Combinatorics
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Young's Diagram and Integer Partitions

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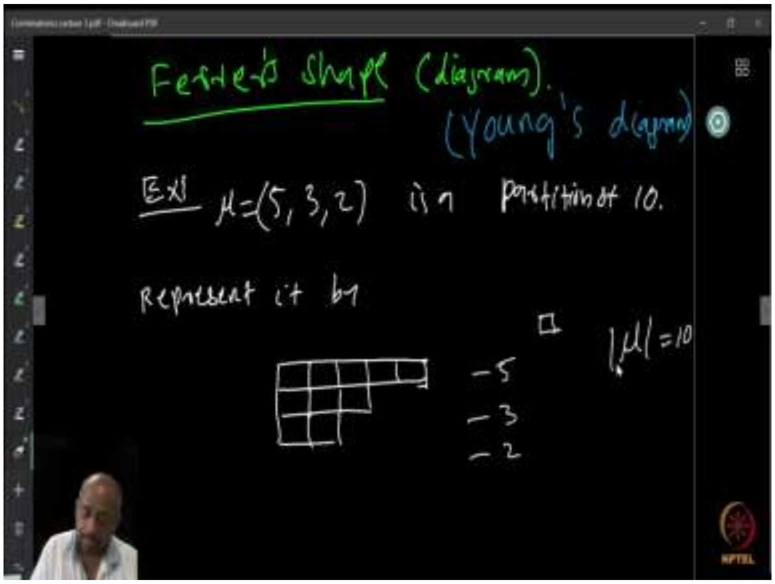

So, what is our new idea? We want to look at partitions in a different way. For that there is a diagrammatic representation of a partition of an integer which is called Ferrer's diagram or Ferrer's shape. This is also called Young's diagram. In many books you will see only Young's diagram because Young has done much more work and we will look into it in future lectures, we will look at the Young's table, tableau.

So, we look at standard young tableau, nonstandard young tableau and weak tableau many of the kind of things are there, we will look at some of these things later, but not at the moment. So, we will look at what is called Ferrer's shape or Young's diagram and it is as follows. Suppose, you are given a partition of let us say a 10. So, let us say a partition of 10 is $\mu = (5, 3, 2)$. So, partition of 10 into 3 parts.

Now, I can represent this partition as follows. So, for, for every unit I will put a unit square. So, the unit square represents 1 let us say. Now, to represent 5, as part of the partition μ, I put 5 unit squares in a row left justified. So, all of them will sit at the leftmost part. So, there are 5 of them, so this correspond to 5 in our partition.

Then in the next row of the shape or in the diagram, I will put exactly 3 unit squares again left justified and in the last part since the integer partition is 2, I will put 2 square at the end. So, I have now 10 unit squares all of them distributed like this, the first line contains 5, next line the next 3, next line the next 2 all left justified. So, the area of this figure is now 10 unit squares and that is why we said that the area of μ is equal to the number that it is partitioning.

So, if you are looking at Ferrer's shape for example, the number of unit squares defines the area and therefore that is the reason we call the size of μ as area of μ. Now, here is the figure that partitions 10. Now, let us look at partitions, there could be several partitions. So, there could be several such figures, they will be distinct.

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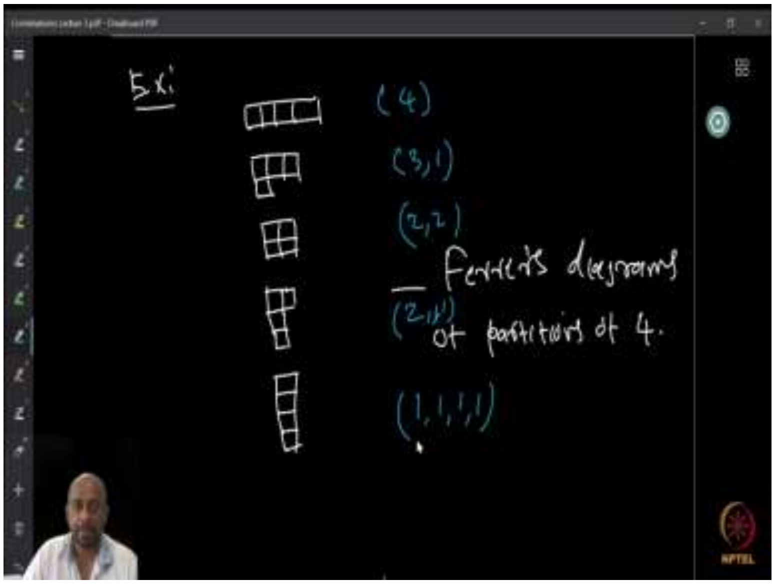

So, here are the partitions of 4, so Ferrer's diagrams of partitions of 4. So, what is the first partition, the first partition represents, the partition (4), the second one represent the partition (3,1), the third diagram represent the partition (2, 2), the fourth diagram represent the partition (2,1,1). And the last diagram represents the partition (1,1,1,1) and these are precisely the 5 different partitions of 4. So, they correspond to these figures. So, the Ferrer's diagram represents the partitions of numbers like this.

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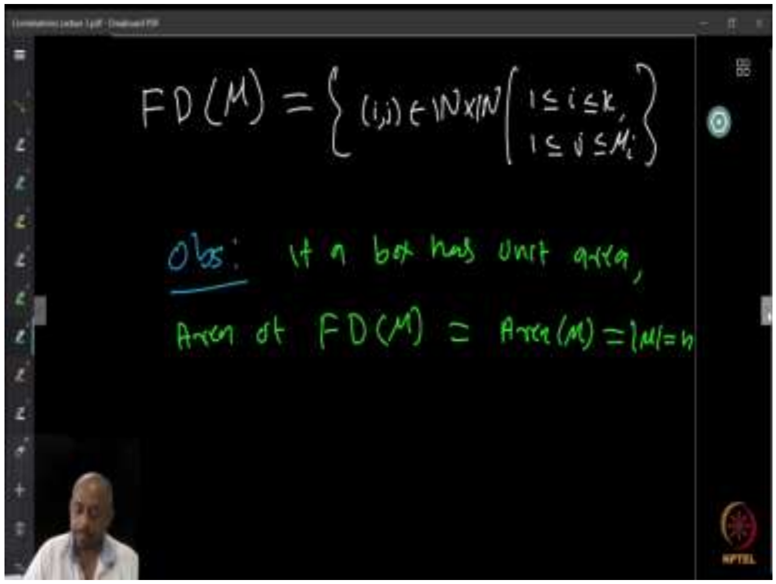

Now, a formal definition of the Ferrer’s diagram can be given as follows.

$$FD(\mu) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq k, 1 \leq j \leq \mu_i\}$$

So, this makes it left justified and top justified. So, it is going to appear in the top left corner of whatever you are looking at. So, the Ferrer’s diagram is going to be exactly like this. So, this is something we already observed that if a box has unit area then the area of Ferrer’s diagram of μ is the area of μ which is the number that it is partitioning.

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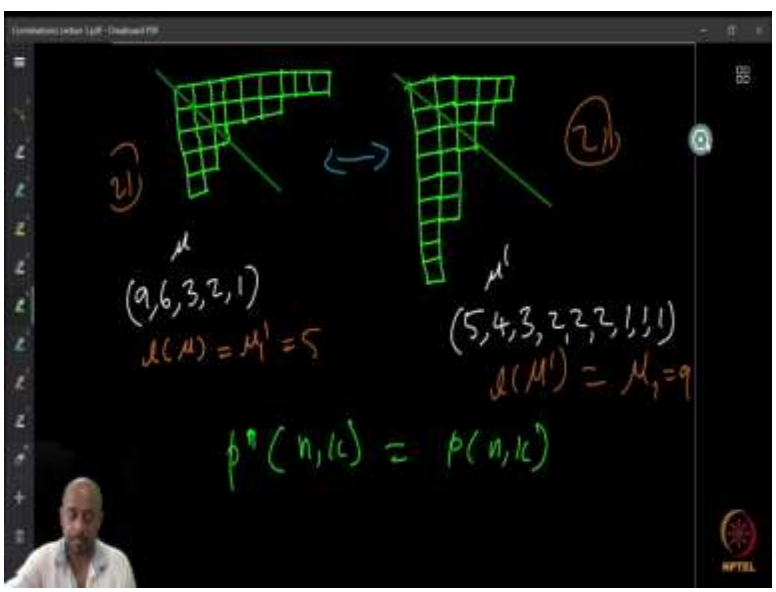

Now, here is another very interesting observation, let us look at the partitions of 21. So, here is a specific partition of 21 that I am giving. So, the partition of 21 is as follows, it is μ =(9,6,3,2,1). So, there are 9 boxes in the first row, 6 boxes in the second row, 3 boxes in the third row, 2 boxes in the fourth row and 1 box in the final row.

Now, here is another partition of 21 again. But now here, so it is the partition let us say μ' =(5,4,3,2,2,2,1,1,1) , where the first row has exactly 5 boxes, second row has 4 elements, the third row has 3, then 2, 2, 2 and 1, 1, 1. Can you find a relation between these two figures? If you look at these two figures you will notice that, suppose you take the line goes from the northwest corner to the southeast corner that line, so 45 degree.

And you put that line on the diagonal like this and then take the reflection of this onto both sides, the reflection with a mirror, you are going to precisely get the same figure that is on the right hand side. So, I am going to get the same figure on the right hand side. So, just verify it yourself these figures are in one to one correspondence with this reflection.

So, you take any figure here to a reflection you will get another figure here and that figure will again be a partition of the same number that you are looking at. So, I take this figure, take a reflection I will get a partition here. Now, can you see something else that is relating these 2. So, what relates these 2 is that in the first case, we had a partition of 21 into 5 parts.

Now, when you take the reflection, this column becomes the row first row here. So therefore, this is a partition where the largest part because the top row is 5, because that is going to be the first row. So, now we see that, now the bijection from, the set that is counted by $p^*(n,k)$ to $p(n,k)$ is just a reflection technique in our diagram.

So, you take this, take the reflection you will get a partition, another partition where the number of parts here becomes the largest part and vice versa, the number of parts here becomes the largest part here. So, what we have observed is that length of μ is the first element μ_1' of the partition μ' , that is $l(\mu) = \mu_1'$. And similarly, and length of μ' is equal to the first element μ_1 of μ ,which is 9. That is $l(\mu') = \mu_1 = 9$. So, this gives a pictorial proof that, that $p^*(n,k)$ = $p(n,k)$

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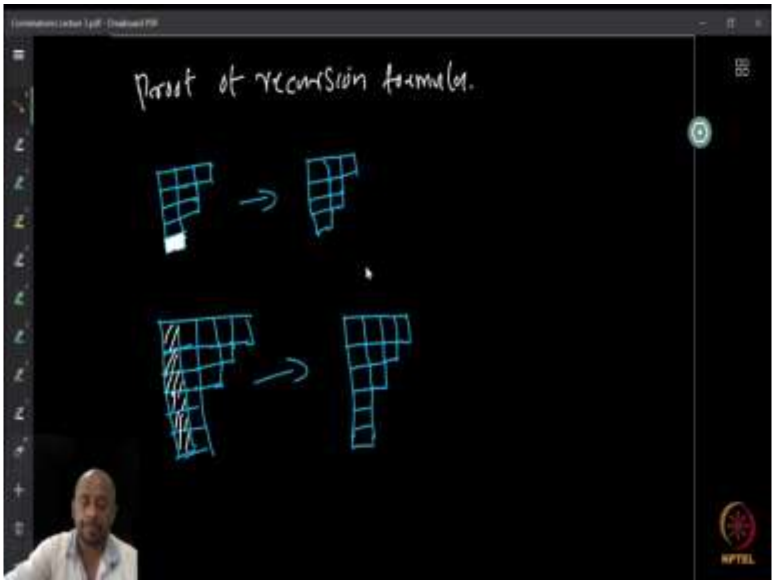

Now, let us look at a couple of more things, we want to prove the recursion formula. So, the recursion formula that we wrote for $p(n, k)$ can be proved using the Ferrer's diagram again. So, what is the proof, so here is the proof. So, this picture represents the proof. So, you have, the Ferrer's diagram. And suppose, the first column is the largest column that in the sense that there is some box, the last row for example, is singleton.

So, this guy is a singleton. So what I do, I just remove this from the picture, and I will get a picture here. Now, what is the property of this picture, this picture is a partition of $n - 1$ because the area has reduced by 1 because I removed 1 unit cell. And the number of parts has reduced by 1 because I removed 1 row. And therefore, it is a partition of, this represent the partition of $n - 1$ into $k - 1$ blocks.

Now, if I have any partition here, I can just add 1 at bottom most part of the first column and I will get a partition here where the last row is a single cell. So, therefore, there is a bijection. Similarly, if you have atleast 2 elements in the last row, then therefore, 2 columns are full, full columns. So, what I can do is that I can remove entire first column from this, then I will get a Ferrer's diagram here, where the, the number of rows are the same because I have only removed the first column as second column is also full.

So, it is a partition of $n - k$ because I removed exactly k rows here. So, it is a partition of $n - k$ into k blocks. So, that is the proof of the recursion and this and these 2 are disjoint because these figures does not have this thing here and they are disjoint and therefore we can add them.

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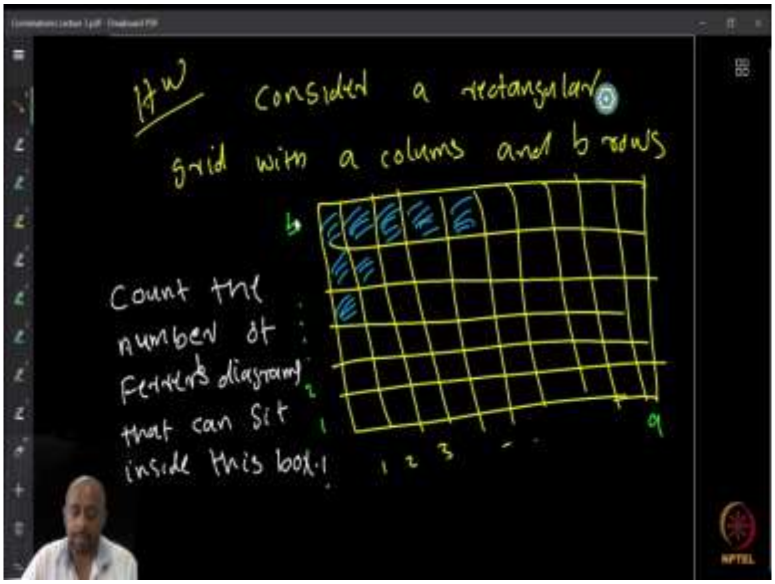

So, here is another homework question for you. So, consider a rectangular grid with a columns and b rows. So, here is the grid, where we have column number 1, column number 2 etcetera, column number a and row number 1 to b. So, take a rectangular grid with a columns and b rows. Now what we are looking at previously was Ferrer's diagrams. Now Ferrer's diagrams are now left hand top justified.

So, let us see how many such Ferrer's diagrams can sit inside this box that we are looking at, so here is the grid shaped box. And in this box, we want to see how many Ferrer's diagrams can be put inside, they fit completely inside. Can you count the number of such Ferrer's diagrams that sit inside this box of size $a \times b$. So, this is a homework and try to find proof by finding a bijection to, some set that you already know how to count.