Combinatronics

Professor. Doctor Narayanan N
Department of Mathematics
Indian Institute of Technology, Madras

Integer Compositions

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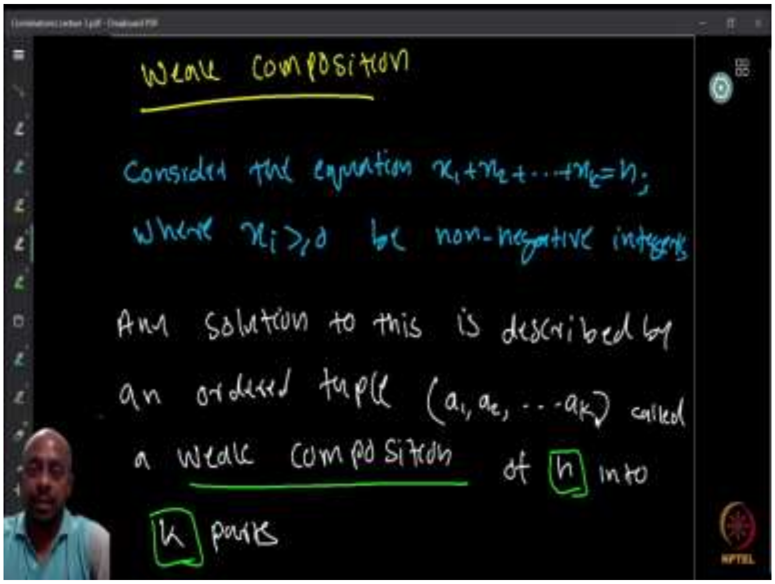

Welcome to everyone to this lectures on Combinatorics, week 3. In the previous two weeks we looked at a pigeonhole principle in the first week and several of its applications and then we went to look at several basic counting techniques in combinatorics that one can use, like the addition rule and product rule, and some other interesting results related to this, and then we looked at some applications.

Now we are going to look at further applications of these basic techniques to develop little more advanced tools or results that we can use in many situations. And that is what we are going to do in this week's lectures. So, we start with a very interesting problem. So, it is the following.

Consider the equation $x_1 + x_2 + \cdots + x_k = n$, where $x_i \geq 0$, be non-negative integeres. Now given this condition, look at any possible solution to this. So, any solution to this equation can be described as a k-tuple, something like $(a_1, a_2, \dots, a_k)$ where a_i is the value taken by x_i.

So, if $x_1 = 5$ and $x_2 = 7$, then we will have this tuple starting with (5,7, …) where $a_1 = 5$ and $a_2 = 7$. Such a tuple is called a weak composition of n into k parts. k parts because like, we are basically writing n as a sum of k non-negative integers. So, this is called a weak composition.

Now before trying to find out, our ultimate aim is to try to count the number of possible weak compositions here.

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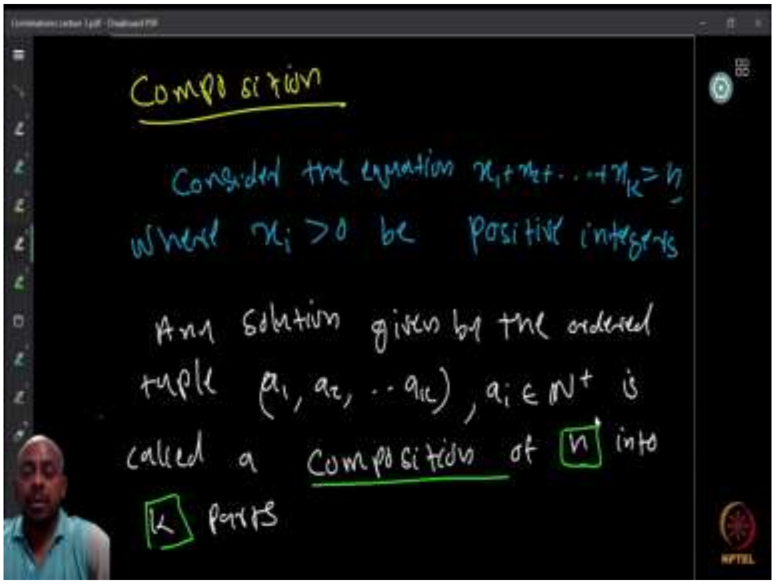

But before that let us also look at a slightly different definition where we allow x_i's to be strictly greater than 0, that x_i is cannot be 0. So if you have equation $x_1 + x_2 + \cdots + x_k = n$, where $x_i > 0$, then again any solution given by the order tuple $(a_1, a_2, \dots, a_k)$ where $a_i \in \mathbb{N}^+$ is called a composition of n into k parts. So, it is not weak composition, it is composition. So, maybe at later point of time it will be clear why it is called weak and the other one has to be not weak. So, but for the time being let us just take the definitions for granted.

Now can you think of any argument to count this number of possible compositions or number of possible weak compositions? So, I tell you that both of these can be solved by some results that we already looked at. So, with this information can you think about this and find a solution by your own.

So, maybe you should spend some time, pause this video, spend some time trying to find out how to count these two things. It will be very useful if you spend some time thinking about this instead of looking at the solution that is presented here. So, here it goes.

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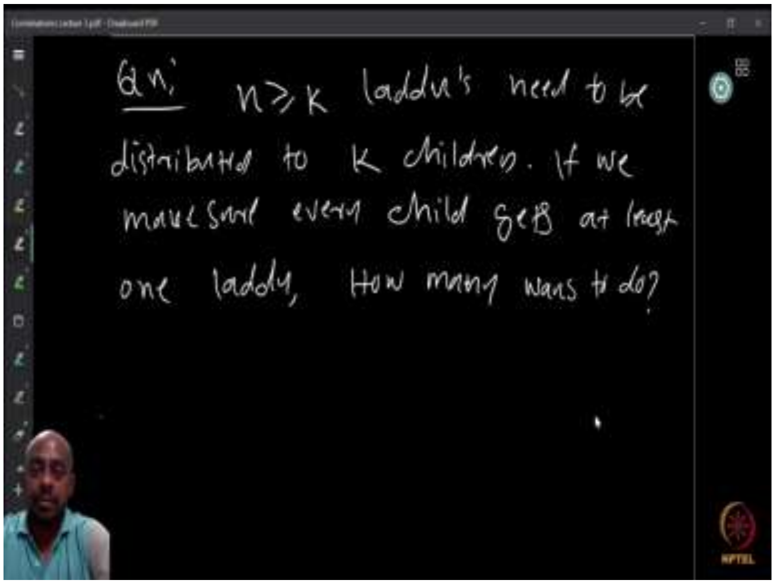

So, we will first try to solve the second question, that is, to count the number of compositions of n into k parts. So, since we want to make our, the lectures or we want to remember the lectures to be sweet, let us convert this question into a more enjoyable form, as follows.

Suppose we are given, $n \geq k$ laddus. So, n laddus, like, identical, same kind of laddus maybe, or Tirupati laddus. They are one of the most famous laddus, tastiest. And they are given, and we want to distribute it to, let us say, k children. So, if you think that you are children then maybe you can hope to get some laddus.

Now so we, we have enough laddus so that we do not have to be very selective. So, we can say that every child is going to get at least one laddu. That, I will guarantee. So, everybody will get at least one laddu, then how many ways you can give the laddus to the k children, that is the question. Because even though I said that everybody get at least one, I did not say everybody will get equal number of laddus. But at least if you get one, you can be happy. So, we will make sure that everybody gets at least one.

So, now the question is that how many ways we can distribute the laddus. Now one can immediately see that this question is precisely the question of composition, because now you can, you can assume the number of laddus received by each of the k children to be the values taken by x_i's and they sum to n, and then the number of ways to do the distribution is precisely the number of solutions to this previous question of composition of n into k parts, because x_i's are to be strictly positive, everybody gets at least one and it should be integers, so everybody gets at least one.

Now, how do you solve this question? Again, think of laddus if it helps, and try to figure out how to do this yourself. If that doesn't help, we will go to the next page.

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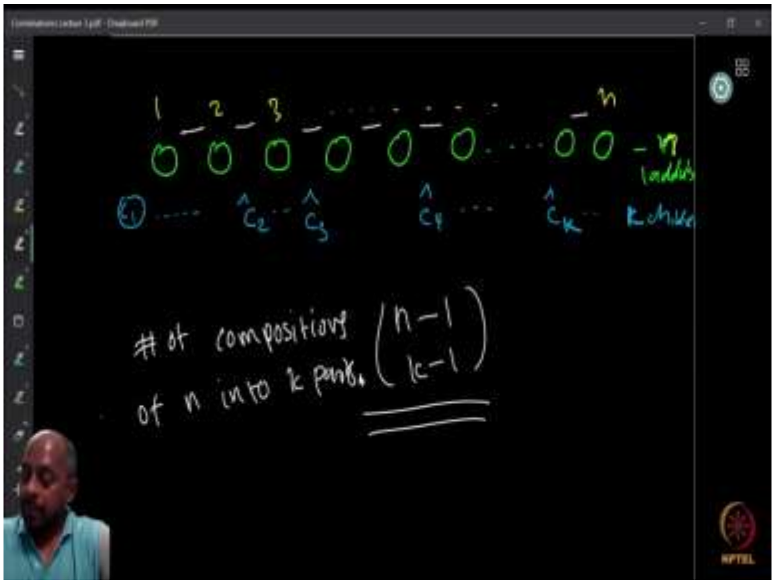

So, let us assume that there is a hall where the children are waiting and then there is a table where all these laddus are kept in a line. So, we have these laddus which are kept in a line. So, laddus 1, 2, 3 et cetera, all the n possible laddus are there. So, all possible laddus are there. Now they are, they are kept in a straight line.

Now, we want to distribute the laddus to the children. So, how do you do? We will say that, okay children, queue up. So, stand in a line. So, the child number 1, number 2 et cetera, stands in one line. So, they stand in a line, and then we will say that, the first child can start at the beginning, the first person starts at the beginning here. Now, we want to use all the laddus. So we will say that you start at the very beginning, the first guy in the queue. Then you start taking some laddus. And once you take some laddus, I do not know, 1 or 2 or something, some number, whatever, maybe five, depending on the person who is standing there.

Now go. So, once he has taken some number of laddus, we will say, now the next guy in the queue comes. So, the child number 2, C_2 will starts at the position which is after the first guy is asked to leave. Again, he will take some laddus and then suddenly the person who control the queue says that, you leave, next guy. So, this way each child will be allowed to go and take something.

Now, since everybody gets we need to make sure that the, the last child gets at least one. So, he should start before the end, and then he should be allowed to take whatever is remaining. Whatever remains should be given to the last child. So, that all the n laddus are distributed to the k children. So, this way we want to make sure that everybody gets at least one laddu.

Now, how many ways this can be done? This is easy because what we have is that we have to decide where each of the children starts. Now the first person doesn't have a choice. So, we do not have to look at that because he always starts at the beginning.

So therefore, we need to only decide the remaining $k-1$ children, 2 to C_k where they are going to start. Now they can only start at some point in between. There, so, there is, there is n laddus which are kept here and therefore now the positions where the children can start are in between these laddus.

So, if there are n laddus in a line how many gaps are there in between? So, there are exactly $n-1$ gaps in between. So, out of these $n-1$ gaps we need to decide which are the positions where the $k-1$ children are going to start taking the laddus. So, once we know that, we know the number of ways you can do the situation, and how many are there? Well since there are $n-1$ gaps available, and we need to select $k-1$ of these gaps where the children can start. So, $\binom{n-1}{k-1}$ possible choices are there. So, these are the number of available choices, and each choice correspond to a composition and every composition corresponds to one of these choices, and therefore this is a one to one correspondence and we get that the number of possible compositions of n into k parts is $\binom{n-1}{k-1}$. So, this is the number of compositions of n into k parts.

Now, what we are going to do is now to look at the first question. Now, what was the first question? We wanted to look at the weak composition. Weak compositions are the solutions $x_i \geq 0$ instead of $x_i > 0$. So, we found out that the number of compositions is precisely the ways of distributing sweets to children. So, we had a great time solving this. Now, we want to solve the other question.

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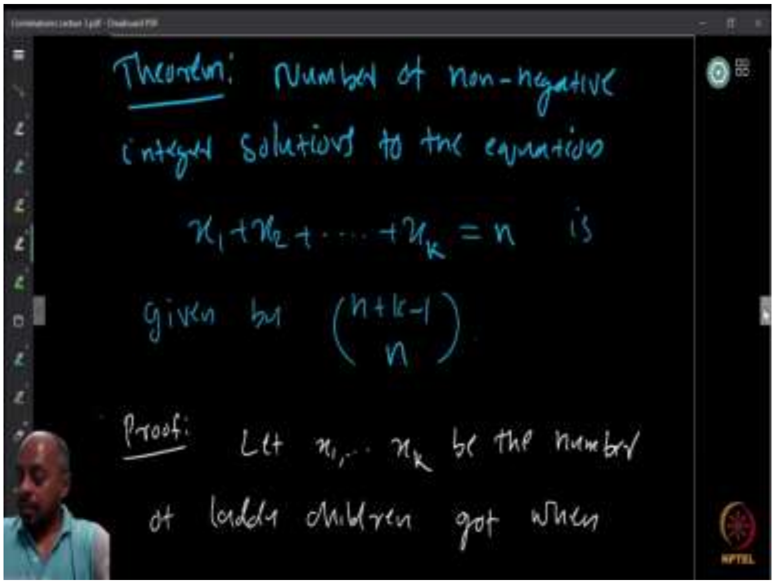

So, here is a theorem which says that the number of non-negative integer solutions to the equation $x_1 + x_2 + \cdots + x_k = n$ is given by $\binom{n+k-1}{n}$. Now, how do we prove this? Now since we already solved a question of composition using sweets, why do not we try the same thing? Maybe we can still work with sweets. So, let us start by assuming that, let, x_1 to x_k be the number of laddus children got when n laddus were distributed to them.

So here also we are doing the same thing. We have n laddus, we are going to distribute them to children but now the person standing at the queue may not be fair. He will say that some children who comes, I do not like you, so you do not get anything. It is possible. So, there is somebody standing there who will say first guy come, you take something, second guy take something, third guy, no, you go, you do not get anything. Then fourth guy. So, this way we can make some of the x_i's to be 0. So when, when the distributor is unfair then this can happen

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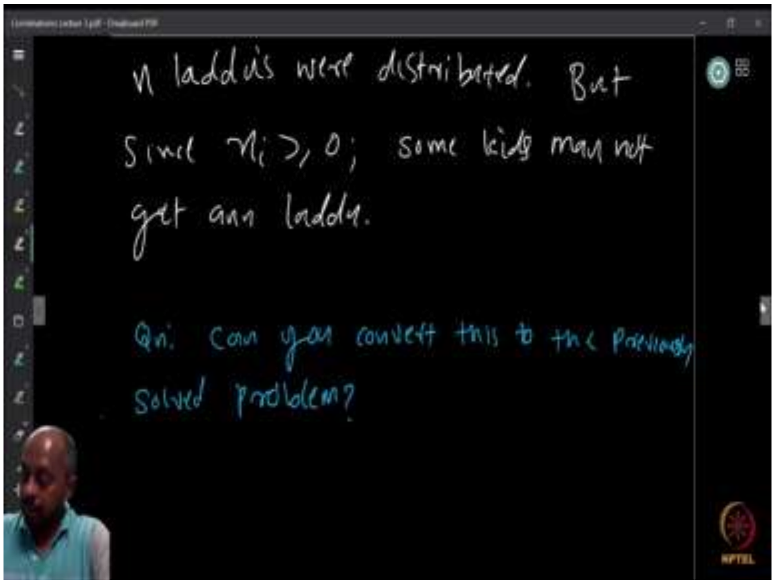

Now, can you convert this to the previously solved problems, any one of the previously solved problems, or the previously solved problem, just before? If you can convert it to a previously solved question, then that is the best thing to do in mathematics. So, you reduce the problem to an already solved problem.

So, if you can do that, then we are done. If you can convert into a previously solved problem, then that is it. So, how do you convert it to a previously solved question? So again, I want you to think about how to convert this to the previous question. So, can you convert the unfair distribution to a fair distribution question? If you can do that then that is good.

This is quite easy actually, because we want to distribute n sweets to k children so that some kids may not get anything. But now we want to convert into the question that everybody gets at least one. Now how do you make sure that everybody gets at least one, when some of them can clearly be 0.

So, what you do is that you assume that, before the distribution to each children, we will go to the child and say that we are going to distribute some sweets but now since we don't have enough can you please give us one laddu each. So, from every child you borrow one laddu. So, you take one laddu from each of the children, now keep it with you. So, now you have k extra laddus. So, you take the k extra laddus and then put that also in the table.

Now, you again send the children back to the queue, and now whenever a child comes he will definitely get at least one because you know he gave one laddu to the distributor. Now, this guy cannot say that you do not get anything, because he will say that I already gave you one laddu, I should get at least that one back. Now what does this mean?

This means that everybody gets at least one, which means that you get $x_i > 0$, but now for every solution to the weak composition or the solution to the distribution where x_i can be greater than or equal to 0, we have a solution for x_i's greater than or equal to 1 or x_i strictly greater than 0, because you just add 1 to each of the x_i's and on the right hand side you have instead of n, you have $n + k$ because we started with k additional laddus.

So therefore, distributing n laddus to k children where everybody gets greater than or equal to 0 laddus is equivalent to saying that, distributing $n + k$ sweets to the same k children where everybody gets at least one, because whenever you have a solution to at least one, you remove one from each of them, you have k less laddus, $n + k - k$ which is n laddus, and some of them can become 0 now, because if x_i was 1 it will become 0.

So, there is a one to one correspondence between the solution, so they should be the same. So, that is it so we have already converted it.

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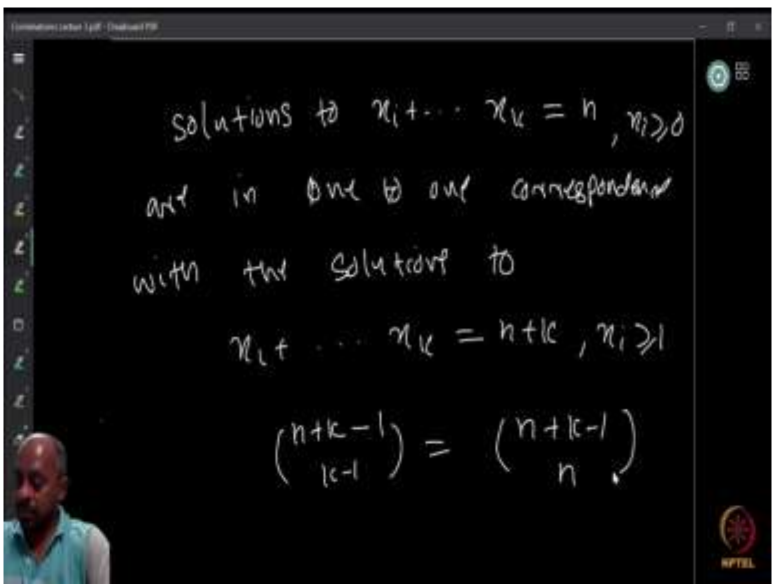

So, we said that okay solutions to $x_1 + \cdots + x_k = n$, $x_i \geq 0$ are in 1 to 1 correspondence with the solutions to $x_1 + \cdots + x_k = n + k$, $x_i > 0$. And we know how to do this because we just did in the previous example.

It is basically, $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$, by the property of binomial coefficients. So therefore, this is what we wanted to prove, $\binom{n+k-1}{n}$. So, this is one way we can prove this.

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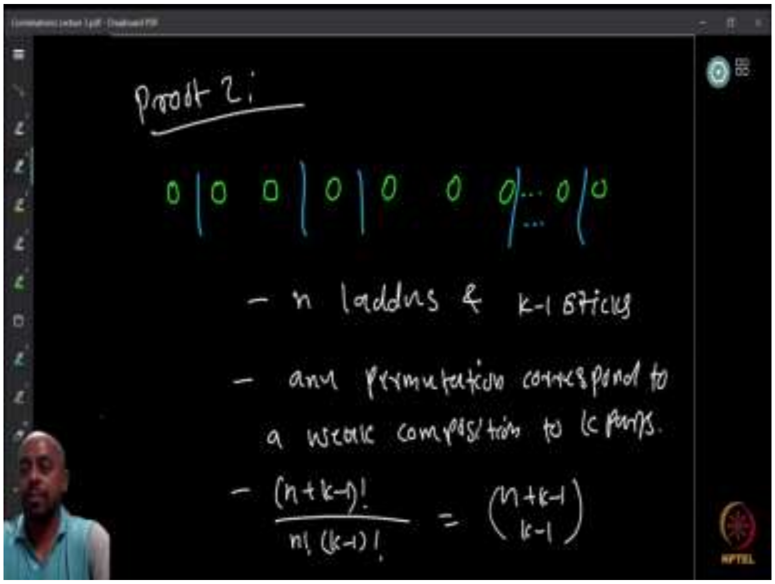

Now let me give you another proof to the first question, weak composition. So, the proof 2 is also quite nice. This is by using one on, another result. So, what we do is that we observe that the number of ways to distribute the laddus to the kids can be made as follows. So, you keep your all n laddus here.

Now, once these laddus are kept what we are going to do is that, we separate the laddus by putting some markers or sticks in between them. So, here are the sticks which separates the laddus. And of course, these (laddus) the sticks can be together so that there is no laddus in between. That is also possible.

So, we keep some bars, or $k-1$ sticks in between to separate the laddus. Now, once the sticks are kept, this decides the composition or the distribution of the laddus because you will say that the first child can take all the laddus till the first stick. So, if the first stick is in the beginning then that guy does not get anything. So, x_i can be 0, x_1 can be 0.

Similarly, after the first stick, so the laddu is taken, the first stick is removed and then say that second child, now go and take the laddus till the second stick. Again, the third child goes and take the laddus till the third stick.

So, this way the $k-1$ children take and the remaining laddus, if there are any, because if the stick is at the last end then there is nothing remaining. So which means that the last guy also may not get anything. So, this is basically you have the laddus and sticks and then you arranged, some of the sticks can be together so that there is no laddus in between.

So, you can have two sticks like this so that after the first guy, the second guy does not get anything. Or maybe like there is something like this, the last two guys does not get anything. So, this is possible. But, now all we have to do is to find the number of possible arrangements of the laddus and sticks. But this we already know, because we already said that, laddus are one type, so there is n objects of type laddus and then $k-1$ objects of type stick. So, we want to arrange the laddus and sticks in any possible way.

So, how many possible arrangements are there? By the multinomial theorem that we studied, if there is n_1 objects of type 1 and n_2 objects of type 2, the number of permutations of them is basically $\frac{n!}{n_1!n_2!}$. So therefore, we have n laddus and $k-1$ sticks so $n+k-1$ total objects are there out of which n of them are of one type and then $k-1$ of them are of different types, so therefore $\frac{(n+k-1)!}{n!(k-1)!} = \binom{n+k-1}{k-1}$.

So therefore, by the multinomial theorem we immediately get that this count, or the number of possible arrangements, or the number of weak compositions is equal to $\binom{n+k-1}{k-1}$. So, that is it. So, we have another proof.

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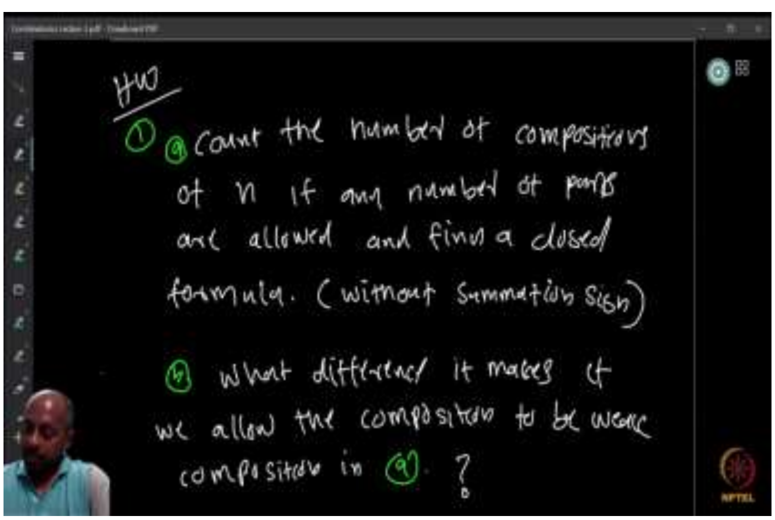

Now, here are some homework questions for you. First is that you count the number of compositions of n, if any number of parts are allowed. So, what do what do I mean by that? So, earlier we said that we have a fixed number of k parts are there. So, we counted for precisely k. Now you want to find out for arbitrary k. It can be 1, it can be 2, it can be 3, et cetera.

Now, can you find a closed formula for this? Now closed formula is a formula without the summation sign. So, this is very easy to write because we already know for any fixed k, you can always write a summation formula like this.

But we do not want that summation form. We want a closed form where you find the formula without using the summation. So, you basically do the summation and give me the formula or find another way to find it directly. So, whichever way you want you can do it.

And the second part of the question is that what difference it makes if we allow the composition to be weak composition? So, in the first part we are saying that count the number of compositions of n, you find the closed formula. Now the second question is that what difference makes to this closed formula if we allow the weak composition instead of the composition.