Combinatorics

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Stirling and Hemachandra recursions

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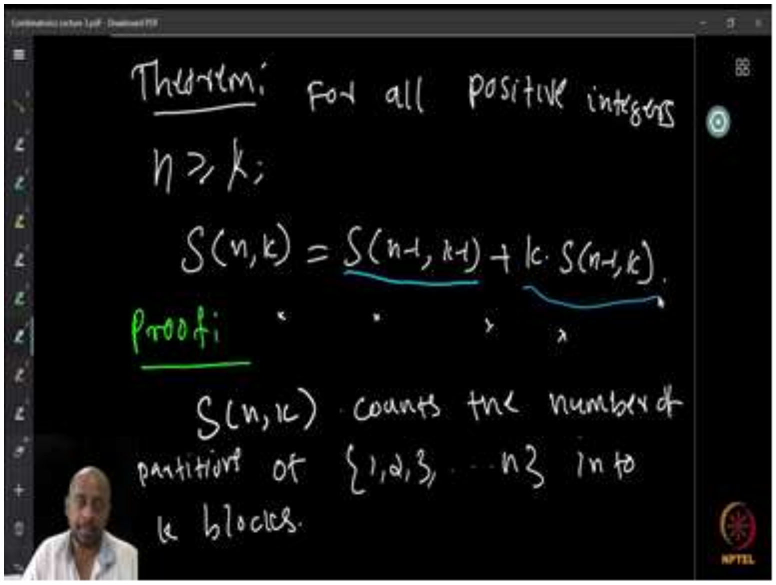

In the previous lecture we were looking at recursions and also we defined $S(n, k)$, the Stirling number of second kind, to be a number that counts the number of partitions of an n-element set into exactly k blocks. So, at the end of the lecture we asked to try to prove the following result the recursion formula that: $S(n, k) = S(n-1, k-1) + k.S(n-1, k)$

I hope that some of you at least have been able to work this out and in any case we are going to see a proof now. So, what we want to do is to show the identity $S(n, k) = S(n-1, k-1) + k.S(n-1, k)$. So, to do this we find a bijection between the set of all partitions of a set, of an n-element set to k blocks to some other sets, or union of sets.

So, how do we do this? Now, if you look at $S(n-1, k-1)$. This says that we have to partition an $(n-1)$-element set to $k-1$ element set. So, somehow, we need to find a way to partition $S(n, k)$ to two sets where one of them has cardinality $S(n-1, k-1)$.

Now, how do we do this? Well, look at a particular set let us say set $\{1, 2, \ldots, n\}$. So, let us say that, $S(n, k)$ counts the number of partitions of the set $\{1, 2, \ldots, n\}$ into k blocks. So, now we have a partition of the set $\{1, 2, \ldots, n\}$ into k blocks. Now, look at the partitions itself.

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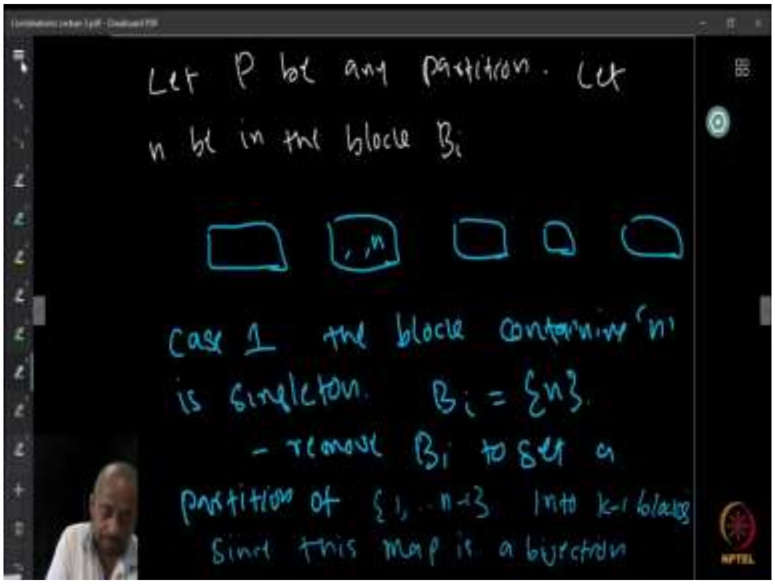

Let P be any partition, so if you take this partition $P,$ we can check the block in which the element n is there, so we what we are going to do is that we are going to look at the block in which the element n is there, because on the right hand side we see that we are only partitioning an $n-1$ element set. So, we can look at the set $\{1, 2, \dots, n-1\}$ and we have to work with that.

So, what we are going to do is that, we make the element n to be special and say that, look at the block in which the element n belongs to. Now, let n be in a block let us say B_i. Now, so we have this partition of the set into k blocks, let us say that we have some blocks and then we have the elements of the set and in one of the blocks there is this element $'n'$. Now we can look at two different cases, one case is that, the block containing $'n'$ is singleton, because each block is a set, now the block containing n is a singleton means that, that that block $B_i = \{n\}$, there is nothing else there.

Now, whenever a block contains n as a singleton, I can remove this block and then what do I get? I will get $(k-1)$ blocks which is a partition of the set $\{1, 2, \dots, n-1\}$. because the element n is not there in any of the other blocks and we have removed the singleton containing n from, that block itself, so we have $k-1$ blocks and we have a partition of the $n-1$ element set $\{1, 2, \dots, n-1\}$ to exactly $k-1$ blocks.

So, therefore for every partition in which n is a singleton, we have a corresponding partition of the set $\{1, 2, \dots, n-1\}$ in which n is not appearing and having exactly $k-1$ blocks. On the other hand if you take any partition of {1, 2, …, $n-1$} into $k-1$ blocks you can add a new block which contains singleton n and this will be a partition in $S(n,k)$.

So, therefore there is a bijection between them. So, therefore all the partition containing n as singleton is in one to one bijection with $S(n-1,k-1)$.. So, therefore this part is already accountable. So, in this case what we do? So, we remove B_i to get a of partition of set {1, 2, …, $n-1$} into $k-1$ blocks. We just observed that this map is a bijection.

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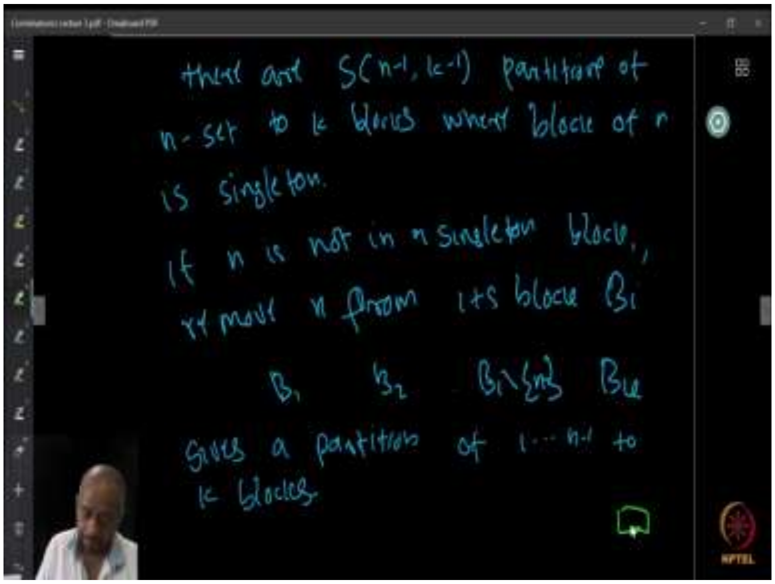

So, now since this map is a bijection there are $S(n-1,k-1)$ partitions of the n-set to k blocks where block containing n is a singleton. So, that counts the first part. Now, what about the remaining, if n is not in a singleton block, then clearly there will be at least two elements in the block containing n.

So, if n is not in a singleton block, remove n from its block B_i. Now, this block is non-empty and it contains at least one element, this block is non empty and therefore, we get a partition of the $n-1$ element set into exactly k blocks. So, if n is not in a singleton block then remove n form its blocks, let us say B_i, so we will get a partition of {1, 2, …, $n-1$} to k blocks.

So, now this same partition we would have got, if instead n was a part of B_1 or B_2 or B_k. So, therefore corresponding to one partition in the $n-1$ element set into k blocks, we can get k different partitions of the n element set into k blocks because you take this partition just add the element into n to any of the k blocks in k possible different ways and we will still get a different partition of the set $\{1, 2, \ldots, n\}$.

So therefore, what we have is the following that corresponding to the partition of an $n-1$ element set into k blocks we get k distinct partitions of n element set into k blocks and in each of this we observed that n is not a singleton, because we have added into one of the k blocks. So therefore, they are disjoint from the earlier set that we were looking at and therefore we can add them.

So therefore, we get $S(n-1, k-1) + k.S(n-1, k)$. And we also saw that this one is a bijection, because for every such set we can produce a set here and for every k partition in the $n-1$ element set gives exactly k different partitions in that and any of those k partitions will give back precisely the single partition in the $n-1$ element set. So therefore, this is a bijection and therefore we have $S(n,k) = S(n-1,k-1) + k.S(n-1,k)$. So, this is our proof.

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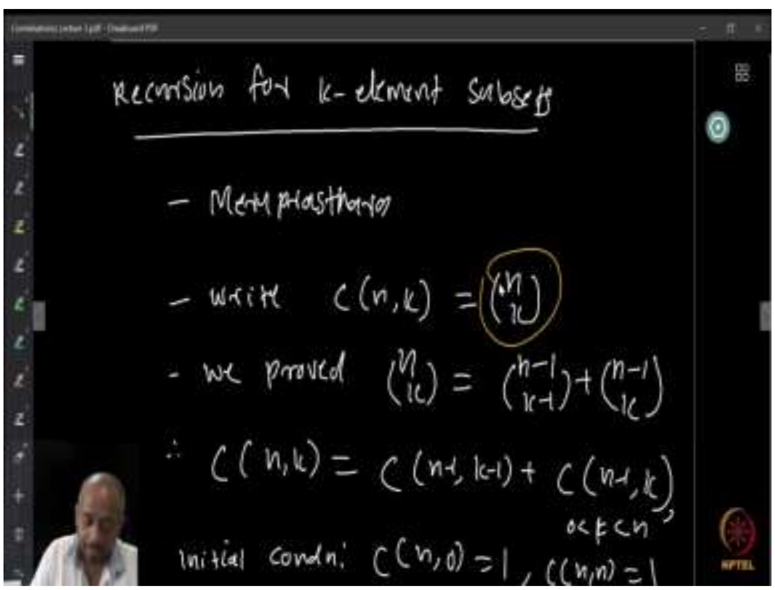

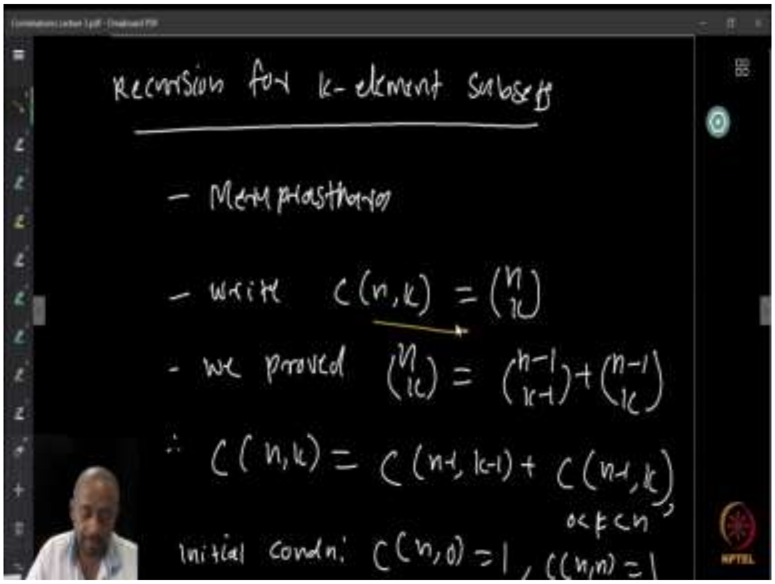

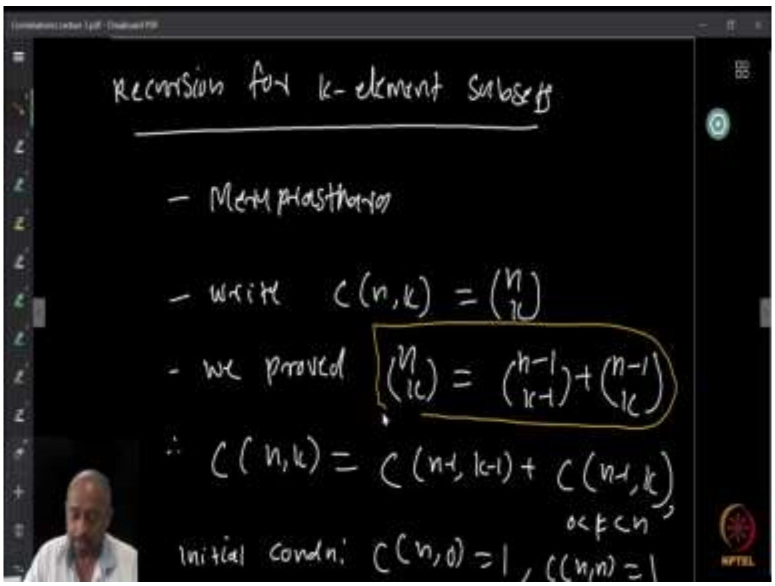

Now, let us look at recursion for k-element subsets, we already have looked at k- element subsets and this recursion relation that we are going to look at is already occurring in Meru prasthara and in fact I asked you to look at several properties of Meru prasthara that you can find or Pascal's triangle that you can find and if you have looked at this carefully you would have already seen one of this property that we are looking at.

The k element of sets of n element set is denoted by $\binom{n}{k}$, but now let us write it as $C(n,k)$, just for writing it as a recursive relation. Now, we already proved in fact that right this relation that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. We used a combinatorial proof, we could also have done using algebraic techniques.

Now, because we have this we can directly use this property to write the recurrence relation: $C(n,k) = C(n-1,k-1) + C(n-1,k)$ for $0 < k < n$, with the initial conditions $C(n,0) = 1$ and $C(n,n) = 1$, these two properties we already observed, these are the boundary conditions and therefore this defines a recursion formula.

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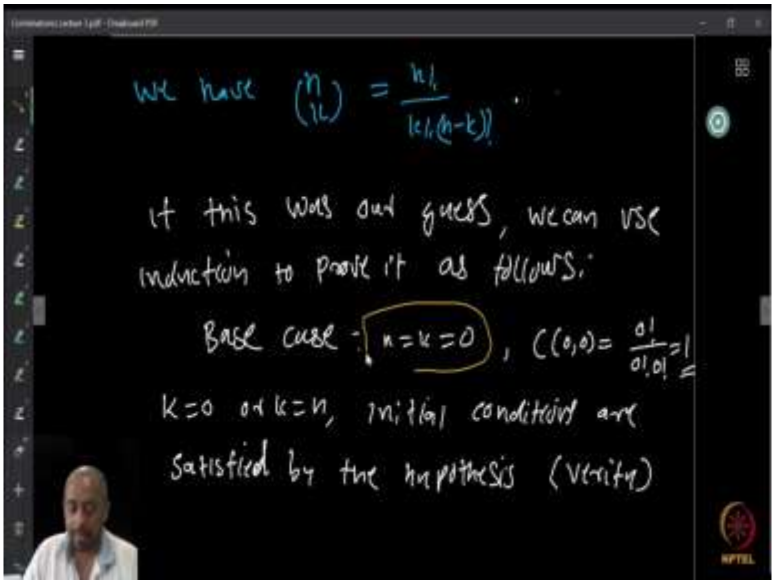

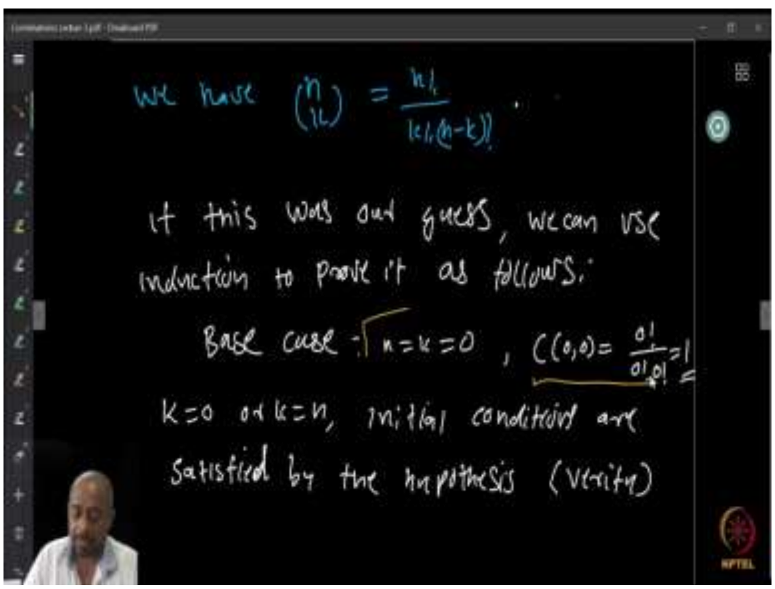

Now, let us suppose that we did not know a formula for $\binom{n}{k}$, we already proved that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. But suppose we did not know this, but suppose we actually tried out several examples and came up with a proof, I mean not came up with came up with a guess that $\binom{n}{k}$, grows something like $\frac{n!}{k!(n-k)!}$, but suppose, we did not have a proof but only a guess.

Now, if we have this guess we can use induction to prove this from the recurrence relation as we did in the earlier cases. How do we do? So, we start the proof as follows, the base case is $n = k = 0$ and we observe that$\mathrm{C}(0,0) = \frac{0!}{0!0!} = 1$.

So, therefore the base case holds and even look at the other initial conditions $k = 0$ or $k = n$ and the initial conditions are satisfied by the induction hypothesis that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, if one of n and k is strictly 0 or basically the boundary condition. So, once it reaches the boundary conditions, the initial conditions are satisfied this we can verify by substituting in the formula. Now, this I know you can verify it yourself.

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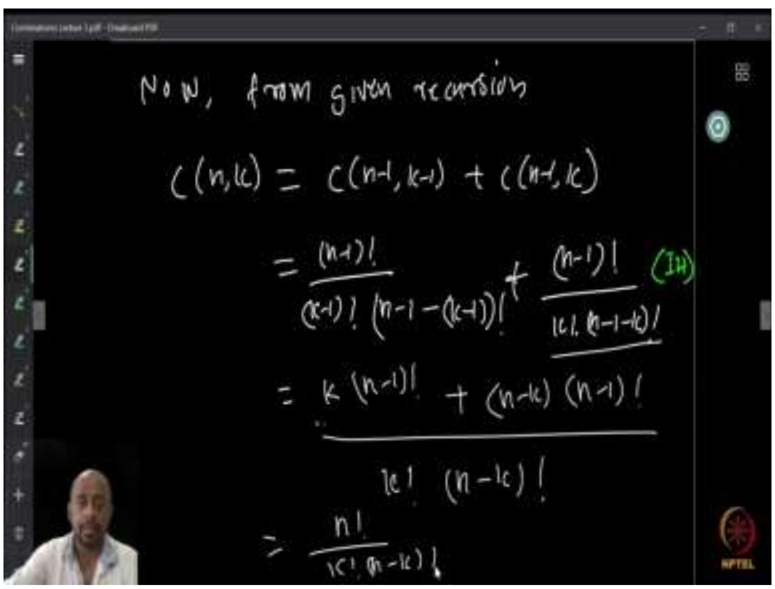

Now, what we do is that we substitute our induction hypothesis into the recurrence relation. So, we have to prove that $C(n,k)$ is equal to something, but now by the recursion formula, we have $C(n,k) = C(n-1,k-1) + C(n-1,k)$

$= \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} + \frac{(n-1)!}{k!(n-1-k)!}$, by induction hypothesis.

$= \frac{k(n-1)!+(n-k)(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!}$.

So the guess works out.

So, therefore we showed that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for every n and k. So, that is it. So, we have a proof using induction as far as we have the guess. Now, one might ask that we already know what is this $\binom{n}{k}$, even the formula then why do we even need the recursion formula?

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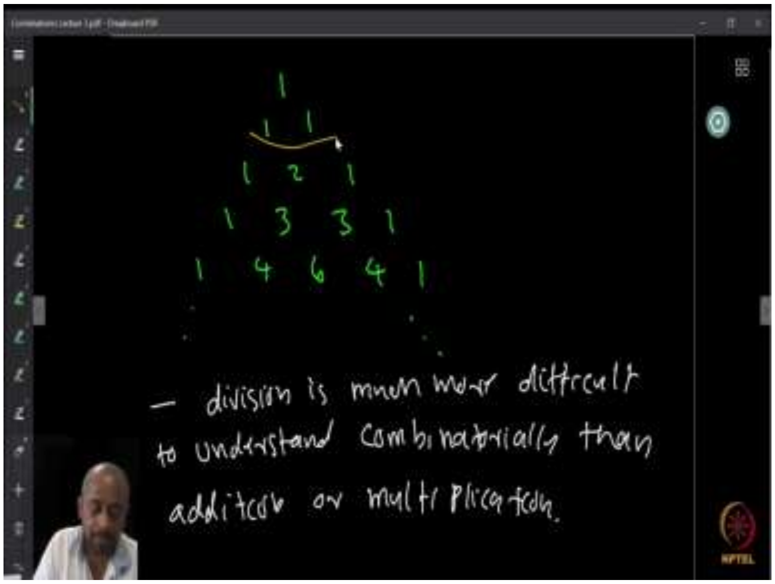

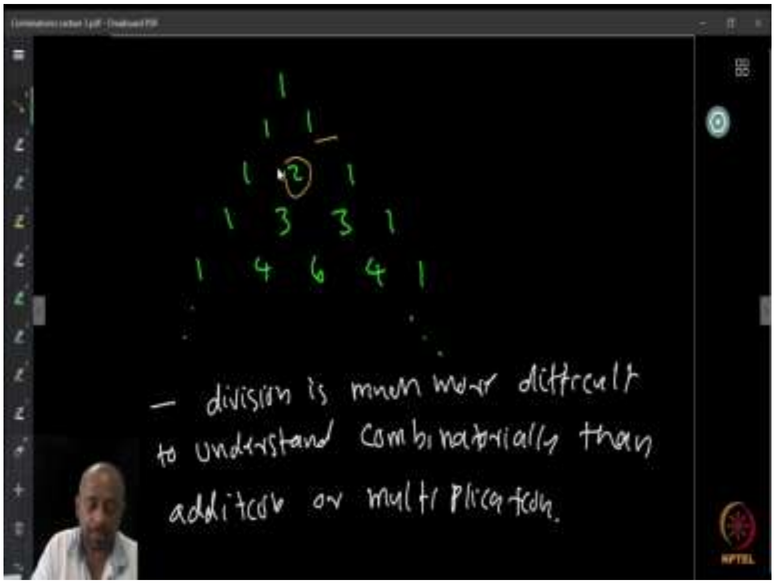

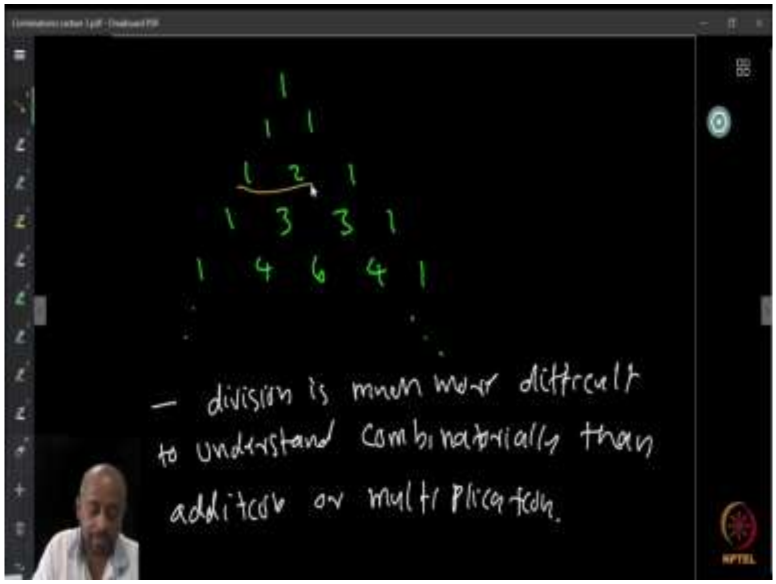
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— division is much more difficult to understand combinatorially than addition or multiplication.
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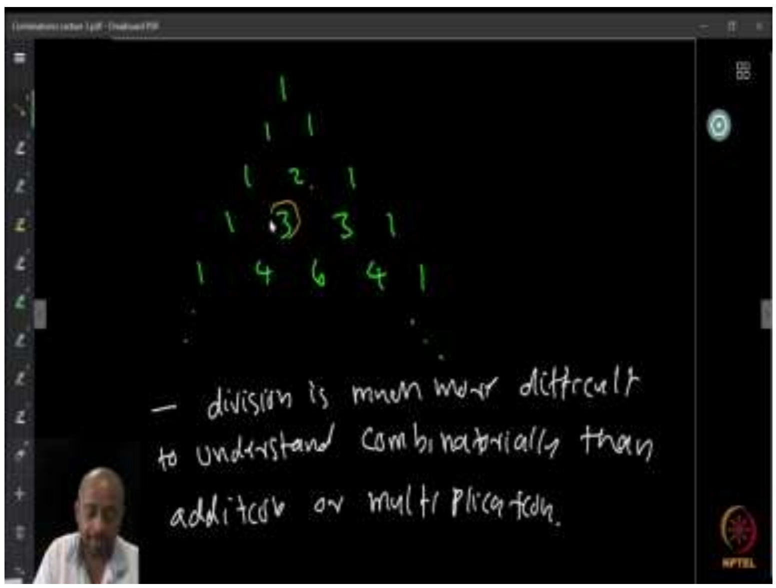
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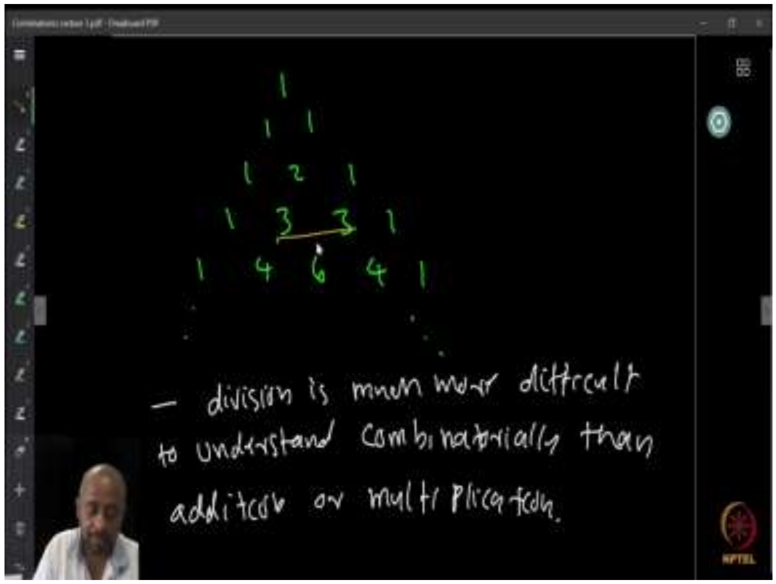
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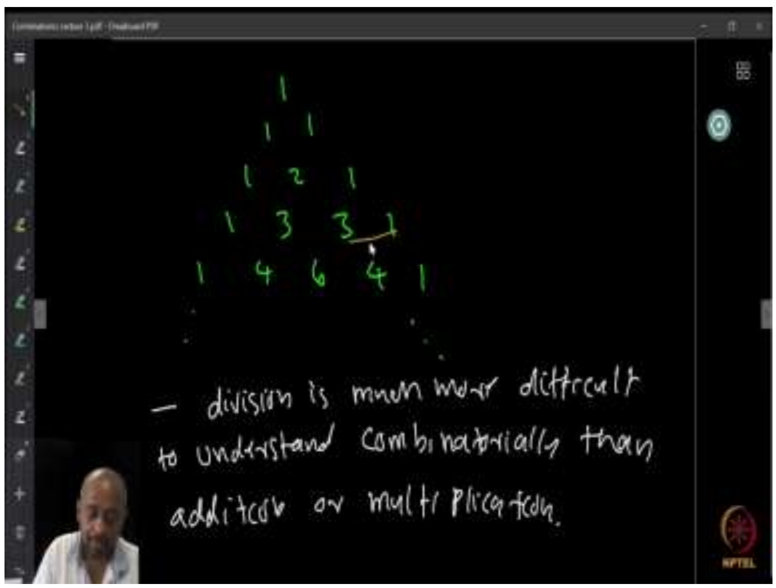
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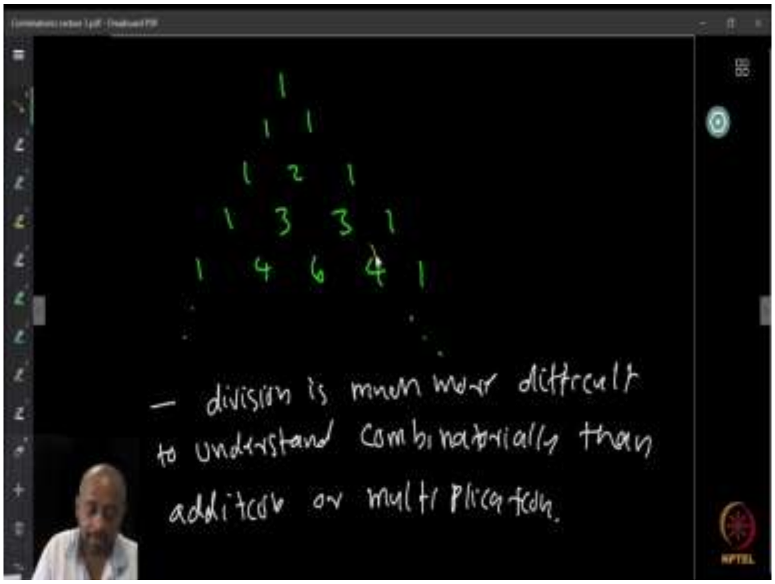

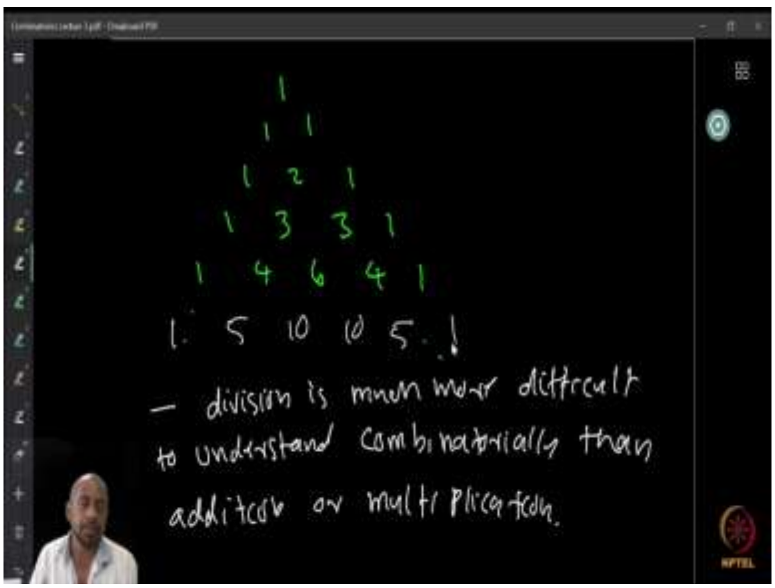

So, the answer is that, division is much more difficult to understand combinatorially than addition or multiplication. And in fact, if we want to write an algorithm for example, we can obtain this as some of earlier known values where in this other case we have to find factorials of larger numbers and then we have to do this product and division.

On the other hand, we can just use summation to find out all the values, for example if you look at the Pascal's triangle or Meru prasthara now you will see that if you sum these two this is the identity that we have just written down these two we will get this one. Similarly, if you add this and this you will get this, if you add this and this you will get this, this will give this et cetera.

So, we can we can even write down further, we will say that okay, what is this here which is going to be 1 then 5 then 10, then 10, 5 1. So, this way we can continue on to write just by adding in the elements of the previous row we can find the next row easily. So, there are other reasons we will come to that maybe sometime in the later part of the course.