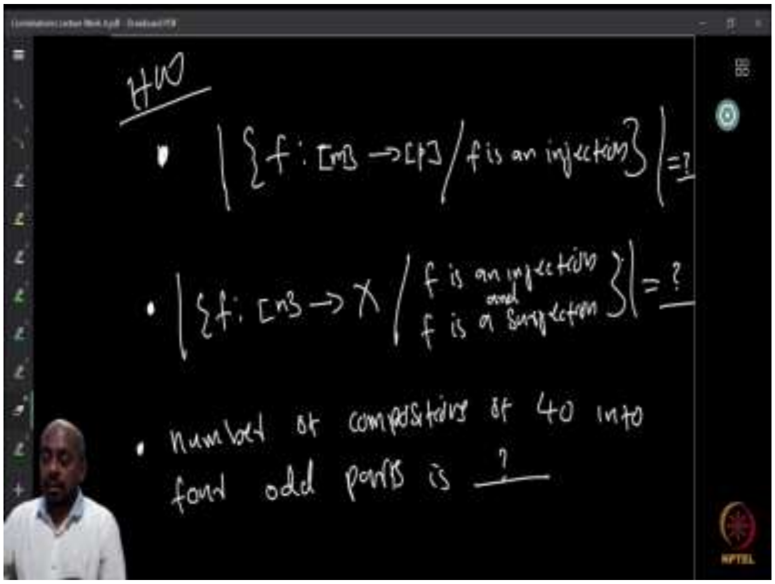


So, let A_1 be the solution in S with the property that $y_1 \geq 5$. Similarly, A_2 is the solution in S with the property that $y_2 \geq 7$, etc. So, then once you define this, so I am not doing all the details, so you can do on your own and complete the proof.

So, now, for each case you have to do some change of variables to reduce that particular variable to be in the nice form. So, what we are going to do is that we are going to do a change of variables again to say that, now $z_1 = y_1 - 5$ and $z_i = y_i$ for every $i > 1$.

Now solutions of $z_1 + z_2 + z_3 + z_4 = 11$ after the change of variable will give you the cardinality of A_1 and then what do you do? Well, this you can find out because you know for this equation, it is basically $\binom{14}{11} = 364$. So, similarly, you can find $|A_2|$, then $|A_3|$ and then $|A_1 \cap A_2|$etcetera and then compute this and give me the final answer. So, this is that, that is the homework for you complete the remaining steps and solve this question.

(Refer Slide Time: 16:49)



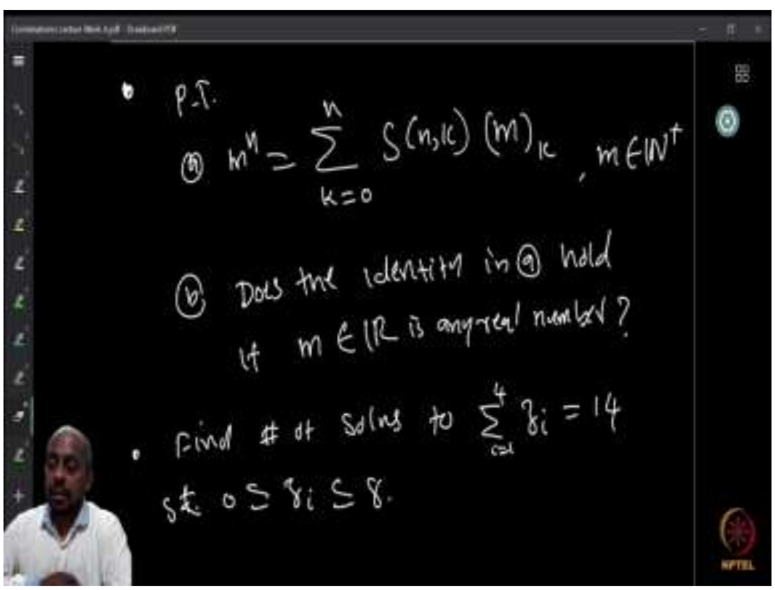
So, now, I will give you a few more homework questions. So, let us look at one by one.

(1) Find $|\{f: [m] \rightarrow [p]: f \text{ } is \text{ } an \text{ } injection\}|$?

(2) Find $|\{f: [n] \rightarrow X: f \text{ } is \text{ } an \text{ } injection \text{ } anf \text{ } f \text{ } is \text{ } a \text{ } surjection\}|$?

(3) Count the number of compositions of 40 into four odd parts.?

(Refer Slide Time: 17:59)



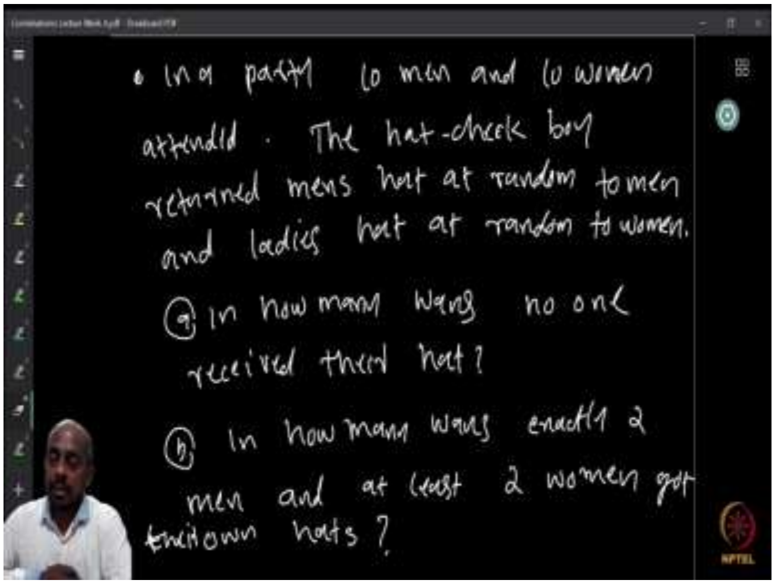
Then next question: Prove that

(a) $m^n = \sum_{k=0}^{n} S(n,k)(m)_k \, , m \in \mathbb{N}^+$

(b) Does the identity in (a) hold if $m \in \mathbb{R}$, is any real number.

- Find the number of solutions to the equation $\sum_{i=1}^{4} z_i$ such that $0 \leq z_i \leq 8$ for every i.

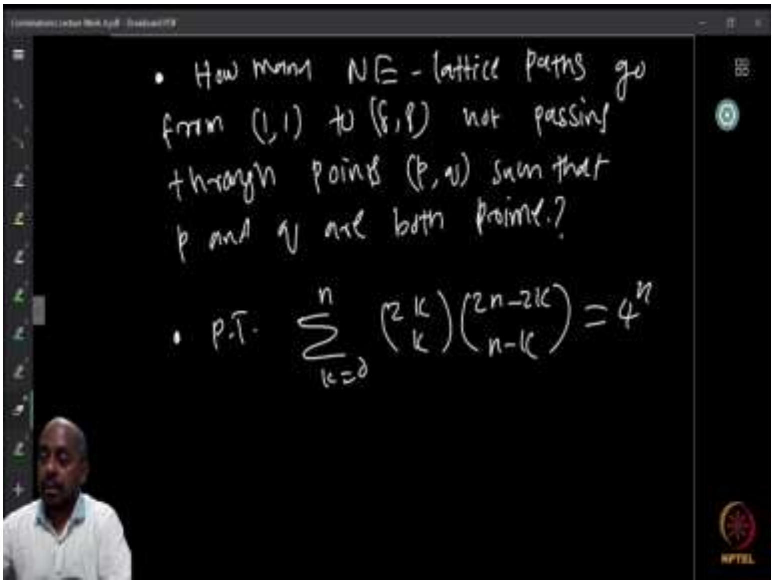
(Refer Slide Time: 19:05)



So, the next question is that:

- In a party say that there are 10 men and 10 women are participating, then there is this hat check person who collects the hats when you enter and then they give it back when you go out. Now suppose this person returned the men's hat to men at random when they go back, but not necessarily the same hat, take one and then give it to the first person who comes. Similarly, for the ladies hats also he returned only to ladies but then again, that is also at random, so from the ladies hats in one stand, he will pick one at random and give it to one of the ladies who is going, this way gives the hats. Then the question is that how many ways no one received their hat and then in how many ways exactly 2 men and at least 2 women got their own hats. So, these are the questions.

(Refer Slide Time: 20:25)



Then, another questions:

- How many north-east lattice paths go from the point (1,1) to the point (8,8) that does not pass through points (p, q) whose coordinates p and $q$are both prime.
- Prove that $\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$.