# Combinatorics

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## Set Partitions and Stirling Numbers

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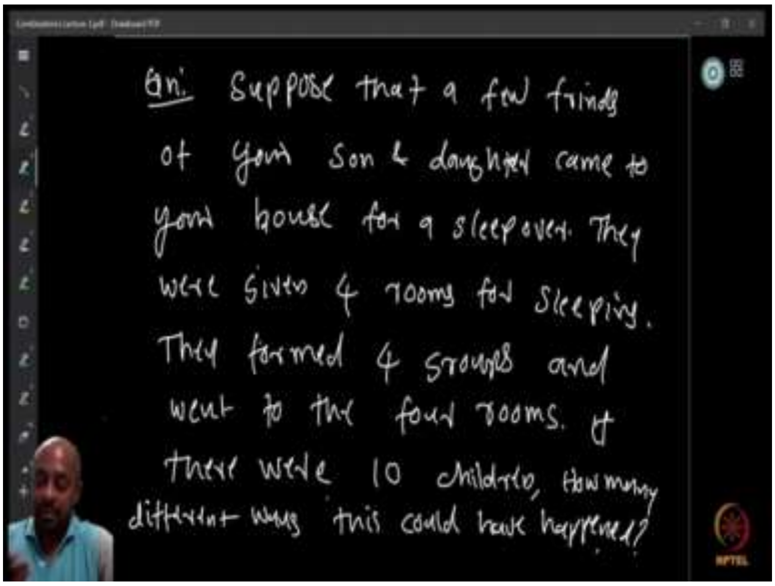


Here is another interesting question that we want to look at. Suppose you have some kids maybe, a son and a daughter maybe and then you invite their friends to come for a sleepover. So, they come and stay for a night. So, their friends arrive and then, there is total 10 kids, including their friends and the children together there are 10 of them.

Now when there 10 children, you cannot put them all into the same room because the rooms are not internally easy to accommodate 10 people. So, you say that okay we do not mind if you all want to put in 1 room but let us not do that. We will give you 4 rooms and you must use the 4 rooms to sleep.

So, we are given 4 rooms to the 10 children. Now you say that okay you go to the 4 rooms and then whichever groups you want form you can, who all want to sleep with others decide and go and find the corresponding rooms and then, let us know and then go to the rooms. Now how many different ways this can happen because, depending on the children's preference whom to group with, there can be many possible ways to do this. So, how many different possible ways this can happen?

So, how do you solve this? If you ask this, then there is something which is not very clear about this question. So, what is not clear is that when we say how many different ways, what exactly do we mean? Because, I can say that, the children let us say A, B and C are going to the room number 1 because I gave 4 different rooms. So, room number 1 is not the same as them going to room number 2 because rooms are distinct.

If I consider the rooms are distinct, then they are not the same, but maybe if you do not consider the rooms, I gave you 4 rooms does not matter. What matters is the way the children are grouped together. I want to just see that who are all going together to sleep. But if I also want to know, who all going to which room, then that is a different question. So, there are 2 different questions one can bring from this.

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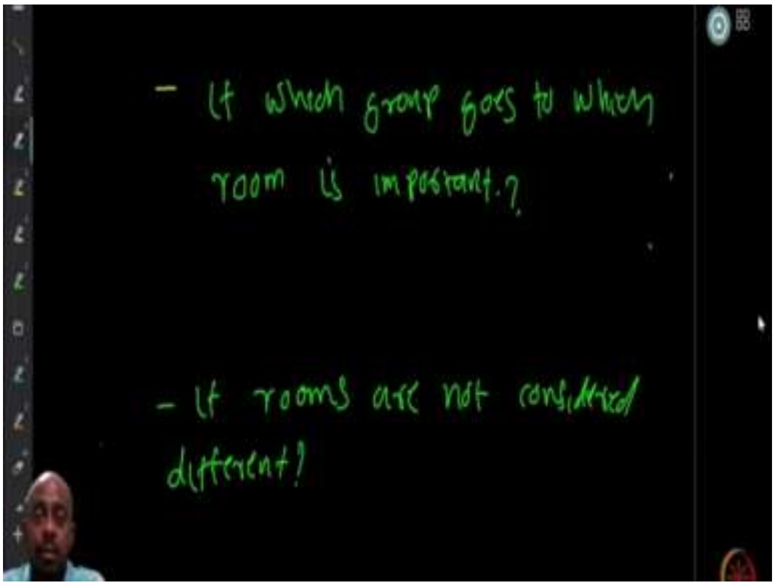


Now, both of them can be interesting question. So, therefore, we want to look at both of them. So what happens if which group of students or children go to which room is important and what happen if the rooms are not considered different, that is which group goes to which room is not important but what kind of groups are formed is only interesting.

Now, if you think about it, you can see that these 2 questions even though they can be very different in their values, they are very related as well. So, can you think of some argument to say that solving one is like solving the other also in some sense?

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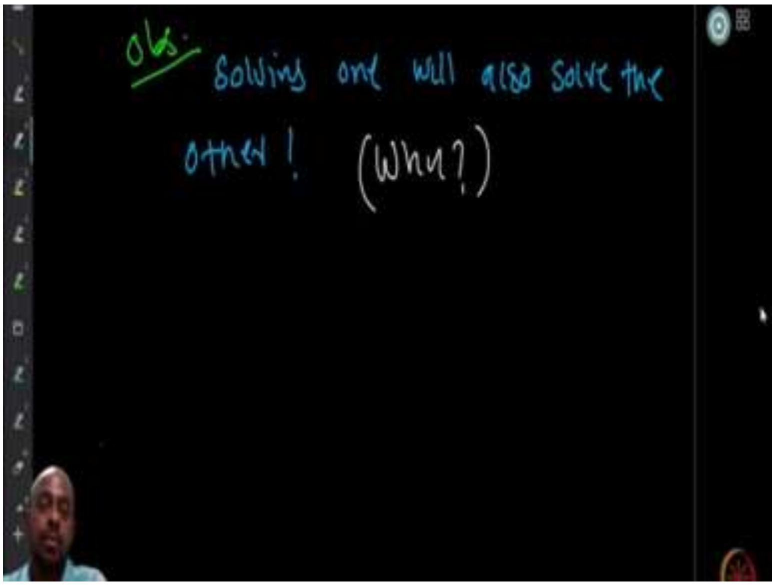


So, this is an easy observation but I want you to think about this. So, think about it and tell me why solving one is as good as solving the other. Once you get an answer to the first one, you know how to solve the second one or vice versa. So again, I do not want to give all these small small whys right answers to all these small 'why's' because if you do not think about this, you are never going to learn things.

So therefore, some of these question when I write this "why" and do not explain why, it should be something, which if you think about for some time, you should be able to come up with. It is not going to be a very difficult thing but something interesting and you should be able to come up with that argument. If you are not able to do that, then there is something that you have to work more. So, hoping that you solve, let us continue.

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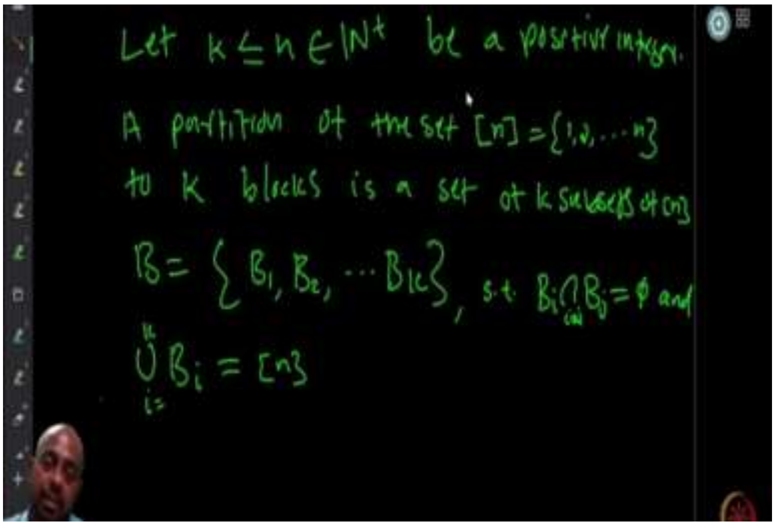

Now, let us look at the more than general statement of what we were looking at.

So let $k \leq n$ be positive integer. A partition of the set, $[n] = \{1, 2, \dots, n\}$ to $k$ blocks is a set of $k$ subsets of the $[n]$. Let us say, $B = \{B_1, B_2, \dots, B_k\}$ where $B_i$'s are blocks such that $B_i \cap B_j = \emptyset$ and $\bigcup_{i=1}^{k} B_i = [n]$.

So, I think I already mentioned this notation $[n]$ is the set $\{1, 2, \dots, n\}$. Now, so what we have defined here is a partition of a set. So, we have a set with $n$ elements I am partitioning into $k$ blocks such that they are pairwise disjoint and the union is the whole set and now we want to find the number of possible such partitions.

Number of partitions of the set $\{1, 2, \dots, n\}$ to the $k$ blocks with this property. If you just look at this definition, it would be very clear that the earlier question that we were looking at regarding the children's sleepover party was precisely the same, in some case.

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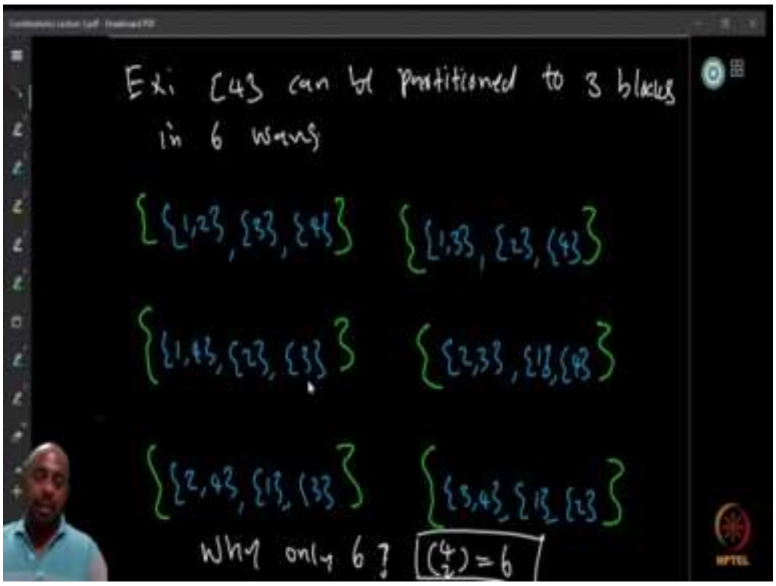


Now, here is an example. So, if you take the set 1, 2, 3, 4 then you can partition this it into 3 blocks in 6 different ways. So, I want to partition into exactly 3 blocks. That can be done in 6 different ways that is the claim. So, let us look at the 6 sets that I am going to give and see whether they are the partitions first. So, here is the sets: {{1,2}, {3}, {4}}, {{1,4}, {2}, {3}}, {{2,4}, {1}, {3}}, {{1,3}, {2}, {4}}, {{2,3}, {1}, {4}}, {{3,4}, {1}, {2}}.

So, these 6, we can see that they are each blocks are disjoint in each of these partitions. So, these are all partitions and you have exactly 3 blocks.

A two element set and two singletons and since we want 3 blocks and we have 4 elements, the only way to partition is to have like 2 elements in one set and 1 element in the second set and again one element in the third set. And this will tell you why we can only have 6 possibilities. So, maybe you argue it slightly more formally. Why only 6 are there? So, 6 are already there but why there are no more. So, give me an argument for that.

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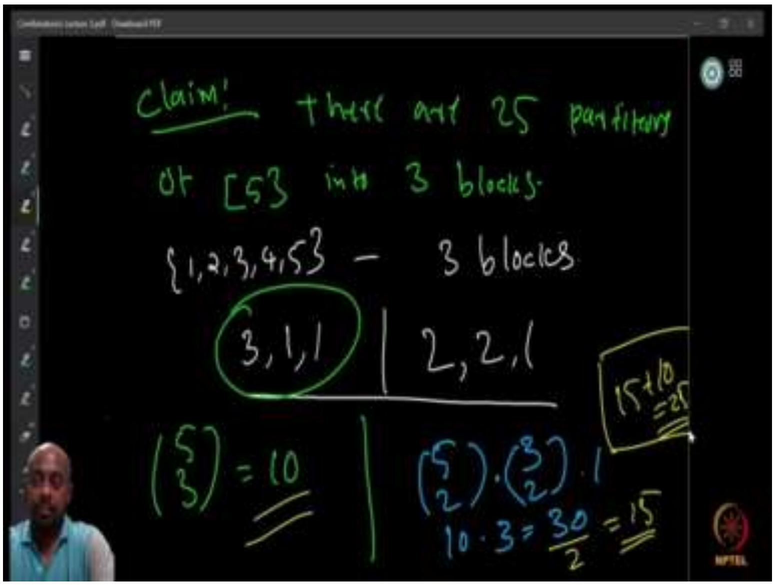


Now, another example.  There are 25 partitions of the set {1, 2, 3, 4, 5} into 3 blocks. We said that, when there are only {1, 2, 3, 4}, number of partitions, into 3 blocks is only 6. Now, when you add 1 more, we have 25 of them. Now how is this exactly 25? Can you give an argument why this is 25? So, maybe you should stop and think about this for some time before going further because if you can do it on your own, that is the best.

Now, let me give an argument if you have thought about it for some time, please continue. So, we are looking at the partition of set {1, 2, 3, 4, 5}. So, there is exactly 5 elements. Since we are looking at exactly 3 blocks, what we can observe is that the only possibilities are, you can have elements, the sets in the partition be of cardinality either 3 and if there is 3, and since we need exactly 3 blocks, if you take 3 elements into one set, the other 2 must be 1, 1 because there is only 5 of them. So, it could be, it should be like something like cardinality 3, 1, 1. So, we should have the blocks of cardinality 3, 1, 1. This is one possibility.

Other possibility is that you have 2, 1 subset has 2 elements, then since there are only 2 more and we have 3 elements so we need to have another 2 and then 1 and you cannot put 4 into 1. You cannot put 1, 1, 1, 1, 1, etcetera because you are only allowed 3 blocks. So, one can verify that the only 2 possibilities are blocks of size 3, 1,1 or 2, 2 , 1. Now once you observe this, we know that since we have exactly 3 blocks and you select the first let us say, let us just concentrate on this part 3, 1, 1.

Suppose you select a 3-element subsets from the 5 elements so which means that how many ways you can do this. You can do it in $\binom{5}{3} = 10$ possible ways.

Then we have only choice is to take the remaining 2 elements as singleton. So, there is 3 elements are already gone. The remaining 2 must be singletones and there is no choice there. So, we have exactly 10 possible ways to do this. On the other hand, if you look at the 2, 2, 1 configuration, then you can see that if you choose the first two of the elements in the first $\binom{5}{2}$ ways, then if you select the remaining 2 elements or 1 element, once you choose 1 element, it is very clear what are the other 2 elements going to be or if you select the 2 elements again, it will be clear what is the remaining 1 element. So, the remaining 2 elements can be chosen in how many ways? $\binom{3}{2}$ possible ways.

Then the last one, there is no choice. But there is a problem. If you use the product rule here, like $\binom{5}{3}$ and $\binom{3}{2}$, then what you will get is that, $\binom{5}{3} \times \binom{3}{2} = 10 \times 3 = 30$, but I claim that this is wrong. Why is this wrong? This is wrong because when we were doing the counting, we did some overcounting. Can you think of why there is some overcounting?

So, the reason there is an overcounting is that when you selected the first 2 element subset in $\binom{5}{3}$ or 10 possible ways and then you selected a 2 element subset from the remaining 3 elements, the 2 element subset that you selected in the first and the remaining 2 elements that you selected in the second could also appear as those 2 elements were selected in the first choice and the other 2 elements are selected in the second choice. So, in these 2 possible ways, it can count and we are counting both of them when you take the multiplication.

So therefore, since we are counting each of this exactly twice, we can divide by 2. So, we will get $\frac{10\times3}{2} = 15$. So, I have 15 possible ways only, not 30 possible ways. And therefore, the total number of such partitions in the blocks is 15 + 10 = 25. So, we need to be very careful when we can we do this kind of counting because overcounting must be avoided or if you do overcount, you have to find a suitable way to deal with that.

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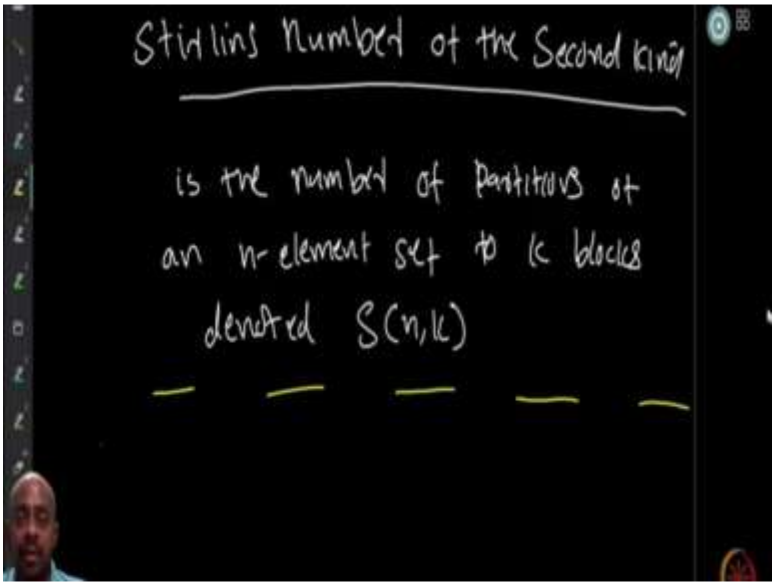


So, if you are looking at $n$- element set and you are looking at $k$ blocks then $S(n,k)$ denotes the number of partitions of an $n$-element set into $k$ blocks s and this is called Stirling number of second kind. So, the number of partitions of a set into $k$ blocks is denoted by $S(n,k)$ and it is called the Stirling number of second kind. So, one can ask what has happened to the first kind?

Why we are not looking at the first kind before we look at the second kind. We will look at the first kind sometime in the future, but I know that is much more difficult to deal with, much more involved computation and we will look at those things in the later part of the course. So, for the time being, we will stick with the Stirling number of the second kind. Now, how do you find $S(n,k)$? Can you find a formula for $S(n,k)$. Can you find a way to compute $S(n,k)$. These are some interesting questions one can ask. So, I want you to think about this question.

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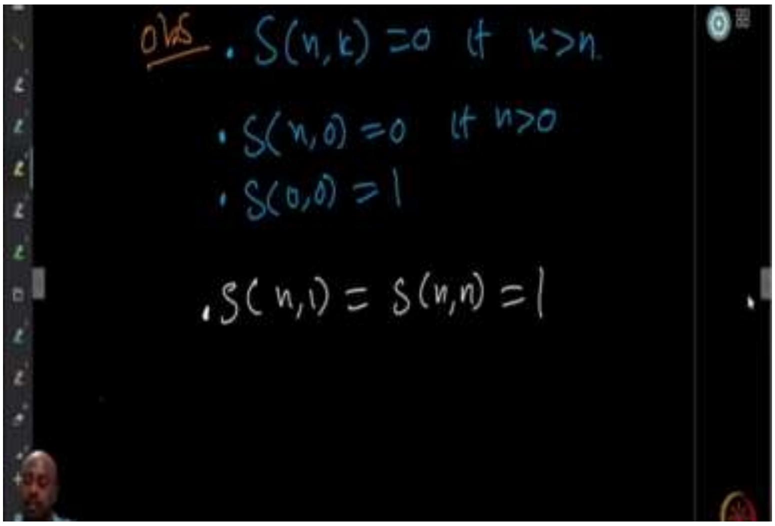

And here are some observations. So, $S(n,k) = 0$, if $k > n$. That is very clear and $S(n,0) = 0$ , $n > 0$. That is also kind of clear why and $S(0,0) = 1$.

Now $S(n,1)$ and $S(n,n)$ are always going to be equal to 1. This is also another observation. You should verify that, whatever I claim, do not take it for granted. When I say something and you saying that why it is that case, it means that you are supposed to figure it out yourself. Think about this and find out why and they are kind of easy question. That is why, I am giving you it as just a statement without any argument. So, figure out why these things are the way I have defined it here.

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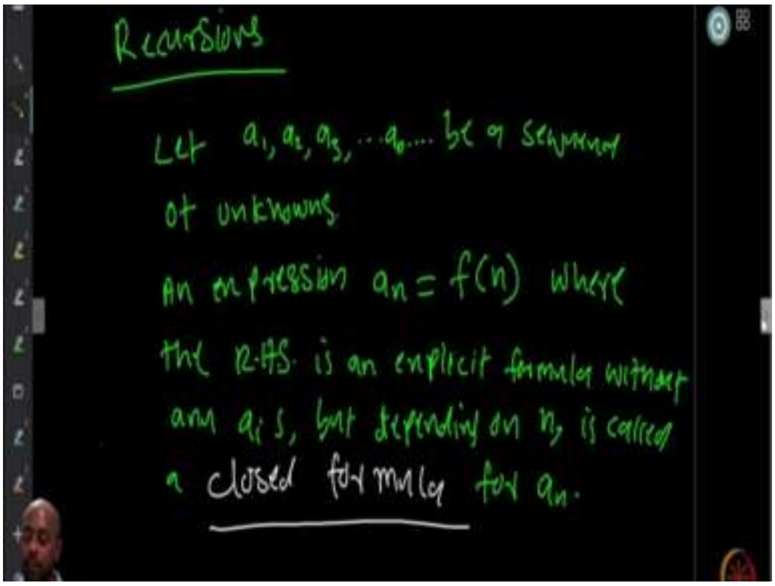


Now to come up with solutions to this, we might need to use some new ideas. So, one of the ideas that is called recursion. Now, suppose you are given a sequence of unknown value. So, let us say $a_1, a_2, a_3, \ldots, a_n, \ldots.$, is the given sequence of unknown values. Now what we want to know is that we want to find out a way to find or describe let us say $a_n$.

So, suppose I write $a_n = f(n)$ where the right hand side expression does not depend on any of the $a_i$'s but it depends on $n$. It can depend on $n$. So if $a_n = f(n)$, where the right hand side is an explicit formula without depending on any of the $a_i$'s, any of the unknowns, but it can depend on $n$ is called a closed formula for $a_n$.

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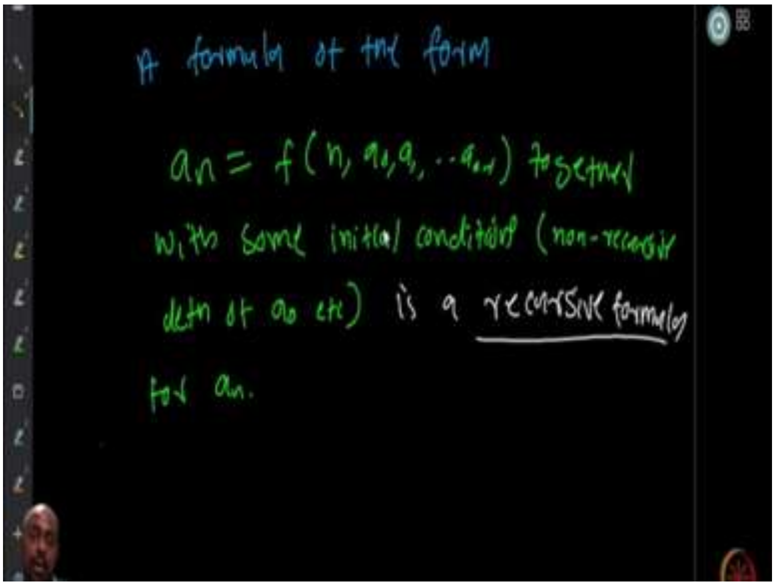


Now, on the other hands, consider a formula of the form, $a_n = f(n, a_0, a_1, \ldots, a_{n-1})$. Index should be less than $n$ and together with some initial conditions like, $a_0$ is something, $a_1$ is something, etcetera. Some of the things can be already given which are not defined recursively. They are given explicitly. Then, such a statement is called a recursive formula for $a_n$.

So, $a_n$ depends on some previously determined unknowns and $n$ also but it does not depend on anything that we have not come across so far. So, such a description is called a recursive formula. So basically, if you know the some of the initial values, you should be able to find the next one. which is not known. Then using that, you can find the next one now in a recursive manner. That is why it is called a recursive formula.

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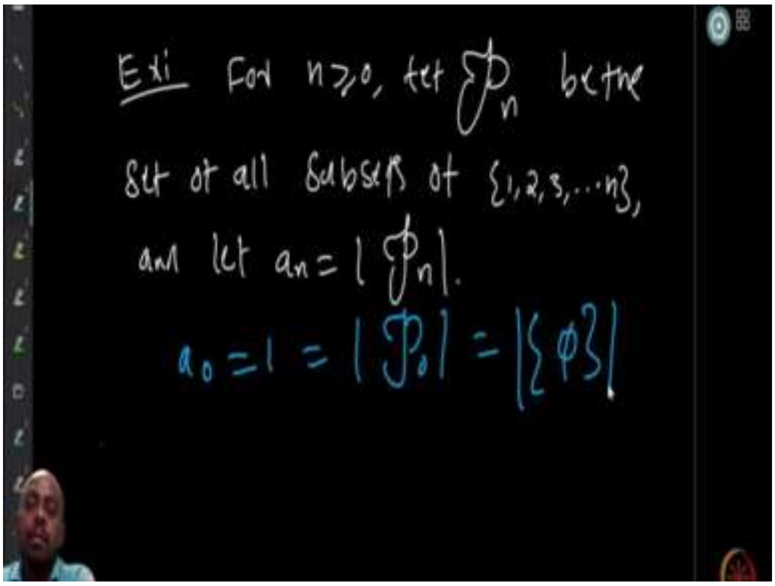

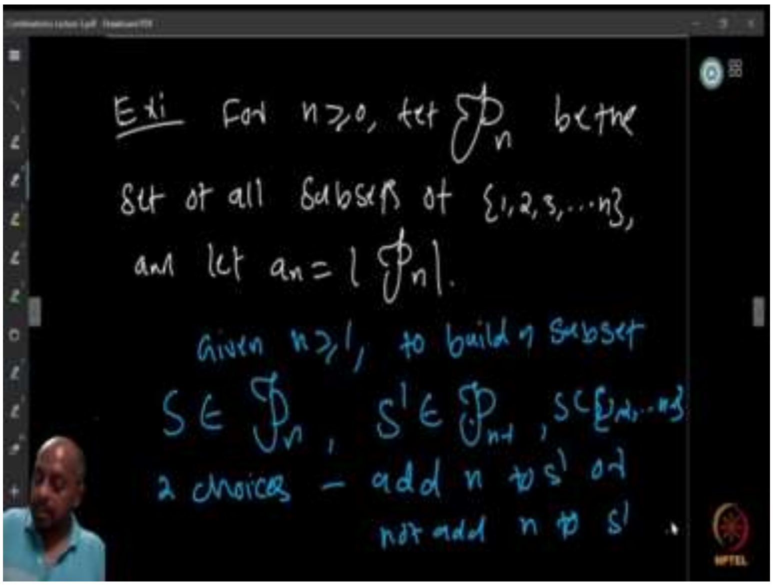


So, let us look at an example. So, for $n \geq 0$, let $\boldsymbol{P}_n$ be the set of all subsets of the set {1, 2,…, n} and let us define $a_n = |\boldsymbol{P}_n|$. So, this is something most familiar to you. So basically, you are looking at the power sets of the sets that we are talking about, set of all subsets of set {1, 2, …, n}. So, power sets of set {1, 2, …, n} and we want to find out what are the cardinalities. We already know how to do it using the product rule we did long time back, but now we want to use and try to solve it using a recursive formula. So, we come up with the recursive formula and try to solve it and get an answer.

So, for solving something like this, what we can do? Well, what we can do is that we can do some observations. So first we observe that $a_0 = 1$, because $a_0 = |\boldsymbol{P}_0|$ and $\boldsymbol{P}_0$ is the set of all subsets of the empty set, which is the sets containing empty sets and its cardinality, because it is singleton, the cardinality is exactly 1. So, $a_0 = 1$ and we get an initial condition.

Now, we need to find the recursive relation, but to find the recursive relation, what we do is the following. So given, let us say that $n$ is greater than or equal to 1 to construct or to build any subset. Subsets, let us say $S \in \boldsymbol{P}_n$ . What we can do is the following.

So, to select such a subset, we will select, we will select any subset $S' \in \boldsymbol{P}_{n-1}$. So, $S'$ is a subset of the set $\{1, 2, \ldots, n-1\}$. So $S$ is some subset of $\{1, 2, \ldots, n-1\}$ and then we will select the element $n$ and say that, now, I have 2 choices.

One is either to add $n$ to this set $S'$ to construct a subset of $\boldsymbol{P}_n$ or I do not add it. I just keep it as it is. So, there are these 2 choices. Either I add this element to that set or I keep the subset at it is. Now, this way I can create all possible subsets of $\boldsymbol{P}_n$, because I take all the subsets of {1, 2, …, $n-1$} and then I add $n$ or I do not add $n$. So, I will get all possible subsets this way. So, there are 2 choices, either add n to S dash or not add n to S dash.

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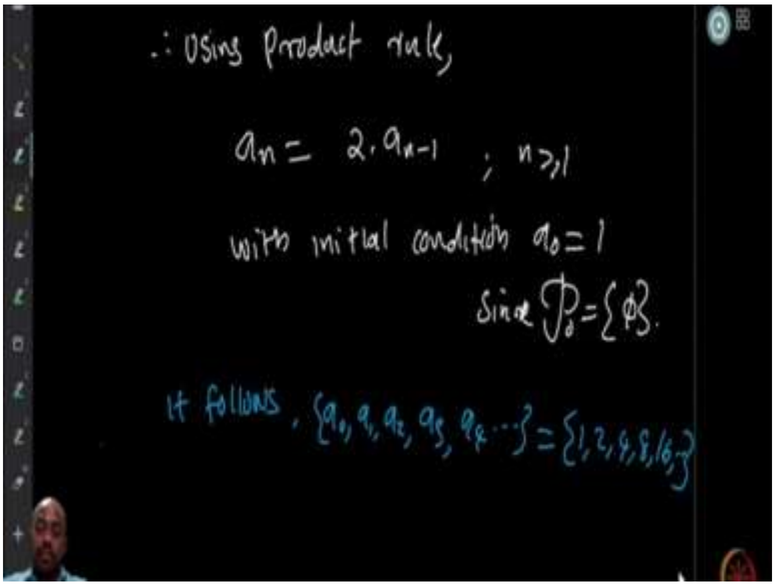


So now, once you know this, using the product rule we can figure out that $a_n = 2.a_{n-1}$, $n \geq 1$. We are doubling it because I take this guys any element here, I add $n$ or I do not add $n$. Both way, I get a subset of $\boldsymbol{P}_n$ and therefore, I get 2 times of this many, which is $2a_{n-1}$. Now with the initial condition that we figured out that is $a_0 = 1$. We can now try to find out the values of $a_n$ as follows.

$a_0 = 1$ and therefore $a_1 = 1 \times 2 = 2$. Then $a_2 = 2 \times 2 = 4$, then you get 8, 16, etcetera for $a_3$, $a_4$, etcetera. So, we find out what are these numbers. So now from this, we can guess the value of $a_n$. We see that $a_n$ is something like $2^n$, and then once we find the guess, we verify this and we can use induction to prove it actually.

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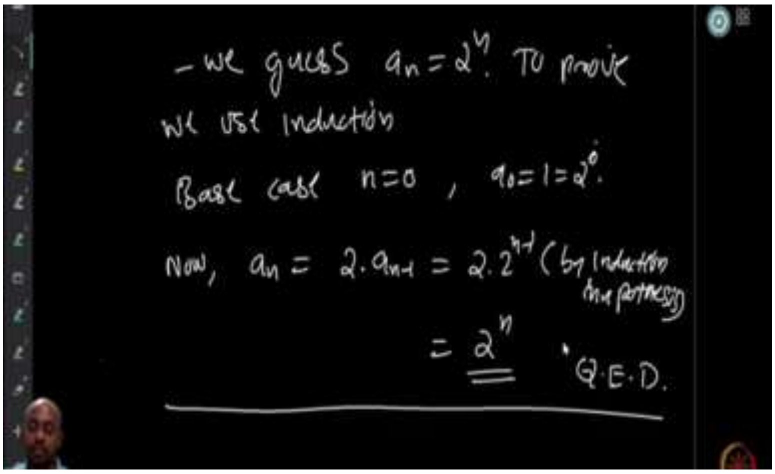

So, how do we do that? So, we do the following. We guess that $a_n = 2^n$ and to prove we use induction. So, the base case is $n = 0$, we already know that $a_0 = 1 = 2^0$. So therefore base case is fine. Now we take the case, $a_n = 2.a_{n-1}$ and by induction hypothesis we have $a_{n-1} = 2^{n-1}$ and therefore. So, $a_n = 2.a_{n-1} = 2.2^{n-1} = 2^n$. Therefore, we have, the proof.

So therefore, by induction, we get that $a_n = 2^n$. So if you can guess, what is the formula, closed formula for $a_n$, most often we can use induction easily because we know, even though it may depend on $a_1$, $a_2$, etcetera, $a_{n-1}$, since we know that each of them has the same formula, we substitute that and then try to figure out why the recursive relation must satisfy this and if it is satisfied, then we get that it is indeed the case. So, this way, we have proved it.

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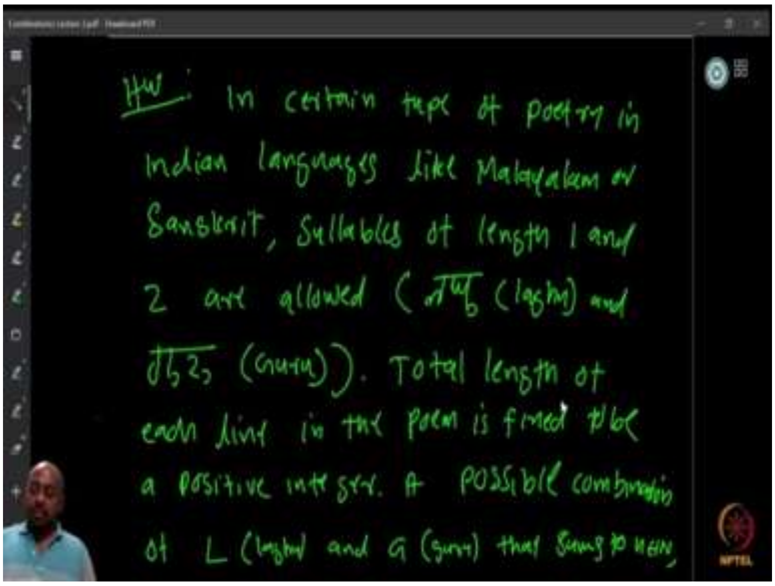


Now, a homework. This is a question that I want you to think very thoroughly about. It is a very interesting question and this question has been solved by the Indian mathematicians many many years before, I mean thousands of years back in fact. So, I want you to think about this, and try to see how you will solve this and how you will form a recurrence relation for this for example.

So, here is the question. So, in a certain type of poetry in Indian languages like Malayalam, Sanskrit poetry, the syllables of length 1 and length 2 are allowed. So, each letter can be either short or long ah, aa, ka, kaa, etcetera. So, we have this long and short syllables. We will say that laghu the short one and guru the long one.

The long one has exactly twice the length of the short one and that is a long-accepted convention. In ah, aa, the second one is twice the length of the first one, it is supposed to be like that even though we do not do it as precisely when we talk, it should be exactly like that. So, the syllables of length 1 and 2 are allowed.

Now, the total length of each line in the poem is fixed to be a positive integer in terms of the syllable length, it should be exactly the same, for certain kind of poetry. We will say that okay, we will allow exactly let us say, length 5 syllables whatever you use, it should be of length 5. So, you can have maybe 5 laghus like 5, 5, 5 like laghu, laghu, laghu, laghu, laghu, aa, aa, aa, aa, aa.

So, there are 5 of them or you can have maybe 2 and 1, 1, 1 or 1, 2, 1, 1, or 2, 2, 1, etcetera. All these possible lengths are allowed. So, once the length of each line is predefined something like this kind of a pattern we can decide.

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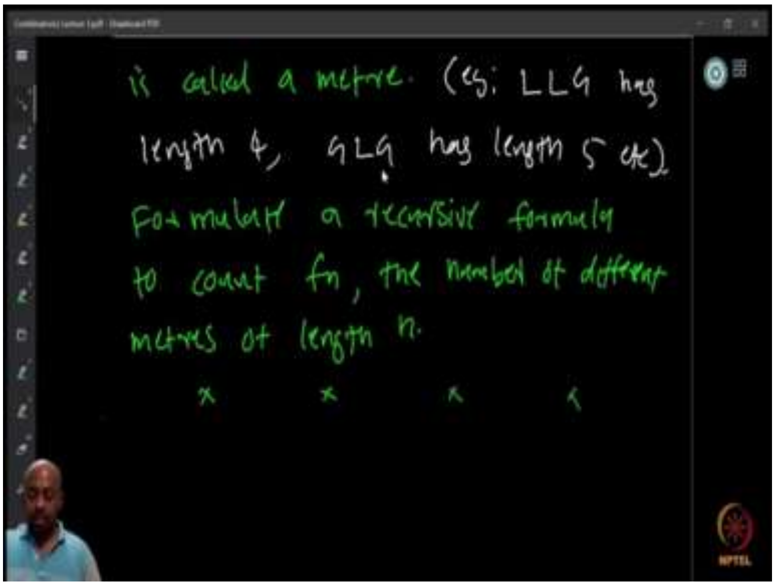


So, once you decide a pattern that such a pattern is called a metre, for example LLG is a metere width length 4 because LL 1, 1 and G has 2. Guru has 2 and leghu has 1. So, 1, 1, 2 it has length 4 and GLG has length 5 because it is 2, 1, 2. Length 2, guru has length 2 then leghu has 1 and again 2. So, formulate a recursive formula to count $f_n$ where $f_n$ is the number of different meters of length $n$.

So, some formula define $f_0, f_1$ etcetera. You find some initial conditions and write a recursive formula to define $f_n$ and then if you can solve the recursive formula, well, even better, but at the moment, I want you to come up with a recursive formula for this. So, this is what I want you to do.

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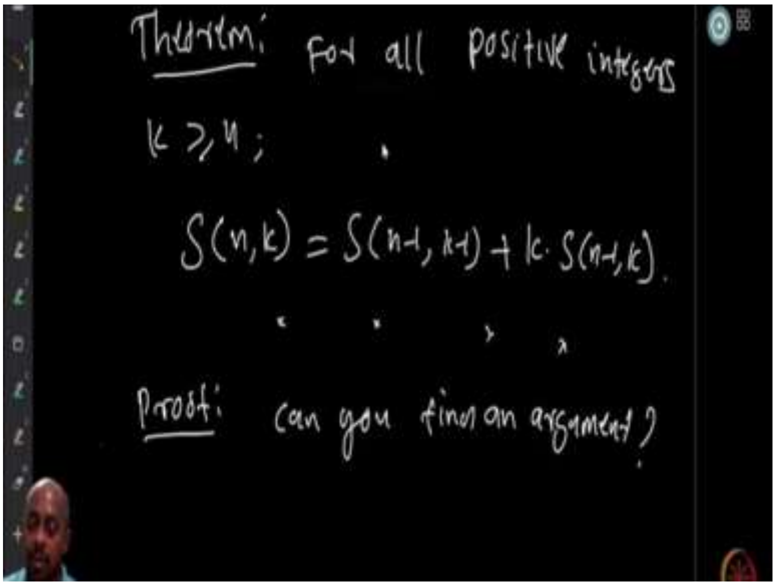


Now, before we finish today, let me state a theorem, and you will think about the proof. So, the theorem is the following.

For all positive integers $k \geq n$, the Stirling number of second kind, $S(n, k) = S(n-1, k-1) + k.S(n-1, k)$.

So, it satisfies this identity. So, can you find an argument why this is the recursive formula for $S(n, k)$? So, this is a recursive formula definition for $S(n, k)$, since here it depends on $S(n-1, k-1)$ and $S(n-1, k)$ and recursively it can depend on the previous one basically. So, this is what I want you to do. So, think about how to do this and we stop for today. We will continue with the proof of this and other results in the next lectures.

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