Combinatronics
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Applying Multinomial Theorem

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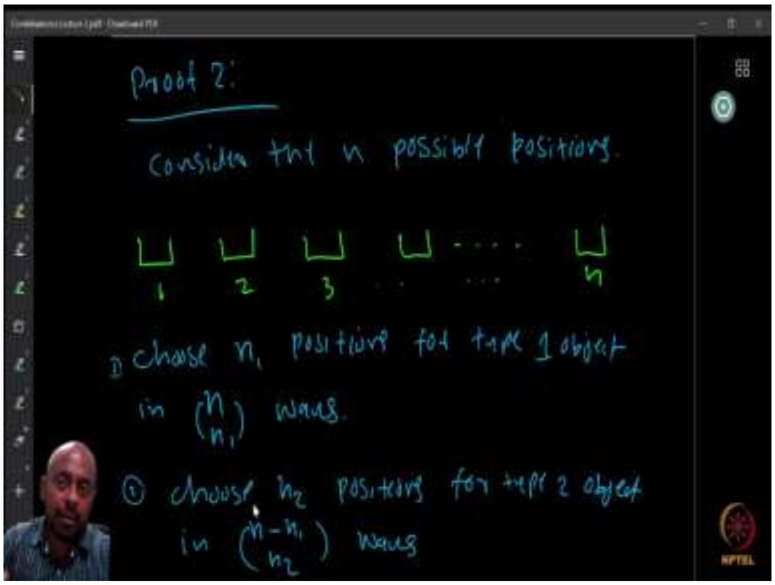

Proof 2. Consider n possible positions for the n objects. So, there are identical objects, I am not going to re-label them or anything. So, out of the n positions, I choose n_1 positions to put the type 1 object. The type 1 object are all identical. So, no matter, I do not want to look at the order. So, I want to just form a subset.

So, I choose the n_1 positions and say that, I just put all the copies of this object in that particular position. So, how many ways I can choose the n_1 positions? There are n positions available. $\binom{n}{n_1}$ ways I can choose the n_1 positions. So, I fix the n_1 positions for the first object in $\binom{n}{n_1}$ ways.

So, I selected this, that positions are not available anymore. So, once I have removed these n_1 positions and put the type 1 object there, I look at the remaining $n - n_1$ positions and out of these, I choose n_2 other positions. In these n_2 positions, I will put the type 2 object. They are identical. I do not care which order. I just put them there. So I choose n_2 positions for type 2 object in $\binom{n-n_1}{n_2}$ ways.

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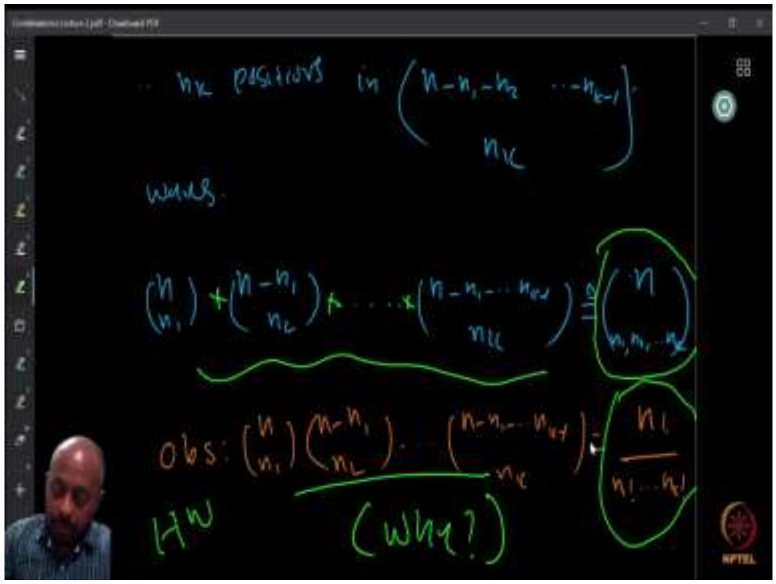

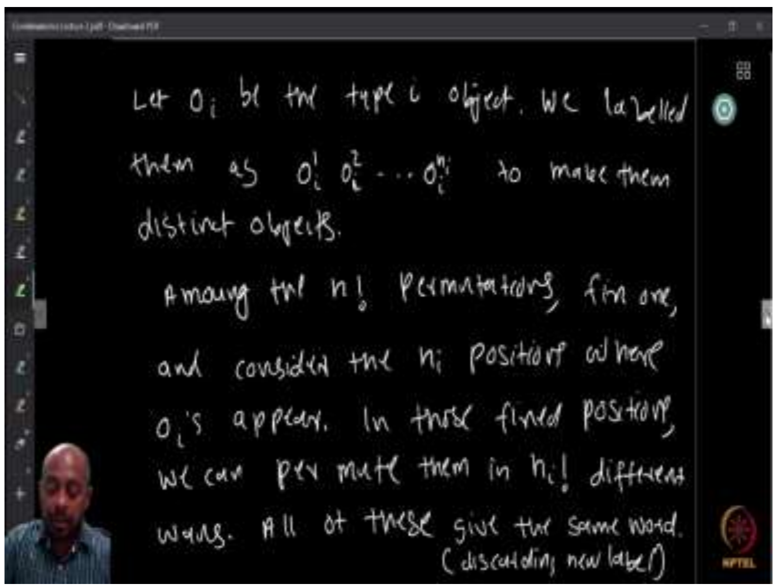

So, continue this way, I choose n_k positions for type k objects in $\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$ ways. Now, the positions that we were selecting were all distinct. So, I selected the first, the number of choices were independent because first I select n_1, now I removed this, and then from the remaining $n - n_1$, I am choosing n_2 position.

So therefore, that choice, the number of the choice is only depending on how many are remaining which is $n - n_1$. And they are independent so therefore I can multiply these and I get $\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \ldots \times \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$. Now this product is precisely finding the number of ways to arrange the terms in the multiset. I am going to arrange the linear arrangements of the objects and that is, I can obtain in this way. So therefore, this thing must be equal to what we were counting. So, this is going to be equal to $\frac{n!}{n_1!\ldots n_k!}$.

So, show this equality $\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \ldots \times \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!\ldots n_k!}$. You can use algebraic method. Combinatorially there is nothing to say. They are equal. The combinatorial proof is clear because we just did the same thing. We selected or we just basically counted the same object.

We counted the number of possible ways to form words or to arrange objects where there were n_1 objects of type, n_2 objects of another type, et cetera, n_k objects of another type. And that we found to be $\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \ldots \times \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$ in one counting. And we found it to be $\frac{n!}{n_1!\ldots n_k!}$ in another counting. And therefore, these two things must be equal. We denote the product $\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \ldots \times \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$ as $\binom{n}{n_1, n_2, \ldots, n_k}$. This is called the multinomial co-efficient as I told you earlier.

So, the notation for multinomial co-efficient is $\binom{n}{n_1, n_2, \ldots, n_k}$ and that is equal to $\frac{n!}{n_1!\ldots n_k!}$. And I want you to show this also as home work, why this product $\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \ldots \times \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$ must be equal to the factor $\frac{n!}{n_1!\ldots n_k!}$. So, find it out. And it is very easy.

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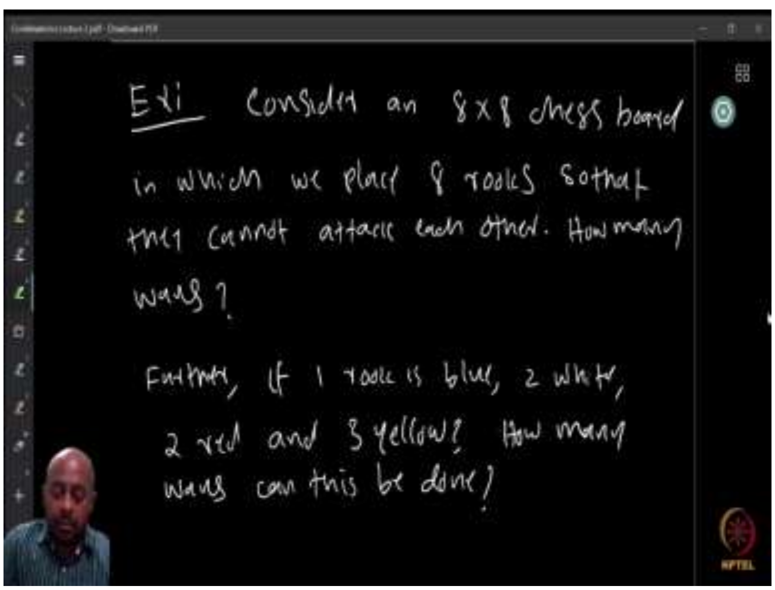

One example. So, any theory we look at, we look at at least some examples. So, consider an 8×8 chessboard, in which we place 8 rooks so that they cannot attack each other. So, how many possible ways are there to keep the rooks so that they cannot attack each other? So if you are not familiar with chess and rooks, I will tell you how the rook moves at least, and attacks.

And as an addendum to the question, I can ask, suppose the rooks are not the same color. We first assumed that all the rooks are identical, the same color. But then we say that okay, suppose one rook is blue color, two are white color, two is red and 3 are yellow. Then, how many ways you can do this? Will the number increases or decrease, whatever?

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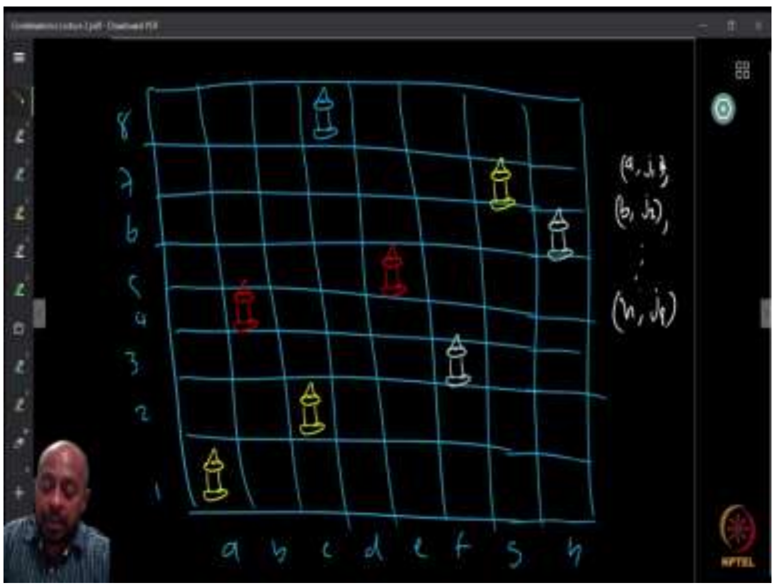

So, here is our chessboard. So, Chessboard is usually labeled with the ranks of the chessboards. Here, a, b, c, d, e, f, g, h, are called the ranks. So, the person will sit, the person with the, let us say, white pieces here.

And then this a, b, c, d, e, f, g, h will be written here where the person with the white color sits and here, the opposite side the person with the black color sits. And these are called ranks. So, the, when the pawn or something moves, so this, which row it is, is called the rank. 1, 2, 3, 4, et cetera, up to 8are called ranks.

So, now, so there are these objects can rooks which can move in the following way. A rook can move in either in its rank or along its file. At one movement it can just move in one direction. So, it can move either this way or it can move this way. Wherever it is sitting, so this guy for example, can move either upward or downward or rightward or leftward. So first, discard this color that I have given to the rooks. The color was for the second part of the question. I do not want to do two times this. So, I gave a color but assume that all of them are of the same color for the time being.

Now, we have these 8 rooks. Now, when two rooks are in the same rank or in the same file, they can attack each other because one can reach the other guy going in that direction. So therefore, if they are in the same rank, they can attack each other. If they are in the same file, then also they can attack each other.

So, I do not want the rooks to be attacking. So, if I have a rook in this position, then I cannot put any rook here, anywhere here or anywhere here. Similarly, so, then I can put the rook in, maybe here if you want, for example. But then, again, if I put one here, or here, whatever, then in this rank, it cannot have, and this file it cannot have anything. So, this way, I want to put 8 rooks so that they do not attack each other.

So, I want to count the number of ways I can do this. So how do you count this ? You can think about this. Try to find your own counting. Then see whether the answer matches with what we will do. Just spend some time and then if you are back, we continue. So, first we observe that because if I look at the files, a, b, c, d, et cetera, h, each file can contain at most 1 rook. Each file can contain at most 1 rook.

Now, therefore, I number the things as follows, (a, j_1). The first rook, wherever it is appearing in the first file in the j_1 th position. In the b file, it can appear in the j_2 th position, maybe. And in the h file it can appear in the j_8 th position. So, I know that any of the 8 rooks, must appear in 8 of these different files so there is no choice in that. They must appear in 8 different files. They cannot appear in the same file. Two of them cannot appear in the same file.

So, once I choose the 8 rooks, I have to decide what are going to be my j_1, j_2, et cetera j_8. Now, can I have (a, j_1) and (a, j_2), where $j_1 = j_2$? If $j_1 = j_2$, both of these rooks are going to be in different files but they are in the same rank. But they cannot be in the same rank. So, j_1 and j_2 must be different.

So, I have $j_1, \dots, j_8$, and out of this $j_1, \dots, j_8$, it must be any distinct 8 numbers from 1 to 8. But that can appear in any order. Any order, I can have, for example in the ‘a’ file if I have the first position, then b file, I cannot have the first position but I can take the fourth position, for example c file, I can take the second position, et cetera h file I can take the sixth position.

So, this is possible, so therefore, all I have the choice is, the choice of the 8 factorial permutations for the ranks. So therefore, 8 factorial choices are there.

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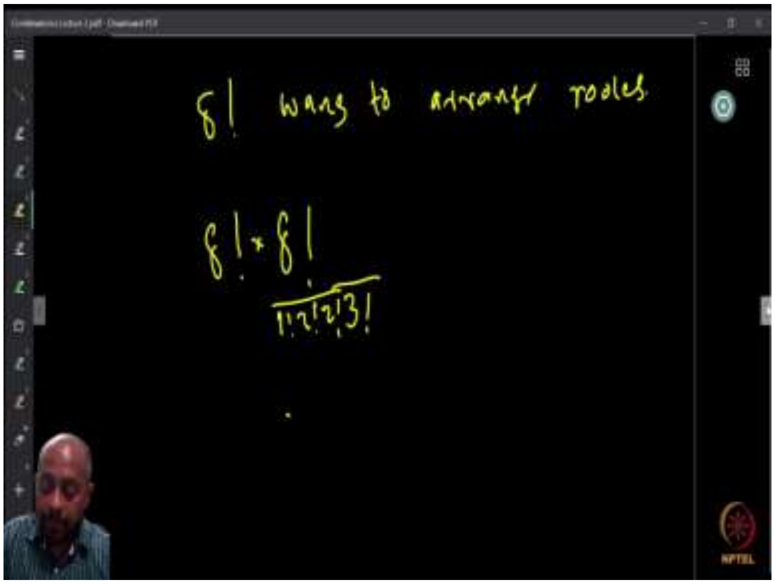

So, we have exactly 8 factorial ways to arrange the rooks, if all the rooks are of the same color. Now, what happens if they have different colors? Now, suppose all of them have, all of them have the same color, we said that there are 8 factorial possibilities. We want to find out what if some of them have, like, one rook is blue, two is white, two is red and three is yellow. Then what you will do?

It turns out that it is easier if you look at another question and solve it first. Suppose all the 8 colors were different. Then you know how to count it, because you have 8 factorial ways to arrange them, but now I can color the rooks in 8 factorial different possible ways because there are 8 colors appearing. So, I have 8 factorial ways to choose the rooks and then I have 8 factorial ways to color them, therefore $8! \times 8!$ possibilities to distribute 8 different colored rooks on 8 different positions here, in the chessboard without them attacking.

Now, since, I know that not all of them are of the same color, now, I will use the multinomial co-efficients, the idea from there because I can say that all these 8 comes from these things. There is one object of, color, blue object, blue colored rook, there is one, exactly, n_1 is 1, then n_2 is 2, n_3 is 2 and n_4 is 3. So, I have the eight rooks but 1, 2, 2 and 3 are appearing.

But, one object is of the same type of 1 factorial, two objects of the same type, two objects of another type and three objects of another type. So therefore, I can divide by $1! \times 2! \times 2! \times 3!$. And this will be my answer for the number of rooks, number of ways to keep the rooks where the colors are like, 1 blue, 2 red, 2 yellow and 3 green, whatever. So, that is the idea.

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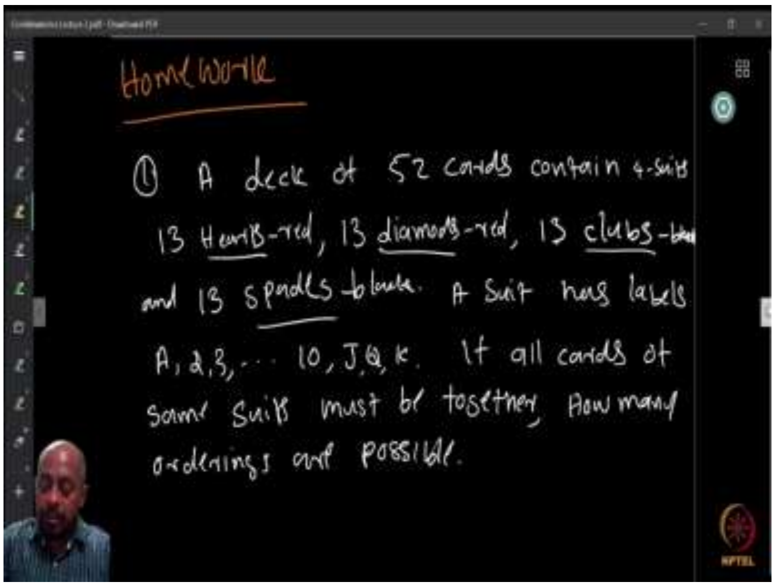

So, time for home work. So, here are some of the home works. So, we have a deck of 52 cards. I hope that all of you are familiar with playing cards. But if not, a deck contains four suits, which is red colored hearts, with the heart symbol. 13 of them. Then13 diamonds which are also red colored. 13 clubs and 13 spades, they are black. Now, of course a suit, heart for example, the 13 cards, will have the labels, like, Ace A, 2, 3, 4, 5, et cetera 10 and Jack J, Queen Q and King K. So, these are the 13 labels for the suits. So, each one has all these numbers.

Now, suppose if all cards of the same suit must be kept together, we can shuffle the cards. So, the 52 cards, we shuffle them, but we have to make sure that all cards of the same suit must be kept together. So, then how many orderings are possible, how many distinct orderings are possible for the cards. You put in one stack, but then we order them in different ways, how many different ways we can do this. So, this is your first home work.

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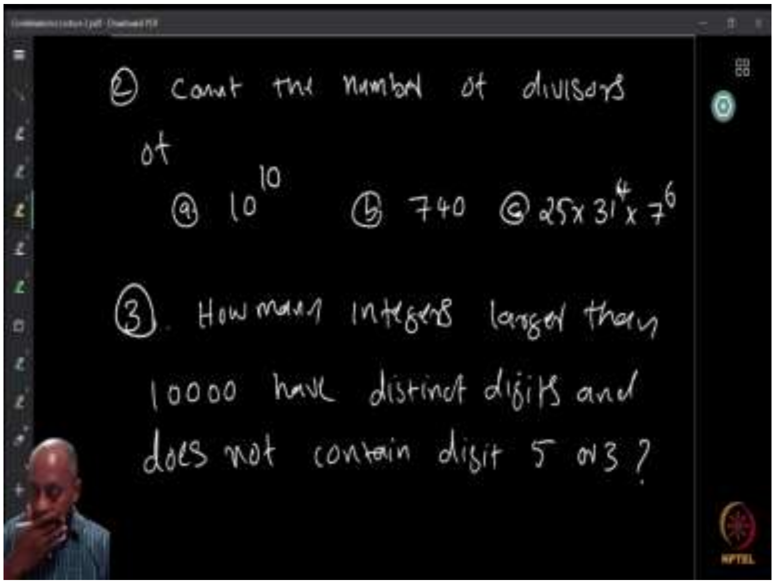

Then, next question is that count the number of divisors of the number. That we already saw some examples like this, 10^{10}, second is 740, and third is $25 \times 31^4 \times 7^6$.

Third question, how many integers larger than 10,000, have distinct digits and does not contain the digit 5 or 3.

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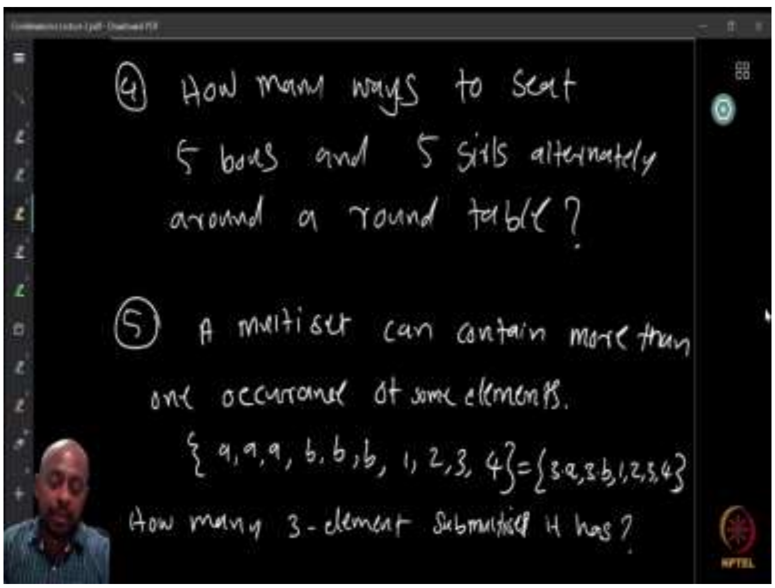

Fourth question is that how many ways to select five boys and five girls and to keep them alternately around a round table. So you want to seat five girls and five boys, and there are 10 seats maybe and then in this round table, we want them to be sitting alternately. So, how many different ways you can do that. So, many of these questions, you may need to use several of the principles that we have learned. You can do all of them with the principle that we have learned but you might have to use a combination of them.

Fifth question, a multiset can contain more than one occurrence of some elements. We already saw multiset but let us say that the multiset contains {a, a, a, b, b, b, 1, 2, 3, 4}. So, a is appearing three times, I denote the multiset in the following way. If a is appearing 3 times I write 3.a and b is appearing 3 times, here I will write 3.b, and then 1, 2, 3, 4, is appearing only once so I will not write anything, just the numbers.

So, {a, a, a, b, b, b, 1, 2, 3, 4} = {3.a, 3.b, 1, 2, 3, 4}. Just for later references, we might use this notation. So, how many 3-elements submultisets it has?

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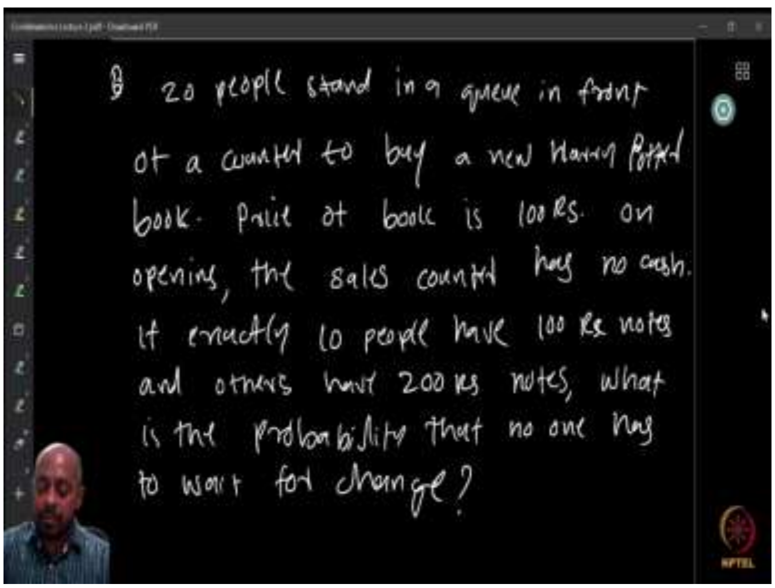

Then, 20 people stand in a queue in front of a counter to buy, let us say, a new book that is published, by, let us say Harry Potter. Some time back, it happened. So, a new book is published. So, when the book is released, people go stand in queue, at the time of the release, they would stand in queue to buy the books. So, let us say that 20 people stand in a queue in front of a counter to buy a new book. Price of the book is, let us say, 100 rupees. So, each book has price 100 rupees. There is only one book for sale for each person in the queue.

Now, on the opening time, in the sales counter, they usually has Zero cash. If you do not have a change, you are in trouble. You have to wait for the change. So, similarly, the sales counter opens with no cash.

So, if exactly 10 people have, let us say 100 rupees note with them, and the others have exactly 200 rupees notes with them, they come with 200 rupees notes. 10 of them come with 100 rupees notes. Now, we do not know which guys have 100 rupees notes and which guys have 200, but we know that 10 of them has 100 rupees and 10 on them has 200 rupees.

So the queue is arbitrary, the people will come and stand there, some of them have 100 and some of them have 200. What is the probability that no one has to wait for change? If somebody gave a 200, and then the cashier does not have change, he will say that okay, wait there, till I get the next guy or somebody who has 100 rupees. When he buys, I will give you the change. So what is the probability that no one has to wait for change? So, find it. It is going to be interesting.

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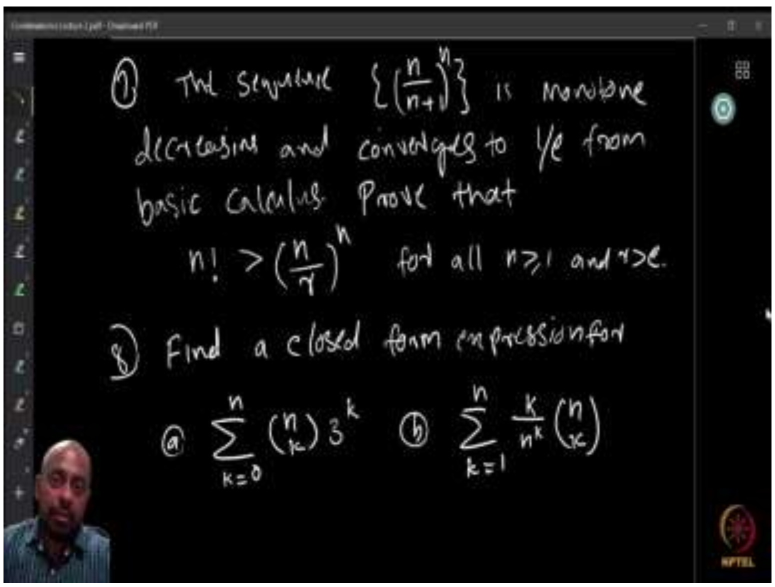

And, seventh question, the sequence, we have, in calculus, we have already seen the sequence, $\left(\frac{n}{n+1}\right)^n, n \geq 1$, is a monotone decreasing sequence and converges to $\frac{1}{e}$ from basic calculus course that we have studied. Now, prove that $n! > \left(\frac{n}{r}\right)^n$ and $r > e$. So, e is the base of the natural logarithms.The next question, find a closed form expression for the given expression,

(1) $\sum_{k=0}^{n} \binom{n}{k} 3^k$ (2) $\sum_{k=1}^{n} \frac{k}{n^k} \binom{n}{k}$

So, find, closed form expressions for both of these using some of the techniques that we have learned.