**Combinatorics**
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**Principle of Inclusion and Exclusion**

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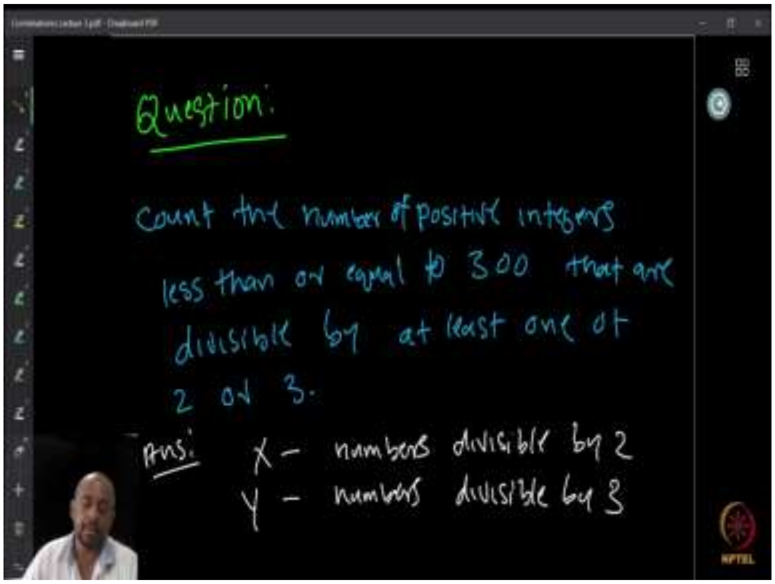


Now, let us look at another question. The flavor is very different from that what we are looking so far. So, we want to count the number of positive integers less than or equal to 300 that are divisible by at least one of 2 or 3. It is a very simple question. Any of you can do it in a minute, I believe. So, just try it yourself. But here is the answer. Let us say that X is the set of numbers divisible by 2, and Y is the set of numbers divisible by 3.

Now, so one way we can do is that, what is the cardinality of X? Well, since every alternate number is divisible by 2, we have exactly 150 numbers. And similarly, every third number is divisible by 3 there are 100 numbers. Now, the problem is that we cannot add X and Y to get the total number of numbers divisible by 2 or 3, because there could be numbers which are appearing in both the sets. So, X and Y might have some intersection, so we will be counting those numbers more than once.

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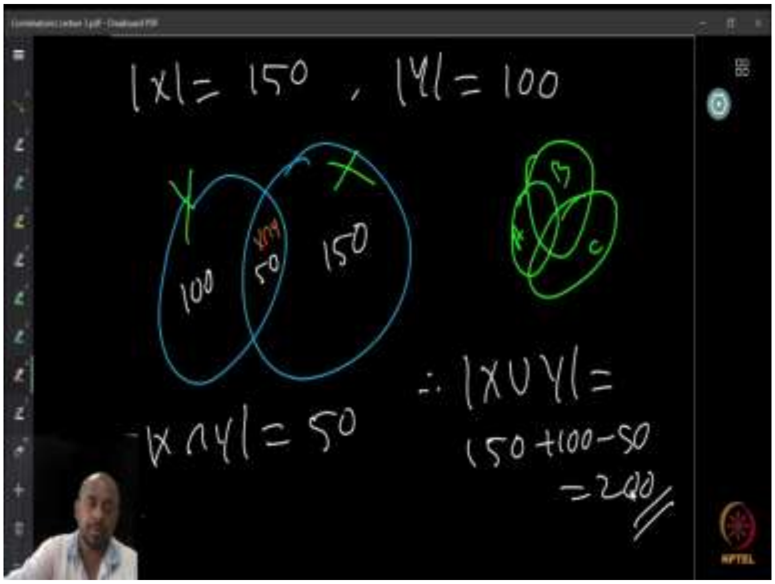


So, to avoid that, we have to discard those numbers which are counted twice. So, what are the numbers which are counted twice, those numbers which are both divisible by 2 and 3. That means they are the numbers divisible by 6 and how many are there well, 300 by 6 which is 50. So now, we can use this property that since we have the numbers that are divisible by 3 are 100, numbers that are divisible by 2 are 150. Their sum is 100 + 150 = 250, but 50 of them are common, they are divisible by both 2 and 3, they are the ones divisible by 6.

And therefore, they these numbers that we have counted as part of X and as part of Y. So, this means that we have to, we have counted them exactly twice, so we can subtract that. So, I will get cardinality of X union Y is 150 + 100 −50 = 200. Now, this is something that all of you know already. Now, can you prove something similar when there are 3 intersecting sets like let us say A, B and C.

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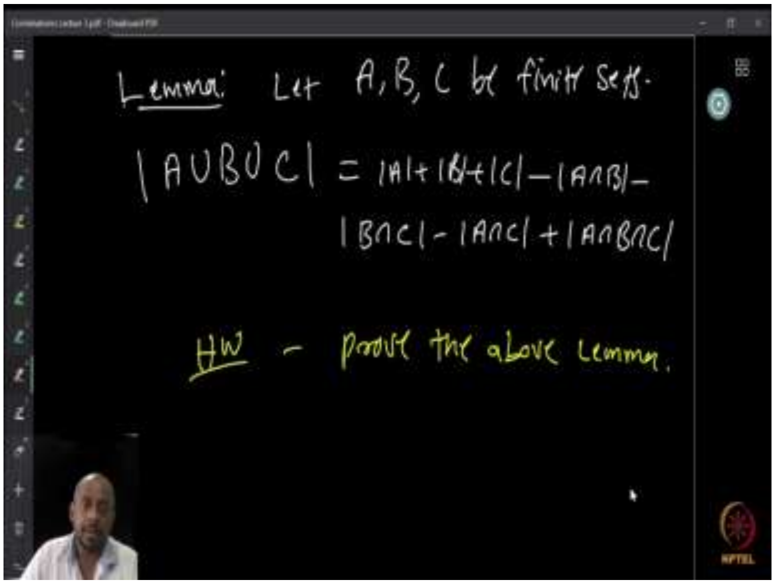


So, the following theorem is a homework.

Let $A, B, C$ be finite sets, then prove that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

Because this numbers we had counted 2 times each, but we have subtracted them more than that. So, therefore, we want to put them back again. So, we cannot subtract more times than the number of time that they were actually over counted. So, as a homework prove the above lemma. Now, we are going to prove a much more general formula for this which is called the principle of inclusion and exclusion.

So, what, what does it mean, we are including the numbers that we want to count first, then the numbers that we are over counting we are excluding by subtracting, again we are including the numbers which we have subtracted too many times, then again, we are putting up the numbers that we have added too many times etcetera.

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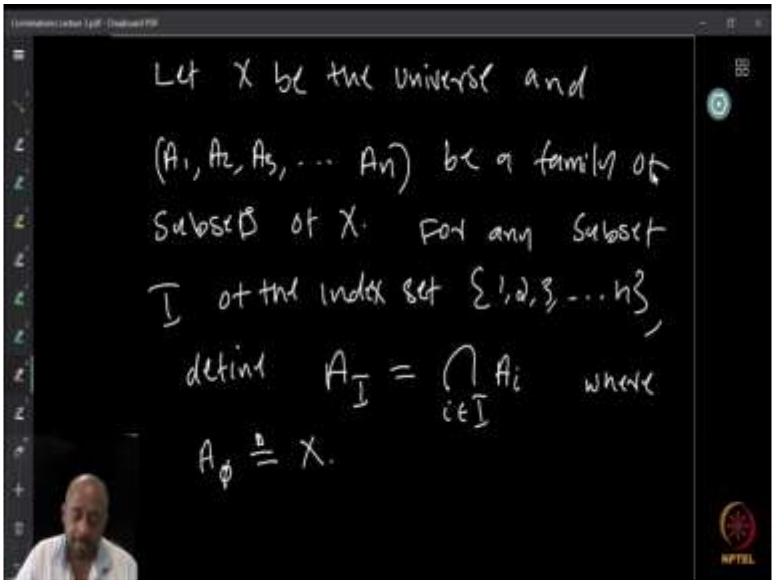


So, for this we need the following conventions. So, let $X$ be the universe, the universe means that the set where all the elements that we are going to talk about in the other sets are going to come from, so everything is appearing in the set $X$. So, $A_1$ $A_2, \dots, A_n$ be a family of subsets of $X$. Now for any subset $I$ of the index set $\{1, 2, \dots, n\}$,We define $A_I = \bigcap_{i \in I} A_i$, where $A_\emptyset = X$

That is $A_I$ is the intersection of all the $A_i$'s where the index i is in the set I. Now, what happens when I is the empty set, well by convention, we define $A_\emptyset$ to be the whole set $X$ itself. Of course, in some books, where they follow different conventions, but for our purpose, we will use this convention, where the reason for the convention is kind of clear that, when you take intersection of sets, the sizes only decrease.

So, if you decrease the number of sets that we are intersecting to 0, then you should have the maximum possible elements which is the universe. So, therefore, we define $A_\emptyset$ set to be the universe $X$.

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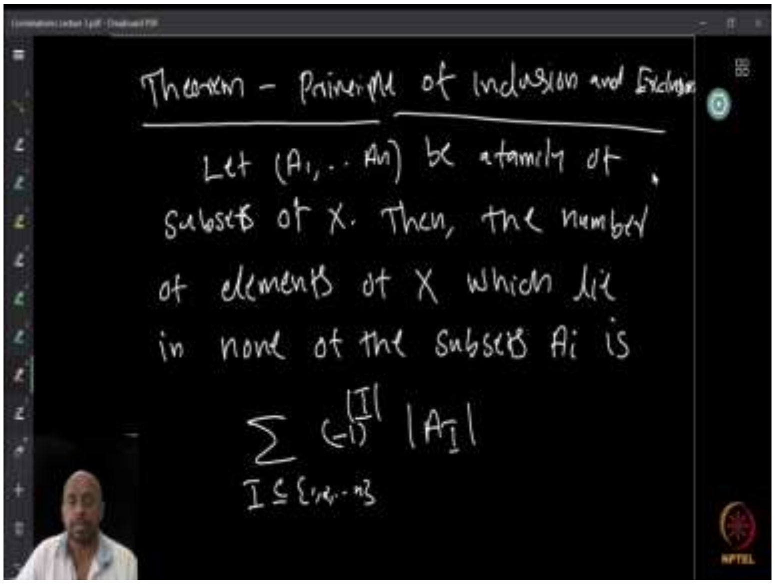


With this convention, we can define the theorem of principle of inclusion and exclusion.

Let $\{A_1\ \ A_2, \ldots, A_n\ \}$ be again a family of subsets of X. Then the number of elements of $X$ which lie in none of the subsets $A_i$ is given by the sum,

$$\sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} |A_I|$$

So, we are counting the elements which are not in any of the sets $A_1\ \ A_2, \ldots, A_n$.

So, we have the universe and we are counting the sets which does not belong to these sets. Now, it can easily be converted into counting the universe because we just take the whole set $X$ and subtract the element that is in none of the things, then we will get the cardinality of the remaining, so that is easy.

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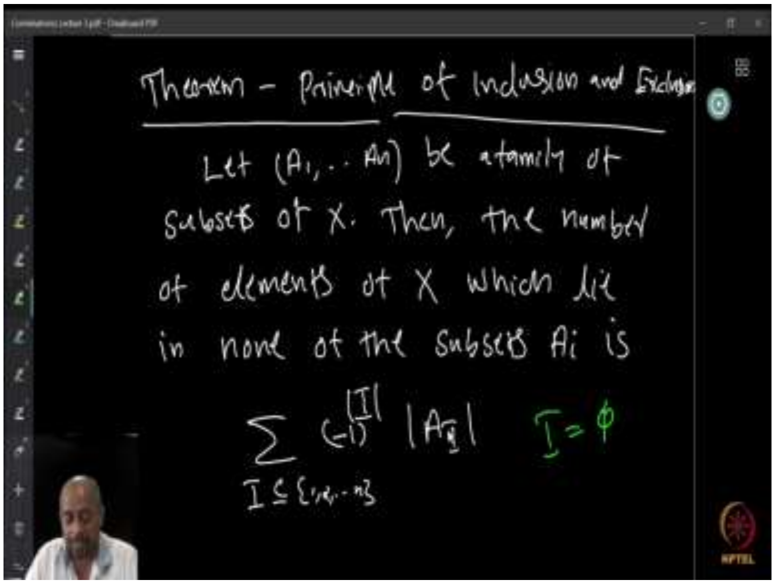


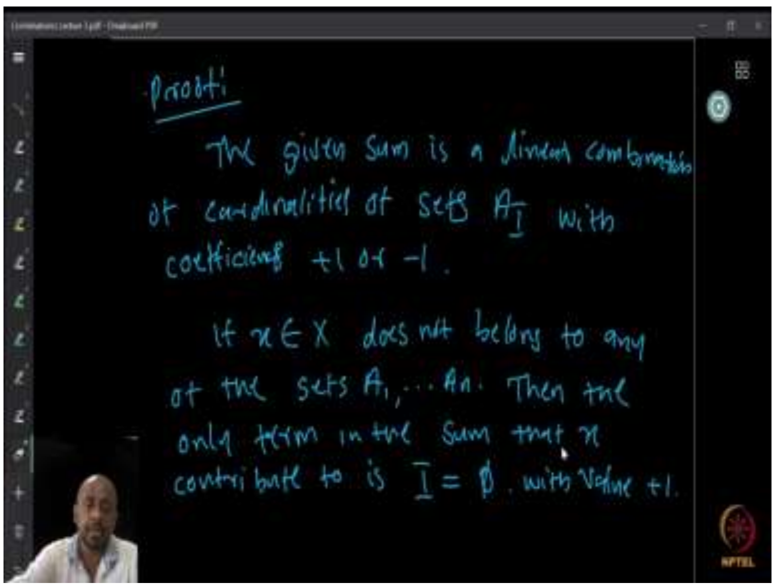


So, here is the proof. Now how are we going to prove this? So, we are going to prove this by counting the elements in the set $X$ which are not part of the $A_I$'s and see what are their contributions to the sum that we are looking at here.

Now, if you can show that, in the sum the, contribution of an $x \in X$ which is not in any of the $A_I$'s is exactly 1 and for every element that appears in one of the $A_I$'s, the contribution is going to be exactly 0, then the sum should give exactly the number of elements which are not in any of the $A_I$'s. So, we are going to show precisely this by taking each element $X$ and say that what is its contribution. So, first we be observed that the given sum is a linear combination of the cardinalities of sets $A_I$ with coefficients either $\pm 1$.

Now, if the element $x$ does not belong to any of the sets $A_1 \;\; A_2, \ldots, A_n$, then the only term in the sum that $x$ it contributes to is $I = \emptyset$. Why is that, because, if $I$ is not empty, then we are looking at cardinalities of $A_I$'s, where $A_I$ is the intersection of the sets in which the element $x$ is appearing. So therefore, $x$ must appear in one of the $A_i$s to be part of any term in the sum, where $I$ is non empty.

So therefore, the only term in which $x$ is not appearing any of the $A_I$ is when $I = \emptyset$. Now, what is the contribution of $x$ when $I = \emptyset$? The contribution of $x$ is $(-1)^{|I|} = (-1)^{|\emptyset|} = (-1)^0 = 1$. And $|A_I|$, where $I$ empty set is defined to be the whole set $X$.

So, the elements of $X$ are all contributing 1 to the sum and in particular the elements which are not part of $A_I$'s will contribute exactly 1 to the sum. And since, the elements not in $A_I$ are not appearing in any of the other terms of the sums, its contribution will be precisely 1. So therefore, if $x$ is not in any of the sets $A_i$, then its contribution will be exactly 1 because it is $(-1)^0$ and cardinality of $A_I$ is going to be the cardinality of $X$. So, each element contributes exactly 1. Now, now, we look at the elements which appear in at least 1 of the $A_i$'s.

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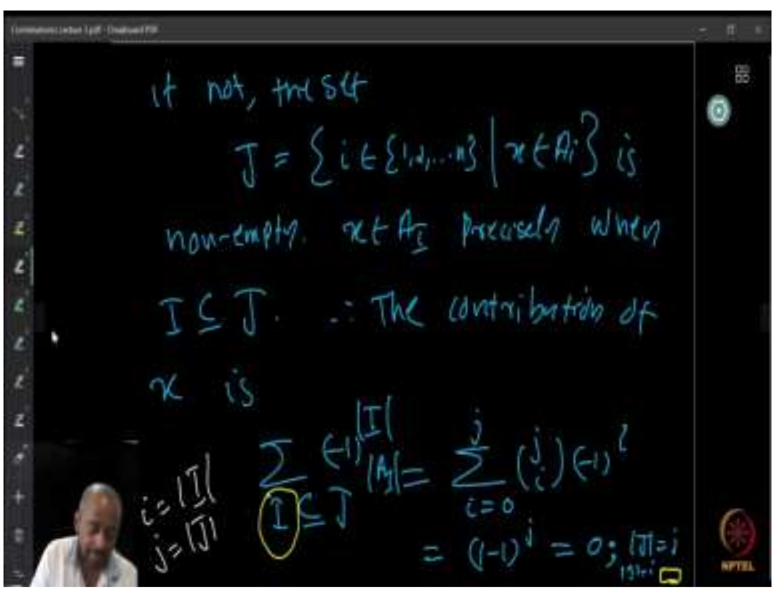

So, if $x$ is appearing in atleast one of the $A_i$'s you consider the set of indices $i$ in which $x$ is appearing. So, let $J = \{i \in \{1,2, \dots, n\}: x \in A_i\}$. Now, this is a non empty set because $x$ appears in at least 1 of the sets. Now $x \in A_I$ precisely when $I \subseteq J$. So, now, the contribution of $x$ to the sum is $\sum_{I \subseteq J} (-1)^{|I|} |A_I|$. Now this is for an arbitrary element $x$, so $J$ can be arbitrary. But what is this sum? What is the cardinality of $A_I$ here? We do not know the cardinality of $A_I$, so we are going to count this in a different way. So, what we are going to do is that for each element $x$, how many times it is going to appear? So, $x$ is appearing in as part of every subset $A_I$ of $J$.

Now, how many such terms are there, well, we are looking at all possible subsets of $J$. But in particular, we are looking at the subsets of $J$ with a fixed cardinality. So, instead of looking at every subset, I collect all the subsets, I collect all the subsets $I$ which has cardinality exactly let us say $i$. So, what I am going to do is that I am going to let $i = |I|$ and then I will rearrange the sum with respect to this $i$.

Now, if I take $|J| = j$, then there are $\binom{j}{i}$possible $i$-element subsets of $J$. And each of the subsets is going to be contributing to the sum because, $x$ is going to be in every subset of $A_I$ including the empty subset as because all the elements are there.

So what is the contribution? It is $(-1)^i$. Now, this is true for every $i$ ranging from 0 to $j$. So therefore we get $\sum_{I \subseteq J} (-1)^{|I|} |A_I| = \sum_{i=0}^{j} \binom{j}{i} (-1)^i$

$= \sum_{i=0}^{j} \binom{j}{i} (-1)^i \; 1^{j-i}$, because $1^{j-i} = 1$

$= (1-1)^j$, by binomial theorem

$= 0$

But this total sum is the contribution of $x$ to the sum, where $x$ is an element of one of the $A_I$'s. So, any element in one of the $A_I$'s contributes exactly 0 to the sum. And as we showed before, every element not in $A_I$ contributes exactly 1 to the sum. So therefore, this sum $\sum_{I \subseteq \{1,2,\dots,n\}} (-1)^{|I|} |A_I|$, basically counts the number of elements that is not in one of the $A_i$'s. So, that is a proof.

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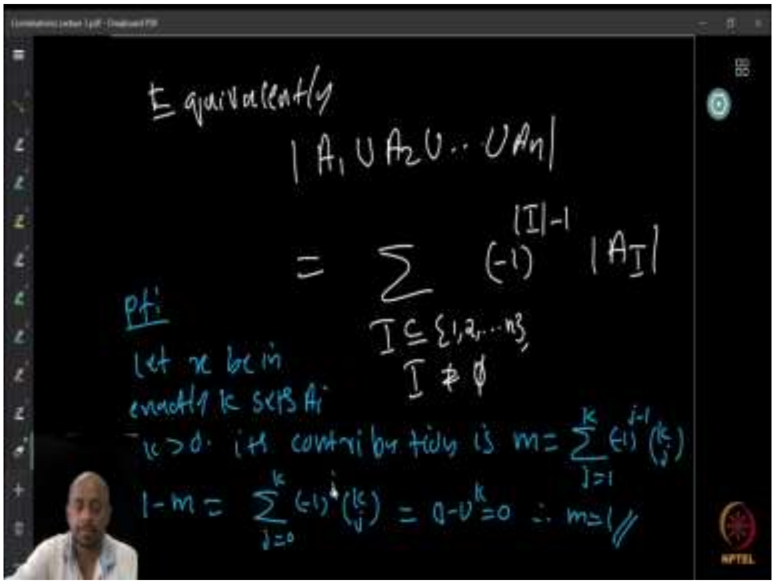

Now equivalently if you want to count the $|A_1 \cup A_2 \cup \dots \cup A_n|$, we can either, take $X$ and subtract the element that are not there, or you can write the formula in a different way as follows.

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{\substack{I \subseteq \{1,2,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} |A_I|$$

So, this will give you the, the cardinality $|A_1 \cup A_2 \cup \dots \cup A_n|$. What is the proof ? Let $x$ be in exactly $k$ sets $A_i$, where $k > 0$. The contribution of $x$ to the sum is let us say $m = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j}$. Because it appears in exactly $k$ sets, every $j$- element subset of this $k$ set contribute something and that is going to be $(-1)^{|I|-1}$, so it is $(-1)^{j-1}$, that is the contribution.

Now, you multiply both sides of the equation $m = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j}$ by -1 and add 1. So, I will get $1 - m = \sum_{j=1}^{k} (-1)^{j} \binom{k}{j} = \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} = (1-1)^k = 0$. Therefore, $m = 1$.

So, the contribution of an element that is in one of the $A_i$, is precisely 1. And the element not in $A_i$, are not appearing because $I$ is non empty. So, therefore, we have the proof. So, both are versions of principle of inclusion and exclusion and you can use either of the formulas as you please. So, now we are going to use this to solve a couple of problems.

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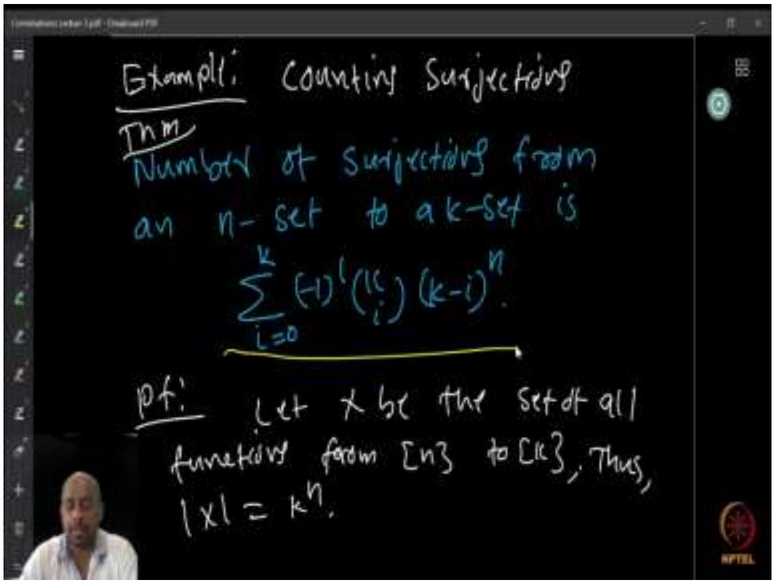


So, earlier we were looking at these surjections. So, the surjections from an $n$-element set to a $k$-element sets how many are there?

So, the surjections from an $n$-element set to a $k$- element sets is given by

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

So, this is the formula that we have. So, we want to prove this formula, so how will you prove this? Again, if you want you know what results that you have to use, what method that you have to use now, inclusion exclusion and can you try to think of a proof of this formula by this. So, pause and think about this and then you can continue.

So, here I am going to give a proof. So, let $X$ be the set of all functions from an $n$ element set {1, 2, … , $n$} to the $n$-elements at {1, 2, … , $k$}. So, what is the cardinality of $X$? It is $k^n$ because every possible functions are there every element has $n$ possibilities. So, each of the $k$ elements can be mapped from any of the $n$ elements on the previous set. So, therefore, there is $k^n$ such functions. So, this is the set of all functions.

So if you look at the functions, you have $k^n$ because we have $k$ possibilities for any of the $n$ elements, that is what we want to say. So, every element can be mapped to any of the $k$, so therefore, $k^n$ possibility. So, this is set of all functions, but we want the surjections. Now, what are surjections? Surjections are those functions where all the elements have a preimage. So, every element in the $k$ element set has a preimage

So, we can do in 2 different ways I am going to use the first theorem. I want to exclude the guys that is not required. So, I will define my sets $A_i$ to be the sets which are, the functions which are not surjections and then I discard the non surjective functions and then I will get the surjection.

So, let me define the set of all functions this way. So, take some fixed element of the set {1,2, …, $k$}, say $i$. And say that, I look at all the functions which map from $[n]$ $to$ $[k]$, where $i$ is not in its image, therefore it is not a surjection.

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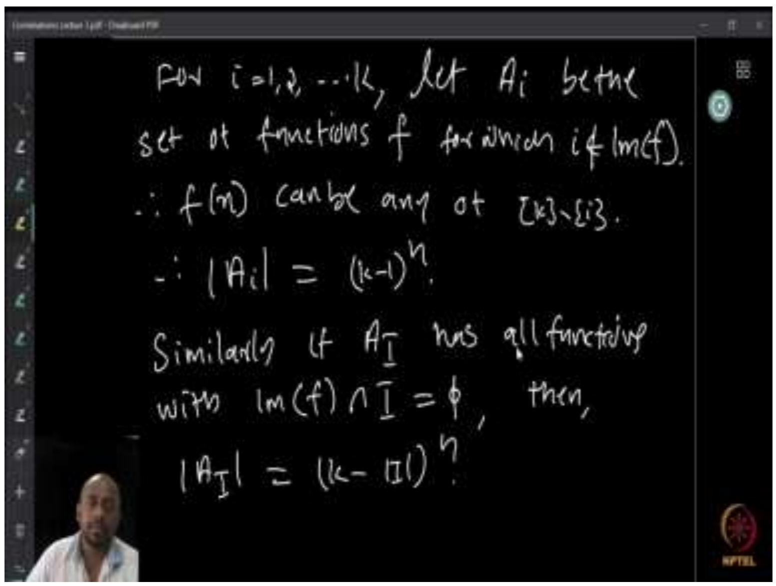


So $A_i$ be the set of functions $f$ for which $i \notin Im(f)$. Now $f(x)$ can be any of the elements other than $i$. So, there are $k - 1$ such elements. So therefore, the number of such possible functions, which does not map to $i, then$ $|A_i| = (k-1)^n$. That is, there are $(k-1)$ elements, so therefore $(k-1)^n$.

Now similarly, if let us say $A_I$ has all the functions where $Im(f) \cap I = \emptyset$ is empty, $|A_I| = (k - |I|)^n$. Because again, if a set of elements are missing say some $t$ elements are missing, then $k - t$ elements are available. So, the function can go from this thing to any of the $k - t$ elements.

So therefore, we have $(k-|I|)^n$. That is a possibility. So, we can also observe that $A_I$ is basically the intersection of all $A_i$, where $i \in I$. So, that is something that you can observe. Therefore, we have all the necessary requirements for applying the principle of inclusion and exclusion.

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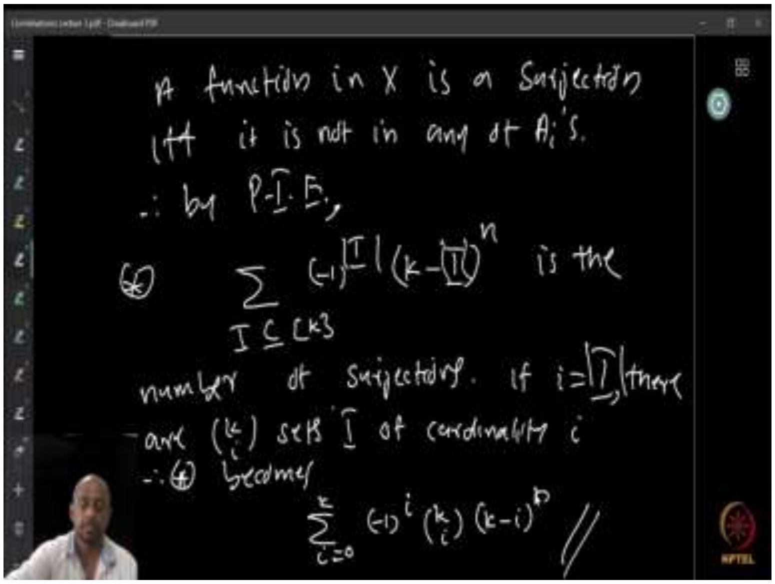


So, let us use it. Now, again a function in $X$ is its surjection if and only if it is not in any of the $A_i$'s. So, by principle of inclusion and exclusion we have

$\sum_{I\subseteq[k]}(-1)^{|I|}(k-I)^n$ is the number of surjections.

Now, if I put small $i=|I|$, there are $\binom{k}{i}$ sets $I$ of cardinality precisely $i$ and therefore, using this observation, we can rewrite the above summation as $\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$. And this is precisely what we wanted to prove, the is a number of subjective function from $n$ element set to $k$ element set.

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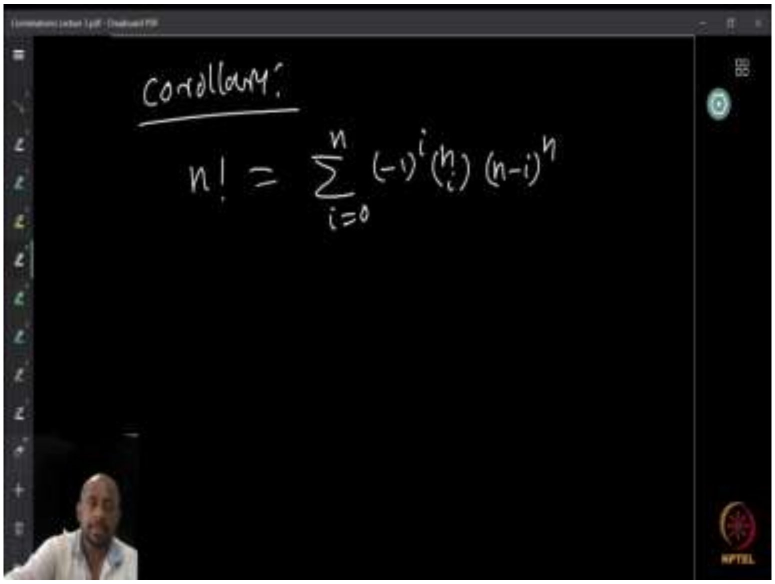


A corollary to this result is that:

$$n! = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n$$

Why is that? Because, when you are looking at surjections from an $n$-element set to an $n$-element set, that is when $k = n$ , they can be thought of as permutations, a surjection from a set to itself.

So therefore, the number of permutations is precisely $n!$ and therefore, the number of surjections from a set to itself must also be $n!$. And this is the formula for the surjections from a set to itself. And therefore, this counts the total number of permutations.

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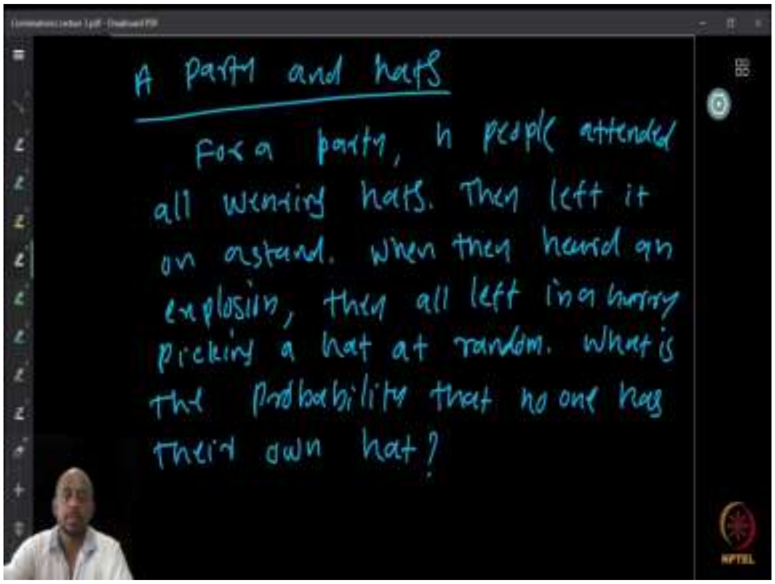


Now, we will stop with the final application where you look at a more interesting question. So, we have a party and let us say n persons are attending the party. All of them came wearing hats. So, everybody came with a hat. Now, when you enter the party hall, you know there is a stand where you can keep your hat. So, you know, they keep their hats at the stands.

Now when the party is going on, something happened in a nearby place and there was some explosion sound. So, everybody got scared. So, they decided that, it is time to go home. So, they all decided to run out. So, they while hurrying out, they pick it up 1 hat and put it on their head and went, they did not even stop the look which hat it is. So, they picked the hats at random.

So, the questions that what is the probability that nobody got there own hat, everybody got a hat that is different from theirs. So, to do this, we can again use principle of inclusion and exclusion. To find this probability. So, how do you do that?

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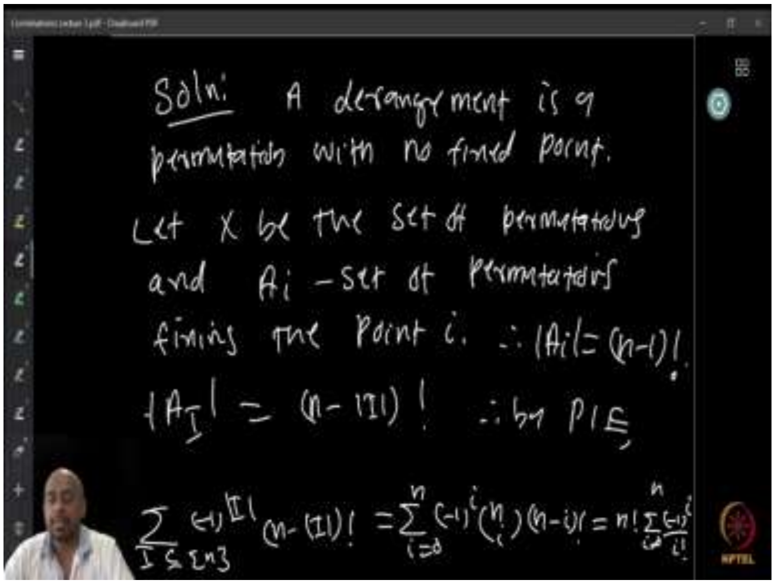


So, what we observe is that a person getting his own hat can be thought of as a permutation where an element is fixed. So, a permutation where there is no fixed point means that nobody got there own hat. Such permutations have names that is called derangement. So, derangement is a permutation where there is no fixed point, that no element is mapped to itself.

Now let X be the set of all permutations and $A_i$ be the set of permutations that fixes the point $i$. It is very similar to the previous one. So, therefore, I am not spending too much time. Now, $|A_i| = (n-1)!$, because it is a permutation of the $n-1$ sets excluding the 1 element that we have fixed because every element can be permuted this guy has to be going to itself.

So, t Now, what is $|A_I|$, where $I$ is the subset of the indices 1 to $n$. Well, if you decide to pick let us say $t$ elements, then the $n-t$ elements can be permuted, those $t$ elements cannot be done anything. So therefore, $|A_I| = (n-|I|)!$. So therefore, by principle of inclusion and exclusion, we get the result that

$$\sum_{I \subseteq [n]} (-1)^{|I|} (n-|I|)! = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

So, this is the number of derangements. But now we want to find the probability that nobody gets the hat is the total number of derangements by the total number of permutations, which is divided by $n!$.

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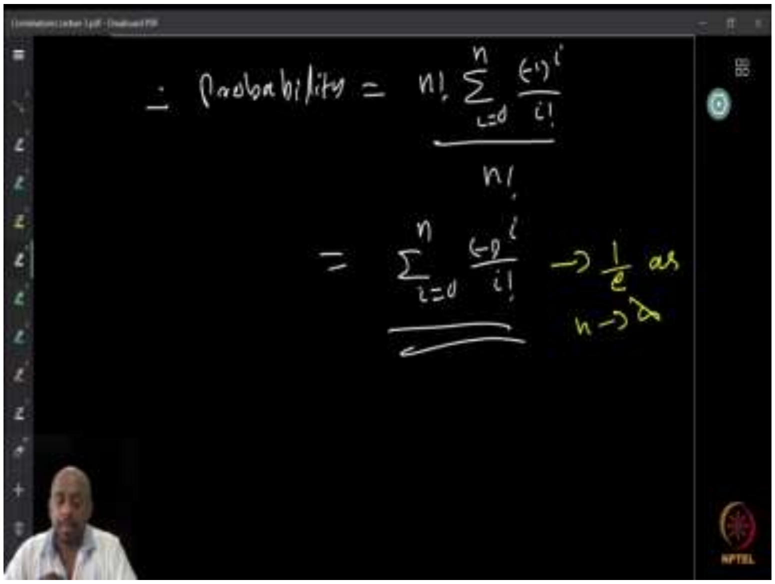

The probability = $\frac{n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}}{n!}$

$$= \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

And if you remember your calculus, you will see that as $n$ goes to infinity, this summation converges to $\frac{1}{e}$ . So the probability basically keeps in increases it, increases and decreases, decreases and increases etc , but eventually it converges to $\frac{1}{e}$. So with this, we will wind up the, the lectures for this week. And see you next week with some more interesting topics.