Combinatorics
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Integer Partitions

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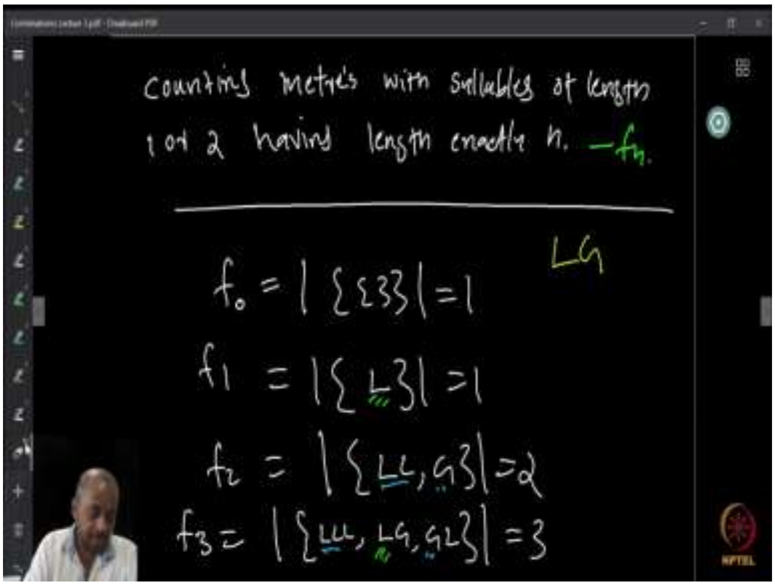
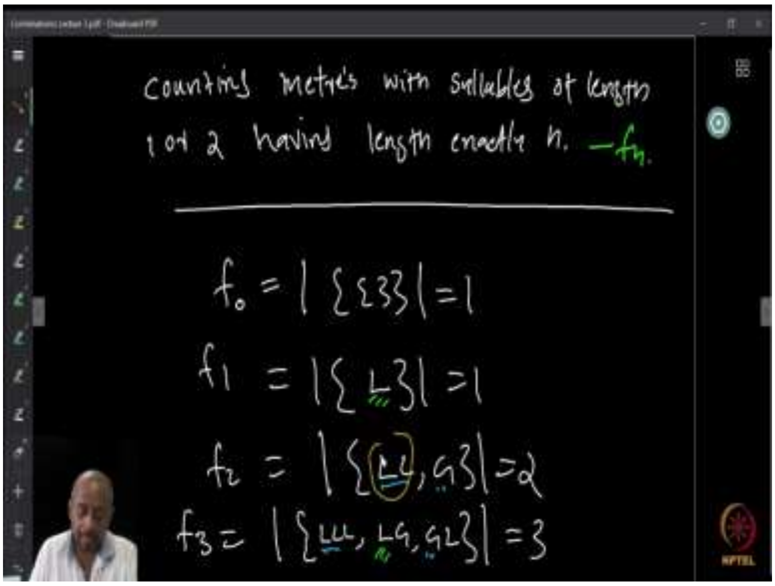
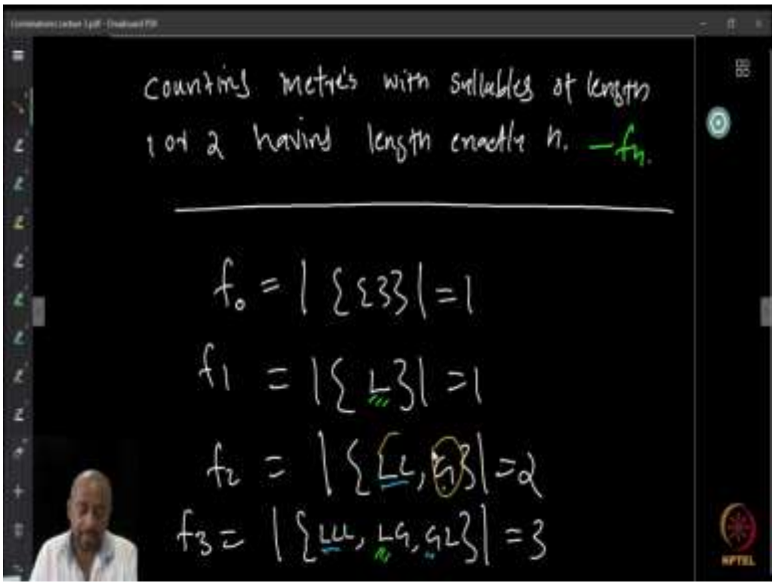
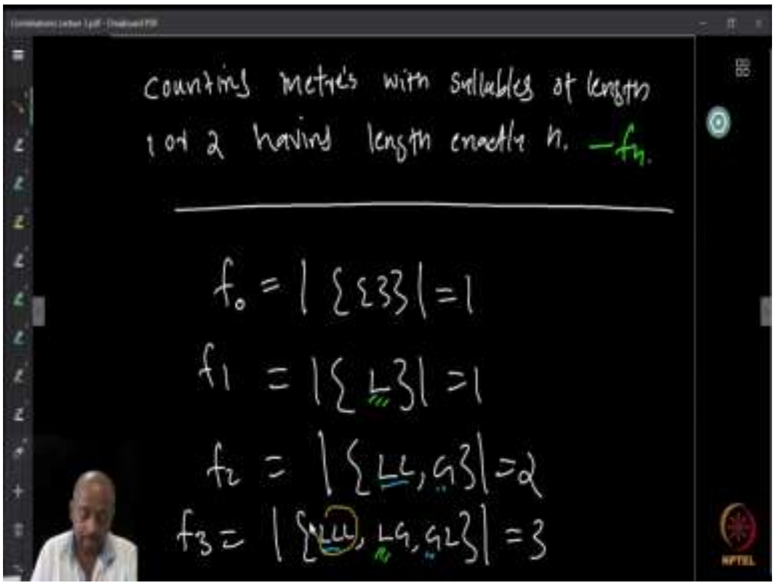





Counting metres with syllables of length
1 or 2 having length exactly n. — f_n.
$f_0 = |\{\{\}\}| = 1$
$f_1 = |\{L\}| = 1$
$f_2 = |\{LL, G\}| = 2$
$f_3 = |\{LLL, LG, GL\}| = 3$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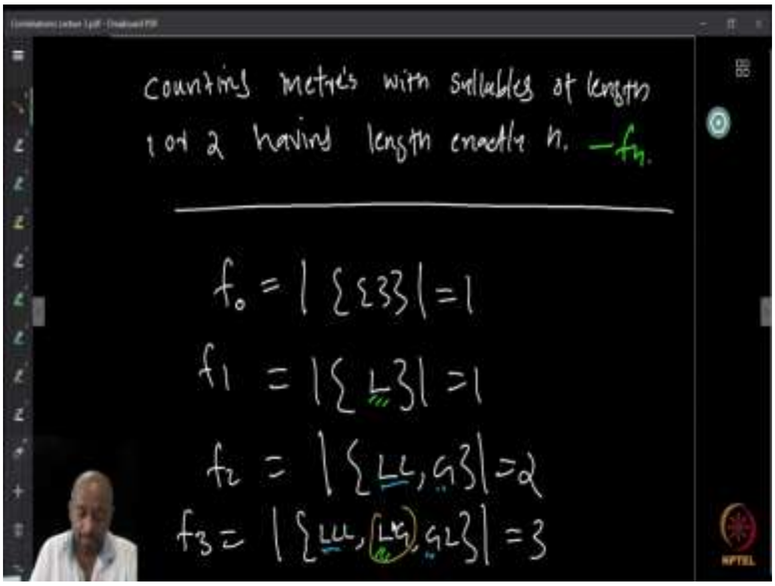
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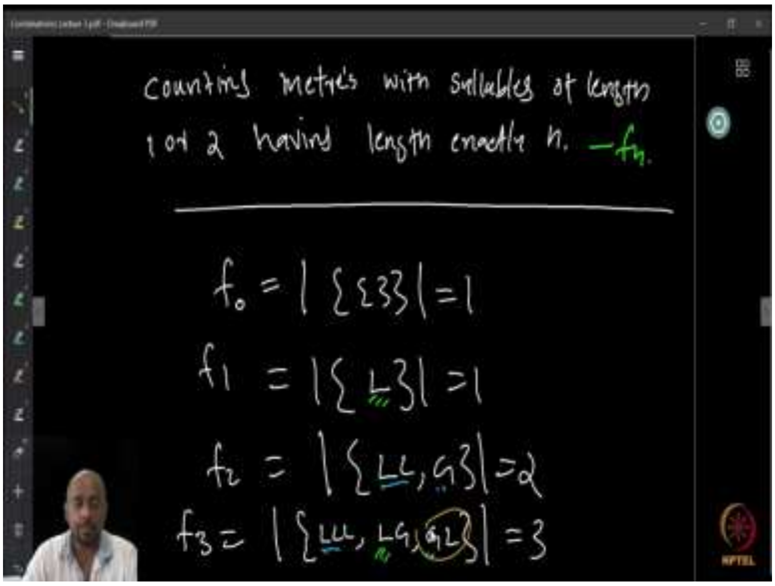
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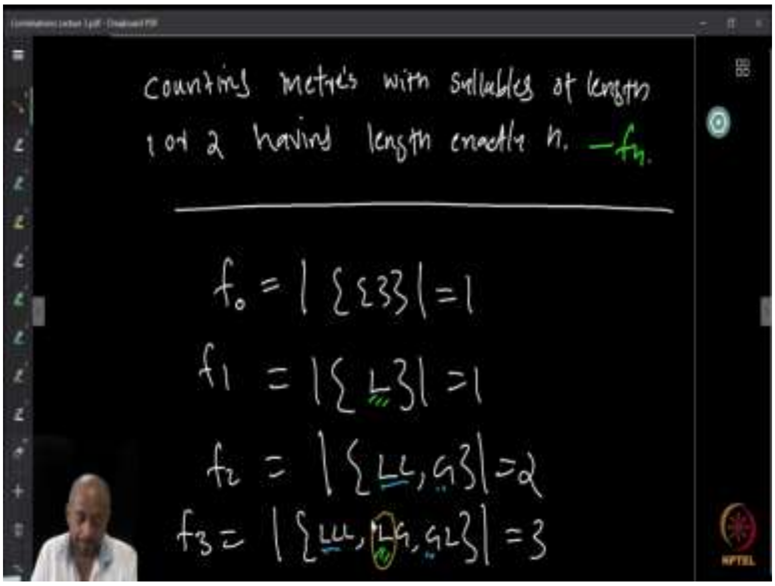
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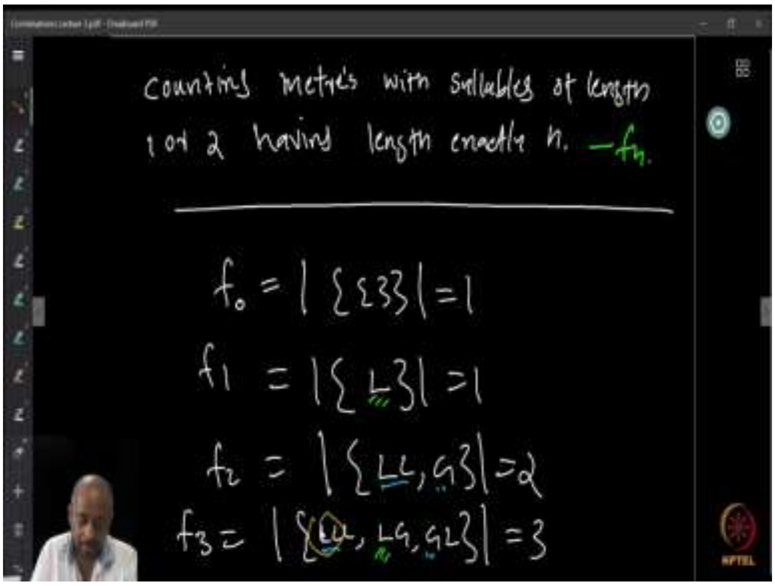
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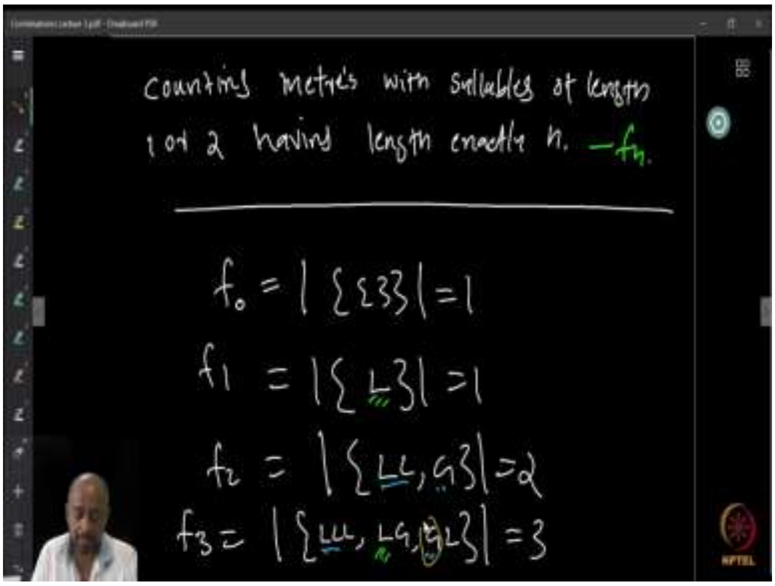

Now, I also gave you another homework question that I asked you to count the Sanskrit metres were the syllables have length exactly 1 or 2 and we want to count we wanted to count those metres which has length exactly n, where the length of the syllable can be either 1 or 2. So, how many distinct metres are possible with this property?

And we already said that, L denotes the laghu or the short syllable and G denotes the guru or the length two syllable. So, to find a recursion formula, what we are going to do is to first try to find a relation between these two things, something to the earlier thing, can we find such a relation?

So, we just look at the possible scenarios. So, let f_0 denote meters of length 0, you know you can only do in one way, basically you do not write anything, there is precisely one way to do that, so therefore $f_0 = 1$. Now, what is f_1? f_1 is metres of length exactly 1. Now, if the meter has exactly length 1 we cannot use syllables of length 2. So, guru cannot be used, only laghu can be used. So, the only possibility is to how the singleton L.

What about if the length is 2? If the length is 2 we can have either LL or G. LL has length 2 and G has also length 2. What about length 3 metres? Well, we can have 3 laghu's LLL or LG or GL. And if you spend a little more time you can figure out there is nothing else possible, so therefore exactly 3 of them are. Now, one observation that we can make is that, if the last syllable that we are going to add is a laghu, we are going to get a metre of length exactly n from a metre of length exactly $n - 1$.

So, if you take any metre of length $n - 1$ add a laghu at the end you will get a metre of length n, similarly if you take a metre of length $n - 2$ and add a guru at the end you will get again a metre of length n, but these two metres are different because the last syllable is a laghu in the first case and the guru in the second case and these are the only possibilities, because every metre of length n must either end in a laghu or in a guru.

Now, if you remove the last L you will get a metre of length $n - 1$ and if you remove the guru you will get a metre of length $n - 2$. So, there is a bijection between the metres of length $n - 1$ and the metres when ends in L, and there is a bijection between the metres of length $n - 2$ and the metres that ends in a G.

We can observe it here like for example, in the set f_3 we have that the ending is a G guru means that the previous one was the length and a one of the things in f_1 which is this precisely one, that is L and if it was a laghu it could be any of the metres in f_2 which is an LL or a G. So, once we have this observation we get the recurrence relation.

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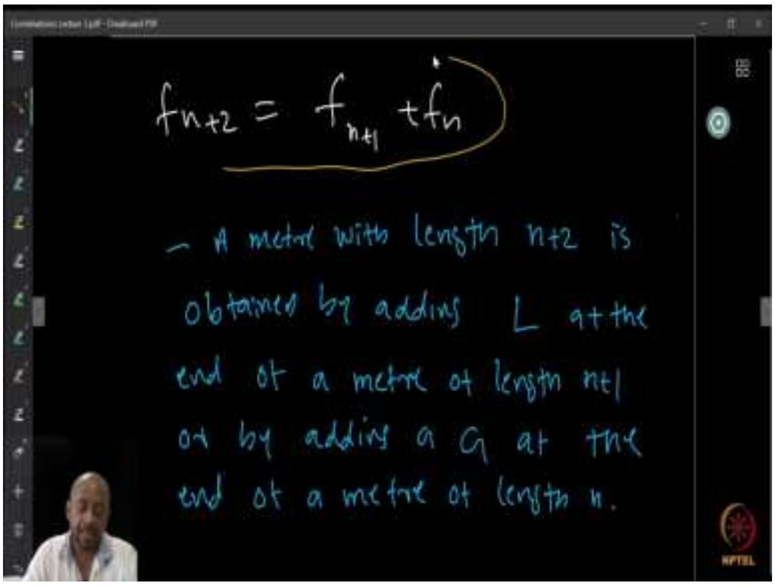

So, $f_{n+2} = f_{n+1} + f_n$. That is, a metre with the length $n + 2$ is obtained by adding L at the end of a metre of length $n + 1$ or by adding G at the end of a metre of length n.

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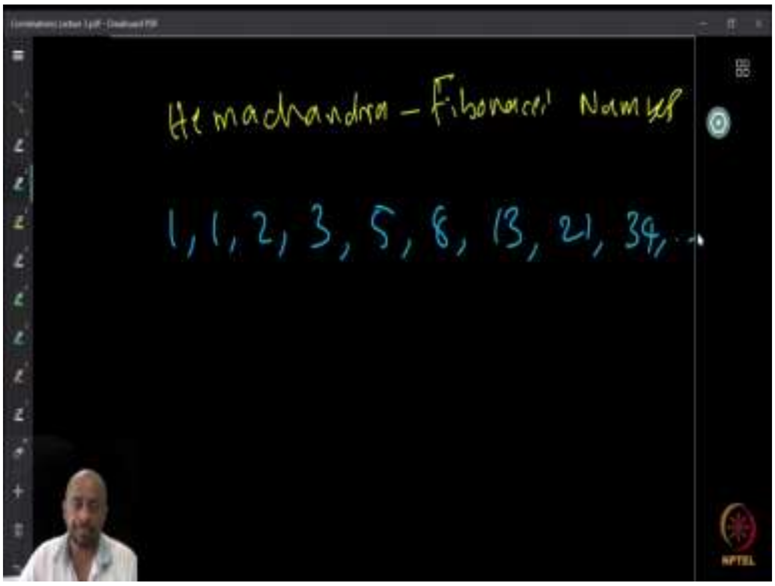

So, these numbers are called the Hemachandra-Fibonacci numbers. In fact earlier it used to be called just Fibonacci numbers, but then they found out that Hemachandra did the work before Fibonacci, he is an Indian mathematician in fact a Jain mathematician if I remember correctly, who lived before Fibonacci and in fact this was known even much before. For example I mentioned the Meru Prasthara Pingla.

So, the Pingla in his work on metres, his works on metres called Chandahshastra, Chandas is a metre, basically describe these numbers, he does not give you the the formula or anything, but then he described this you know he observed this kind of a pattern. And there was some work in between Pingla who was in maybe 500 or 600 BC, and to, after around maybe 1000 years, maybe like Hemachandra and then later Fibonacci came, in between there were some other work in which apparently these formulas were discovered, but we do not have the original, we only have the reference given by Hemachandra who talks about some other mathematicians who did discover this formula.

But we still give the credit to Hemachandra, because we do not have the original manuscript, anyway, so these are called Hemachandra or Fibonacci numbers. So the Fibonacci numbers or Hemachandra numbers occurs several times in combinatorics, it occurs naturally in many places in the nature, because the numbers will be something like 1, 1, 2, 3, 5, 8, 13, 21, 34 et cetera.

And if you look at many are most of the flowers will have their number of petals will be in these and then you know there are many other things and we can try to find out a formula to calculate the Fibonacci numbers we will do it later, when we look at generating functions, there are also other ways to calculate the formulas from the recursion relation maybe you can try you to find the formula for Fibonacci or Hemachandra numbers by using your own methods, just give it a try, it will be fun.

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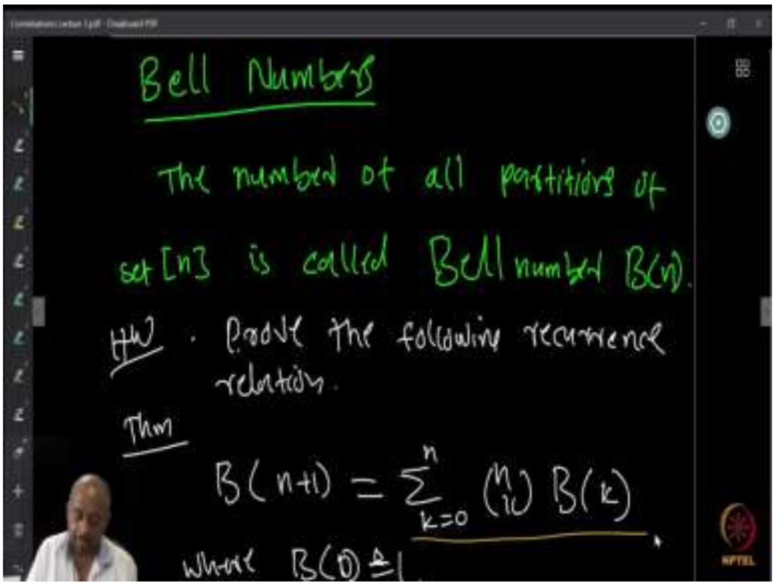

Now, so we are looking at partitions of n into exactly k parts, suppose we do not worry about exactly k, but we want to partition into any number of part. Now, how many are there? So, of course we know that you sum over all possible k ranging from 1 to n, we will get all possible partitions. So, therefore if you sum over all the Stirling numbers of second kind, $S(n,k)$ where k ranging from 1 to n, again we will get the number of all partitions.

So, therefore this number called Bell number can be calculated by the summation. Now, what I want you to do is to prove the following recurrence relation of Bell numbers. So, bell numbers follow the following recurrence relation that is a theorem:

$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k)$, where $B(0) = 1$.

So, $B(n)$ is the number of all partitions of the set $[n]$ to any number of blocks, so that is the definition and it should satisfy that recurrence relation. So, similar to the argument that we did in case of $S(n,k)$ prove that $B(n)$ also satisfies a recurrence relation that is given here. So, that is your homework.

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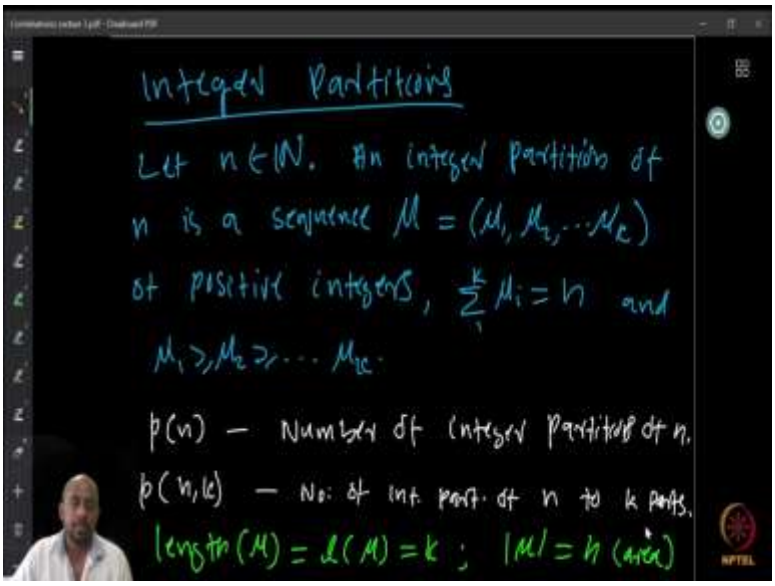

Now, we already looked at what is called compositions of integers. So, we have weak compositions and we have compositions, something very closely related is integer partition. An integer partition of an integer n is a sequence $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ of positive integers such that $\sum_1^n \mu_i = n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$.

So, the only difference is that the last part, so we wanted a composition of integer to be the positive integers where let us say λ_1 to λ_k such that $\sum_1^n \lambda_i = n$. But now we are saying that okay we also want to write this in decreasing order, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$.

Or in other words what we are saying is that we do not really worry about the order, so in the composition we are saying that if 1 is appearing at the first position and 2 is appearing in the second position is different from 2 is appearing at the first position and 1 is appearing at the second position.

But here we are saying that we will make all them all of the uniform by writing in the decreasing order. So, therefore we are not really worried about the order there, so that is what it makes a difference. So, integer partitions are precisely partition of the integer to k smaller integers such that their sum is exactly n.

Now, the total number of distinct partitions of a given integer n is denoted by $p(n)$ and $p(n, k)$ is the number of integer partitions of n into exactly k parts. Now, given a integer partition μ the length of the partition μ is the number of distinct parts which is k. If $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ the length of μ, $l(\mu)$ is the number of distinct parts which is k. The area of μ, denoted by $|\mu|$ is n, the number that μ is partitioning. So, that is called area. So, why it is called area? We will see very soon. So, let us look at some examples.

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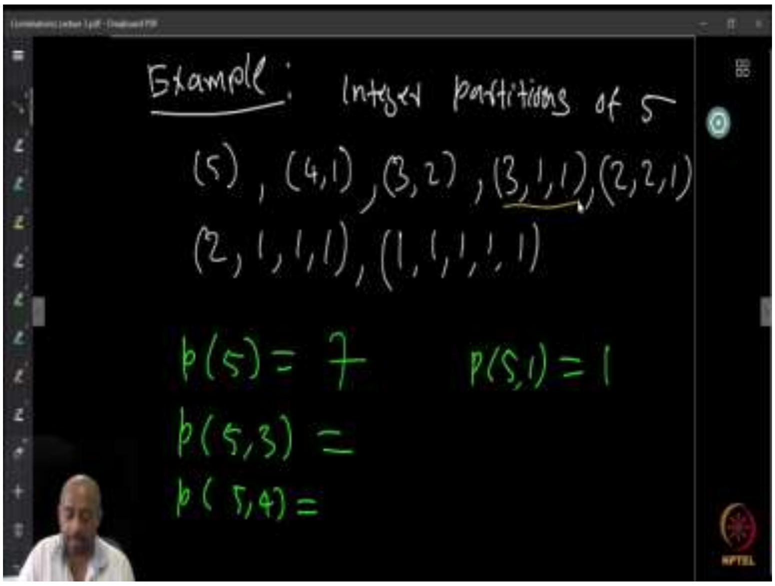

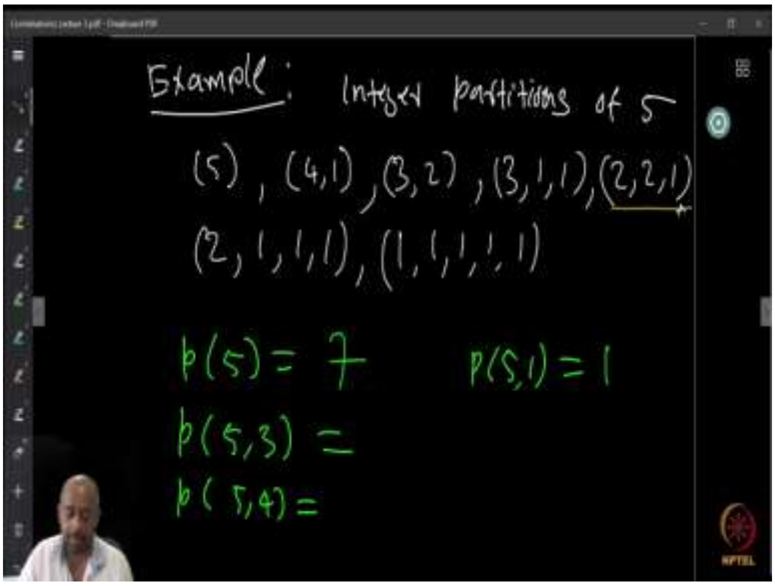

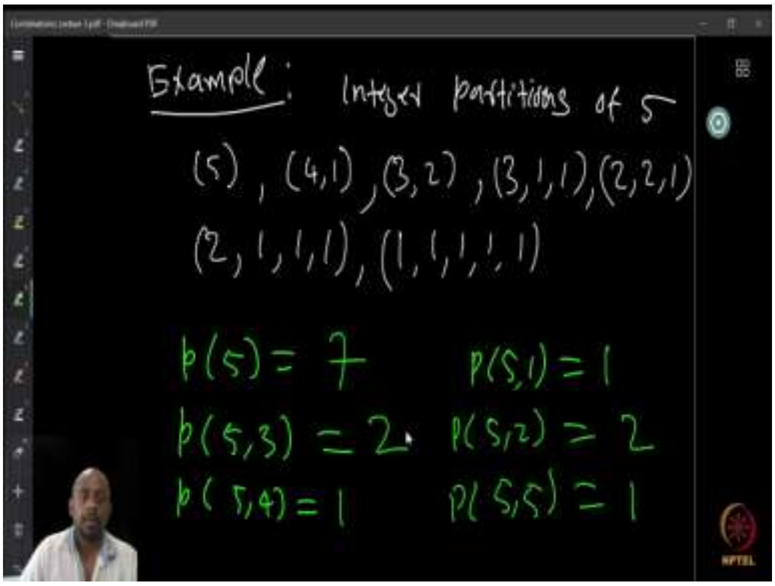

So, let us take the number let us say 5 and we want to find the partitions of 5. So, what are the partitions of 5? We can write it as (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).

These are possible ways of writing 5 as decreasing numbers, sum of decreasing numbers, how many possible ways are there? So, if I have not missed anything my $p(5) = 7$. That is the total number of partitions of 5 there are 7 of them. Now, what about $p(5,1)$? So how many partitions of 5 are there into exactly 1 part? There is only 1, (5) itself. What about $p(5,3)$? So, we need exactly 3 parts. So, what are the exactly 3 parts? So, there are two ways namely (3,1,1) and (2,2,1).

What about $p(5,4)$? Again, if you look at this there is only (2,1,1,1), and there is nothing else I can write, so there is exactly 1. Similarly, you can ask for $p(5,2)$, $p(5,5)$ and if you add all of this things it should be again 7. That is $p(5,1) = 1$, $p(5,2) = 2$, $p(5,3) = 2$, $p(5,4) = 1$, $p(5,5) = 1$ and 1+2+2+1+1=7

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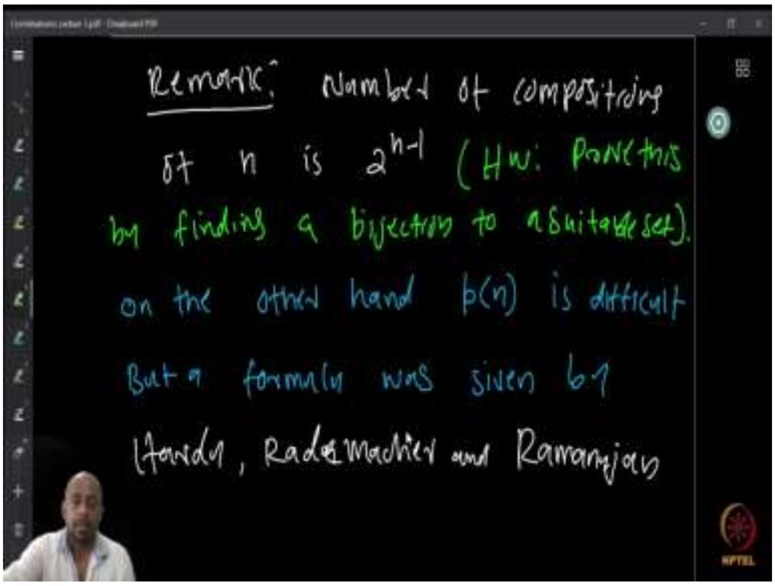

Now, just a remark, the number of combinations of n is 2^{n-1}. Now, I do not remember whether we proved it or not, I think we have not, so I can give it as a homework. So, homework, prove that the number of combinations n is 2^{n-1} by finding a bijection to a suitable set.

So, define a set which has exactly 2^{n-1} elements and then show that there is a bijection from the set of all combinations of n to the set with 2^{n-1} elements. So, that we will prove that, number of combinations is 2^{n-1}..

Now, for the number of compositions we have a nice formula 2^{n-1}, on the other hand $p(n)$ is kind of difficult to find a nice formula, but actually one exists and this formula was given by Hardy Rademacher and Ramanujan. So, the formula is as follows I will just state it.

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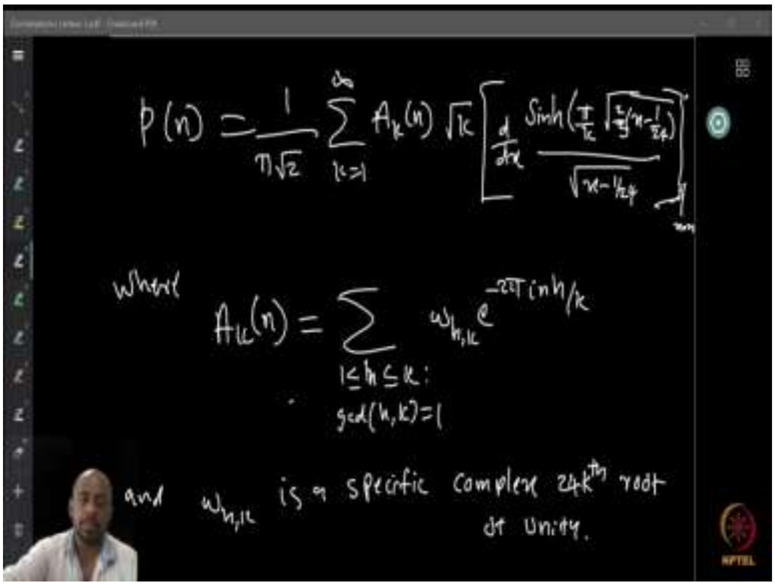

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)}\right)}{\sqrt{\left(x - \frac{1}{24}\right)}} \right]_{|x=n}$$

Where $A_k(n) = \sum_{\substack{1 \le h \le k, \\ \gcd(h,k)=1}} w_{h,k} e^{-\frac{2\pi i n h}{k}}$

And $w_{h,k}$ is a specific complex 24th root of unity.

So, this is the formula that is given by Hardy Rademacher and Ramanujan and this is you know this finds it exactly, $p(n)$ is equal to this which is itself in very surprising. And, as you can see it is not very easy to calculate. Now, so there are some approximations of this from the expansion of this formula, but that is not again something that we are going to look at.

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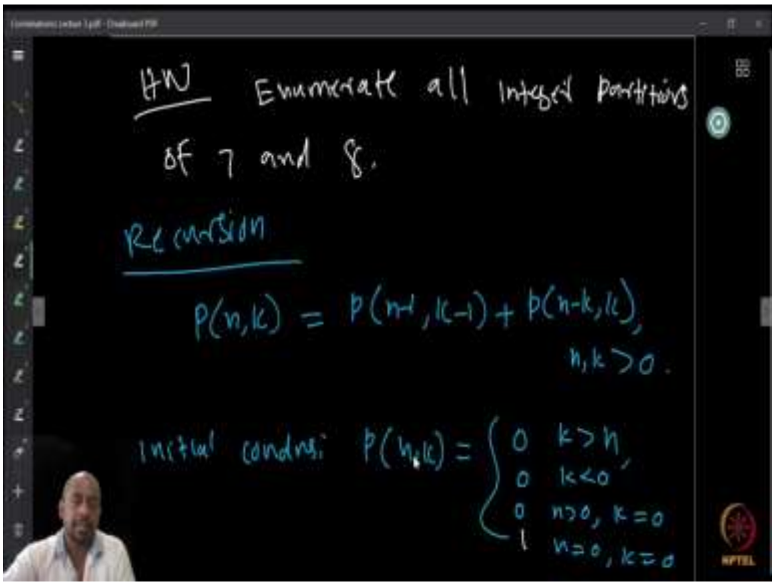

Now, as a homework you can enumerate all integer partitions of 7 and 8, so basically write down the integer partition of 7 and 8, how many are there, count the number of them and also write each one individually. So, this is nice homework just to count.

Now, we can also prove a recursion formula for counting $p(n,k)$, where the integer partitions of n into exactly k parts. So $p(n,k) = p(n-1,k-1) + p(n-k,k)$, where n and k are positive, with the initial conditions $p(n,k) = \begin{cases} 0, & k > n \\ 0, & k < 0 \\ 0, & n > 0,\ k = 0 \\ 1, & n = 0, k = 0 \end{cases}$

So, using this boundary conditions, we can try to prove the to find the bounds. If you get an idea how to work this out, you are welcome to try.

Now, we are going to prove the recursion relation by the following. So, we are going to prove the recursion formula by again writing $p(n,k)$ which is the number of all partitions of n into k parts, so what we do is that we look at the set which is the set of all partitions of n into exactly k parts, of course its cardinality will be $p(n,k)$.

So, we will denote $P(n,k)$ as the set of all such partitions. And then we are going to write $P(n,k)$ the set, as a disjoint union of two sets and show that they are in bijection with partitions of $n-1$ into $k-1$ parts and partitions of $n-k$ into k parts. So, I want you to think about this and try to do it yourself, I gave you the idea, but it would be better if you actually tried to do it yourself.

So, try to spend some time thinking on this and find out how to find a bijection to this on the that set on the left side and the sets that we are counting on the right hand side. So, here we are going to do the following. So, what we do is that, we will write two sets, let us say Q and R, let us write two sets Q and R where the set Q counts some particular type of partitions and R counts another type of partitions, what are these type of partitions that it is going to count?

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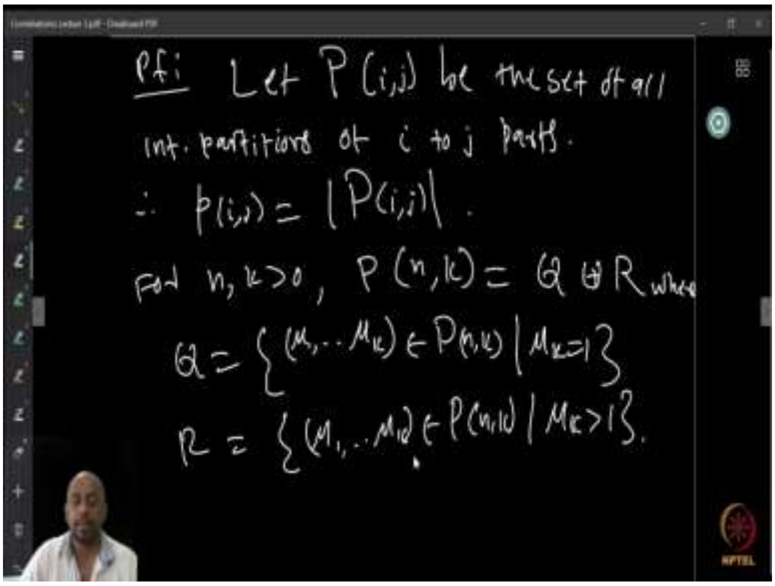

So, let $P(i,j)$ be the set of all integer partitions of i into j parts. Therefore cardinality of $P(i,j)$ is $p(i,j)$. Now for n and k positive, let us write $P(n,k)$ as Q disjoint union R, where $Q = \{(\mu_1, \ldots, \mu_k) \in P(n,k) \colon \mu_k = 1\}$ and $R = \{(\mu_1, \ldots, \mu_k) \in P(n,k) \colon \mu_k > 1\}$

So, now since we are writing in the decreasing order of numbers, when the last part are different, then the sets must be disjoint, but now since we are looking at all possible partitions where $\mu_k = 1$ and $\mu_k > 1$, they will cover all partitions into exactly k parts, so therefore $Q \cup R$ will have exactly $P(i,j)$. So that is clear.

Now, we want to find a bijection from Q to the set of all partitions in $P(n-1,\ k-1)$, now that is easy, because well what you do is that you take the partition $(\mu_1, \ldots, \mu_k)$, and $\mu_k = 1$, so what you do is you just remove μ_k, so, you will get a partition $(\mu_1, \ldots, \mu_{k-1})$, but this is a partition of $n-1$ into $k-1$ parts, because we have just subtracted 1, because $\mu_k = 1$ we just subtracted 1, and we have removed one part, so therefore we have exactly how many parts? We have exactly a partition of $n-1$ into $k-1$ parts.

Now, if you take any partition of $n-1$ into $k-1$ parts, we can add a 1 at the end and we will get a partition in Q, because it is ending in μ_k and it has exactly k parts and the sum is equal to

n. So, therefore there is a bijection. Similarly, if you take R, and we know that μ_k is strictly greater than 1, this means that all of the μ_i's strictly greater than 1.

Now, since all of the μ_i's strictly greater than 1, I can subtract 1 from each of them, if I subtract 1 from each of the μ_i's, the sum of the new μ_i's (say $\mu_i{}'$)are going to be exactly $n-k$ because we subtracted one from each of the k, μ_i's. So, it is a partition of $n-k$ into k parts.

But now again this is also a bijection, why? Because you take any partition of $n-k$ into k parts exactly k parts, I can add 1 to each of the parts and I will get a partition of $n-k$ and because I added 1 to the last part, that last part will be strictly greater than 1. And therefore, we have, that belongs to R say partition of n into k parts. So, therefore this gives the entire bijection, so we have $p(n,k) = p(n-1,k-1) + p(n-k,k)$

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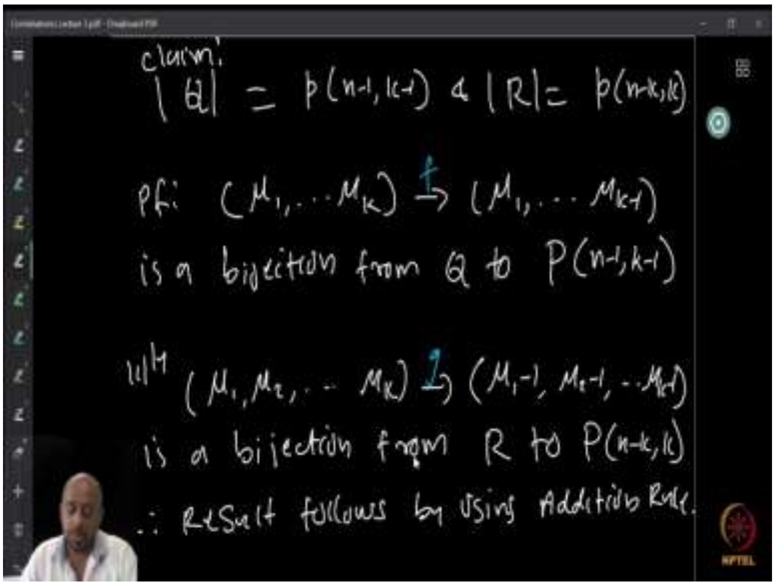

So, to write it more formally, so we claim that $|Q| = p(n-1,k-1)$ and $|R| = p(n-1,k)$. Prove this as follows, $(\mu_1, \ldots, \mu_k) \xrightarrow{f} (\mu_1, \ldots, \mu_{k-1})$ is a bijection from Q to $P(n-1,k-1)$, it is a bijection because I can just add one at the end from any element in $P(n-1,k-1)$, to get an element in Q and vice versa, I can remove 1 at the end to get a an element of $P(n-1,k-1)$.

Similarly, $(\mu_1, \ldots, \mu_k) \xrightarrow{g} (\mu_1 - 1, \ldots, \mu_{k-1} - 1)$ is a bijection from R to $P(n-1,k)$, because I can add 1 to each of the partitions in $P(n-1,k-1)$, and I will get a partition of n into exactly k parts again, where the last part is strictly greater than 1. So, that is it. So, we now using the addition rule because they have disjoint sets Q and R, we get the recursion relation.

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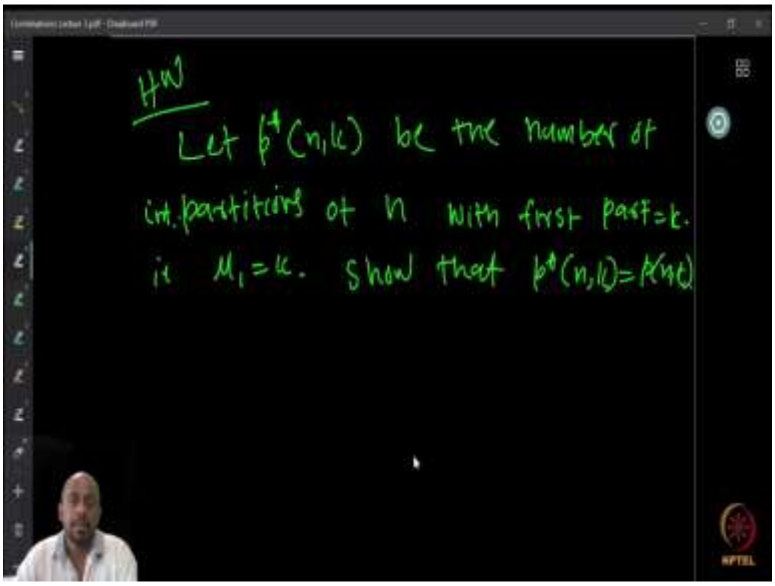

So, here is a very interesting homework, its kind of surprising, because I am going to define $p^*(n,k)$ to be the number of integer partitions of n with the first part being exactly k. So, $\mu_1 = k$, so we have μ_1, μ_2 et cetera, so the largest part is k, so $p^*(n,k)$ counts the number of integer partitions of n, where the largest part is k. So $p(n,k)$ was the number of integer partitions of n into exactly k parts, now we are saying that the largest part is k.

Now, show that $p^*(n,k) = p(n,k)$. So, basically the number of partitions where the largest part is k is in bijection with the number of partitions where the number of parts is k. Now, this is very surprising, but you know it is still true. So I want you to find this proof and when you do the proof make sure that you use only the results that we have used so far.

Of course, you can use some new ideas, but that is okay, but we will use only the results that we have come across so far, because we are going to come up with a very different proof for the same which is a pictorial proof. So, that is going to be presented but we do not want that proof to be use here. So, try to come up with a proof on your own