Combinatronics
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Lecture No. 02
Multinomial Theorem

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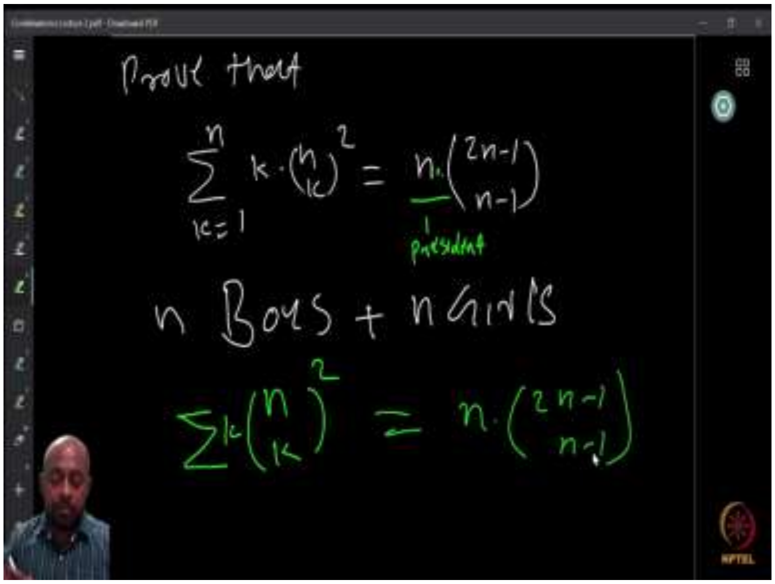

Here is another question, that I want to find a combinatorial proof. So, prove that,

$$\sum_{k=1}^{n} k.\binom{n}{k}^2 = n.\binom{2n-1}{n-1}$$

Of course, you can find an algebraic proof and I would say that, you can try to find an algebraic proof, it is going to be a little difficult, I think. But still you can find it. But it is going to be a little cumbersome to work out with these things. But try it, and try to find out why this must be equal. But then, we want to find a combinatorial proof.

So, what is going to be our combinatorial proof? So, think about, I mean, finding a proof for yourself and after spending some time you come back here and then continue. I recommend that you think about this because unless you think about this, it is going to be difficult to solve questions. Because the subject of combinatorials is such that, like, you will see lots of problems, each of them, the solution is kind of easy once you see it.

But till you see it, coming up with the idea, how to frame this is going to be tricky. You need to like, strike the idea in your mind before you can answer. It is not like a standard procedure because each question has a slightly different flavor which affects the way you have to think. So, you need to develop a habit of thinking on this. And that is why I am asking you repeatedly, again and again, that you think on this before you proceed.

So, what is the combinatorial proof I am going to give you? Here is it. So, what is on the right hand side? I have n and $\binom{2n-1}{n-1}$. So, I can immediately see that $\binom{2n-1}{n-1}$ is choosing $(n-1)$ members from a $(2n-1)$ -element set. So, basically forming an $(n-1)$ -element subset of a $(2n-1)$ -element set.

Now, I am going to multiply that with n, which means that there is some choice involved by our product rule. There was some choice of n distinct persons. That is, we were able to choose n distinct persons. And that is independent of the choice of the, $n-1$ guys from the $(2n-1)$ guys.

So, what I am going to do is the following. I am going to say that there are, let us say, let us say n boys and there are n girls. So, we have $2n$ persons here, n boys and n girls. Now, from this, I want to form, let us say, a club or a committee, whatever you want, call it.

So, I want to select a club with n people inside and I want to make sure that the club has a president and the president in a girl. They usually do a better job as president than the boys. So therefore, we will choose a lady president and then we will have remaining members. So now, we can see, what is on the right side, because from the$n$ girls, I can choose president for the club in n possible ways. So, I have n possibilities to choose the president.

Now, my requirement for the club was that the president must be a girl. But then, the remaining, there is no condition. So therefore, once I choose the president, the remaining members, the president is already a member, so remaining $(n-1)$ members I have to choose. But $(n-1)$ can be any of the people. So therefore, I can choose any of the $(n-1)$ person from the $(2n-1)$ remaining people expect for this one girl, the president. There is n boys and $(n-1)$ girls.

So, out of the $(2n-1)$ persons, I can choose $(n-1)$ members, So, I choose the president and I select $(n-1)$ members depending on what is n. Unless $n=1$, it is not empty. So, you choose $(n-1)$ elements of subset of the $(2n-1)$ -element set.

So, you select these guys in $\binom{2n-1}{n-1}$ possible ways. And then since the number of choices of the president and the remaining $(n-1)$ guys were independent, we could multiply them. So, I get $n.\binom{2n-1}{n-1}$.

Now the claim is that the left-hand side counts precisely the same thing. So, why does the left-hand side count precisely the same thing? So, what happens on the left-hand side, I have $\sum_{k=1}^{n} k.\binom{n}{k}^2$. So, let me count this club forming business in a slightly different way. How I am going to count this?

So first, I will form the club by selecting k girls from the n possible candidates, right? So, I choose k girls from the n possible guys. So, I say that these girls are going to be the members. Now, once I choose the k girls, I will select one of them to be the president because I want a girl to be the president. So therefore, out of the k girls, I have exactly k choices to make one of them as a president.

So, I choose k girls and then select one of them to be president. And this is independent. So, I choose the k girls first, and then independently, whichever k girls I choose, one of the k I can chosen as the president. So, $\binom{n}{k}$ ways I can choose the k girls to be in club and then select president. So, $k.\binom{n}{k}$.

Then, once I choose the k lady members, I select the remaining $(n-k)$ boys from the n boys. So, from the remaining n boys I need to select $(n-k)$ boys. But instead of selecting the $(n-k)$ boys to be in the club, I select the k guys who are not going to be in the club and throw them out. That is the same choice I am making. So, I make the choice of k members who are not going to be in the club, throw them out, and then select the remaining $(n-k)$ guys to be in the club.

So, this is a way I can form a club with $(n-k)$ boys and k girls and one of these as the president. But this can be any of the case. k can be 1, k can be 2, et cetera, k can be n. And each of them give distinct clubs so therefore, the number of ways to form the club is $\sum_{k=1}^{n} k.\binom{n}{k}.\binom{n}{k} = \sum_{k=1}^{n} k.\binom{n}{k}^2$

So, but now, this is all possible ways it can happen that there is a lady president and there is the remaining members were chosen. All of them appear here so therefore these two quantities must be equal. So, that is why we have the equality.. So, what we counted on the right is precisely what we counted on the left so therefore they must be equal. That is , $\sum_{k=1}^{n} k.\binom{n}{k}^2 = n.\binom{2n-1}{n-1}$.So, this is another combinatorial proof.

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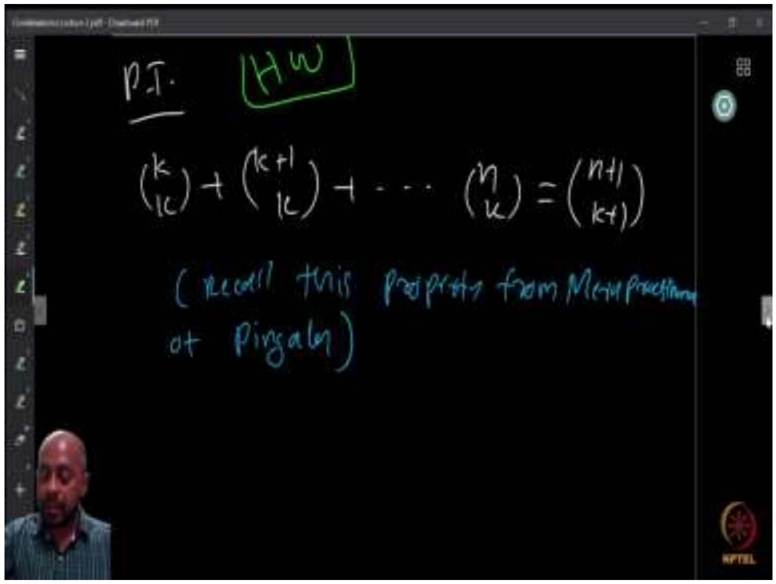

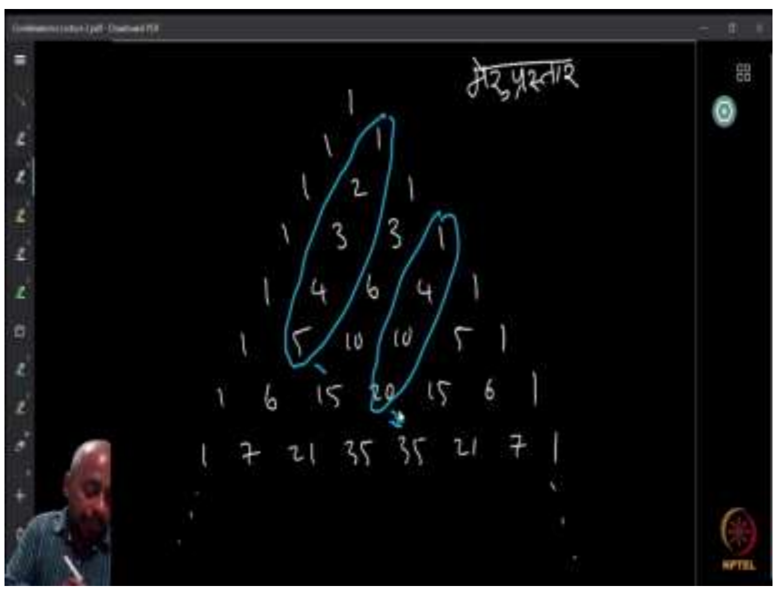

Now, so, recall the special property that we observed from the Meru Prasthara of Pingala. We saw this nice property. So, we had this property, that the sum of these guys, or this, or some of these entire guys will be this. Similarly, sum of any of these will be exactly this. So, using this observation, we can write it in a formal way.

I asked you to perform it yourself. Maybe you have already done it, but if you have not done, here it is. Prove that,

$$\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$$

We just went one step bottom and one step to the right. That is what, $\binom{n+1}{k+1}$. We can see it in the in the triangle. We can see this pattern, why this happens to be like this. So, $\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}$ precisely the diagonal that we are looking at and this $\binom{n+1}{k+1}$ is the off-diagonal entry that we were showing. So, this property, I want you to prove now, combinatorially. Come up with a combinatorial proof. Why this must be the case. So, this is a home work for you.

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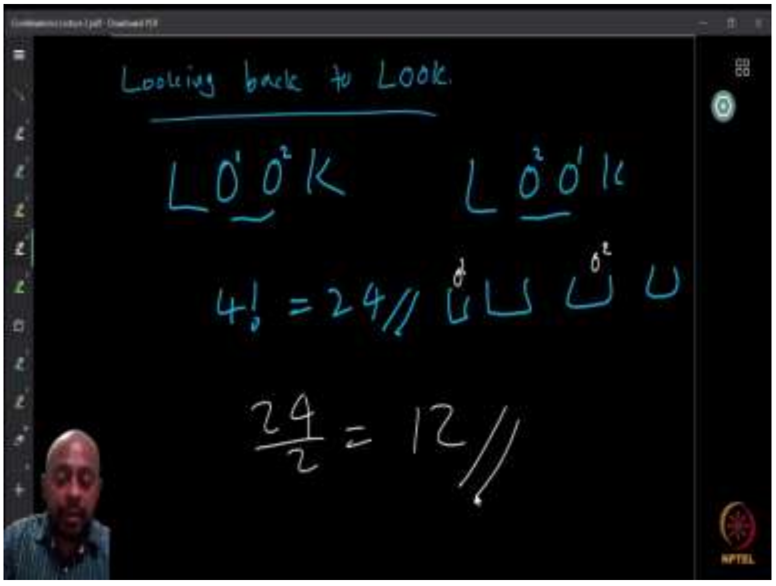

Now, looking back to the anagrams of LOOK. I asked you to solve this yourself, if you can. Few ones have already solved it. You can definitely use brute force to solve it, if nothing else works. You can just write down all the words that you can come up with and count it. So, if you have counted it, you must have found the answer will be 12. So, what is that number? So, how did we find this? So, I would suggest that you try to find it out by whichever way that you prefer.

Now, I am going to use a different method. Maybe not different, but a method as follows. So, the reason we were not able to use the product rule was that, now, when we were having that L O O K, there were two O's. Now, the two O's, when I choose, there will be several orders that I make when I put them in different permutations. But some of them could undoubtedly be the same, which says that the choices were not really independent.

The choice of one of the O's was interfering the choice of the other, because the other comes here or this one comes here, it is the same exactly. So, because of that lack of independence, we could not directly use the product rule. If you want to use the product rule directly, we should be able to bring in more independents. Now, how can we bring independents?

So, you can bring independents by saying that the O's are actually different. So, I will say that I will have the O's, but I will mark one of these is O^1 and the other one is O^2. Then the anagrams are going to be exactly 4!.

Because all permutations are going to be different because if I put, for example L O O K with O^2 here and O^1 here, this new labeled O, L O^2 O^1 K is different from this labeled L O^1 O^2 K because this is the first O and this is the second O but where is the second O, it is coming first and the first O coming second.

So therefore, I can count now using product rule. So, there is going to be 4! = 24 different words with the new O's, the labeled O's. But, now, what we observe is that because, when we remove the labels, this one and this one counts to be the same. They give the same word. So, let us see, how many over counting we are going to do when I do this 4! = 24.

So, we observe that, once you fix the position of the O's, the four different positions of the O's. There is an O, appearing here, O^1 and O^2 . If I swap their positions, it is not going to make any difference in the word, right? Because when they remove the label, they are going to be the same. Can you do anything else? I cannot move the O to somewhere else because that is going to create a difficulty? It is going to be a different word.

So, if I swap, there is no problem. When I swap, how many possibilities are there? Either 1 goes to 2 and 2 goes to 1 or, the same thing. There is only two possibilities. So therefore, there are exactly two ways to get this O^1, O^2 or O^2, O^1 by swapping them. And these are the only over counting.

So therefore, every word I counted exactly twice because O^1 O^2 was coming and O^2, O^1 was coming in. So, O^1 comes first and O^2 comes second or O^2 comes first and O^1 comes second, in the same positions, they lead to the same word. So therefore, I can use the division principle, now to say that, because I over-counted every word exactly twice, I can divide by 2. So, 24 / 2 = 12. So, there are 12 different numbers. You can verify by finding the 12 different numbers.

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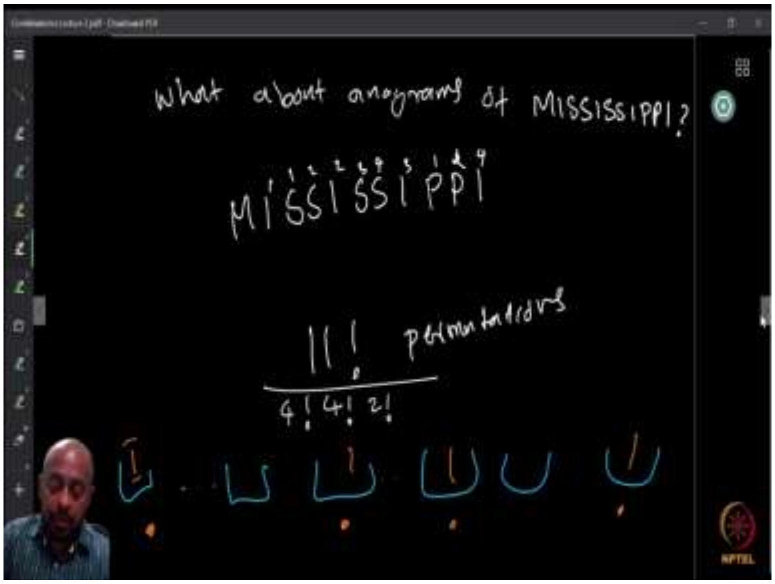

Now, we found out the counting for LOOK. Now, I give you a bigger word that is MISSISSIPPI. Can you count the word MISSISSIPI and its anagrams? Now, I think, once you get the idea of the other one, this is easy.

So, suppose I put labels to make sure that all the words are distinct. So, I will put label, let us say, I is appearing 1, 2, 3, 4 times. We will say I^1, I^2, I^3 and I^4. S is appearing 4 times so S^1, S^2, S^3, and S^4. P is appearing two times so P^1 and P^2. Then, M is appearing only once so I will not do anything. If you want fun you can make this M^1. It doesn’t matter.

And then, I say that these are distinct words, they are distinct letters now. I take all possible permutations of this. I get words. So, 11 letters are there so therefore 11! permutations are there. Then I observe that but whatever the permutation is given, the I’s are occurring four times. But they were all four different I’s because I put labels.

Now, if I switch the position of the two of the I’s, it does not make any difference to the word. In fact, if I switch the positions of any of these I’s, between themselves, it does not make, but if I permute all the ‘I’ within their position, so each of them have a fixed position in the word that we created. So, there are 11, I will not draw the 11. So, I will just mark these special positions where 'I' was occurring in the word.

It was occurring here, maybe, and here, here and maybe here. So, the 'I' was appearing here. I^1, I^2, I^3, I^4, maybe. Now, I do not change the positions in the word but between the I^1, I^2, I^3, I^4, I can permute them anyway I want. So, the four I's can be permuted in 4! different ways. The two O's, we could permute in two different ways which is actual 2 factorial.

So, the four I's, I can permute in 4 factorial different ways. And each of the 4! permutations, that I obtained by labeling them, are all going to give the same unlabeled word where I's are all identical, I will get the same word. So, I have to divide this 11! by 4!, to compensate the fact that I over counted all the permutations of I's in 24 different ways. So, I over-counted them in 4 factorial many ways.

Similarly, there are 4 different S's. So, I can apply the same rule. So, I can use, again, division rule to say that once I unlabel S, all these four factorials these will give you the same words. So, 24 of them will give the same so I can divide by 4 factorial again. Then, I have two different P's so I can divide by 2 factorial different ways. There is only one M, so I can divide by 1 factorial, if you want but we did not over count there so therefore, I will say that $\frac{11!}{4!.4!.2!}$ different permutations or anagrams of the word MISSISSIPPI are there.

Now, it is very interesting to see that 11 factorial can be divided by 4 factorial two times, 2 factorial once and still we will get an integer, because we are counting, counting things. Number of anagrams. It must be integer. There is no other possibility. So, anyway, that is just a remark.

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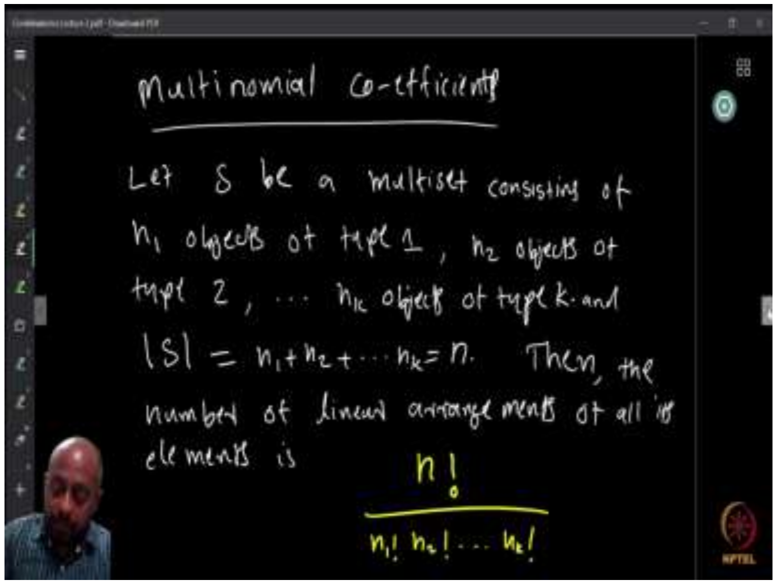

So, this leads us to something called Multinomial Co-efficients. So, what we were looking at. Now, we will use division principle and multiplication principle only but we can now come with a slightly more general, slightly refined tool, but, more basic, still basic. So, this is called multinomial co-efficients. So, what is a multinomial co-efficient?

Let S be a multiset consisting of n_1 objects of type 1, n_2 objects of type 2, … , n_k objects of type k and $|S| = n_1 + n_2 + \cdots + n_k = n$. Then the number of linear arrangements of all its elements is $\frac{n!}{n_1!n_2!\ldots n_k!}$.

So, S is the multiset. It has cardinality $n_1 + n_2 + \cdots + n_k$ which I called n. And these n elements, there are these repetitions, of n_1 of particular 1 type, n_2 of another type, etcetera, n_k of another type. Then, the number of linear arrangements of all of these elements is precisely $\frac{n!}{n_1!n_2!\ldots n_k!}$.

So, this is called multinomial co-efficient. We want to see why this is precisely this. We already saw the proof in some sense. We argued it, but let us make it little more formal.

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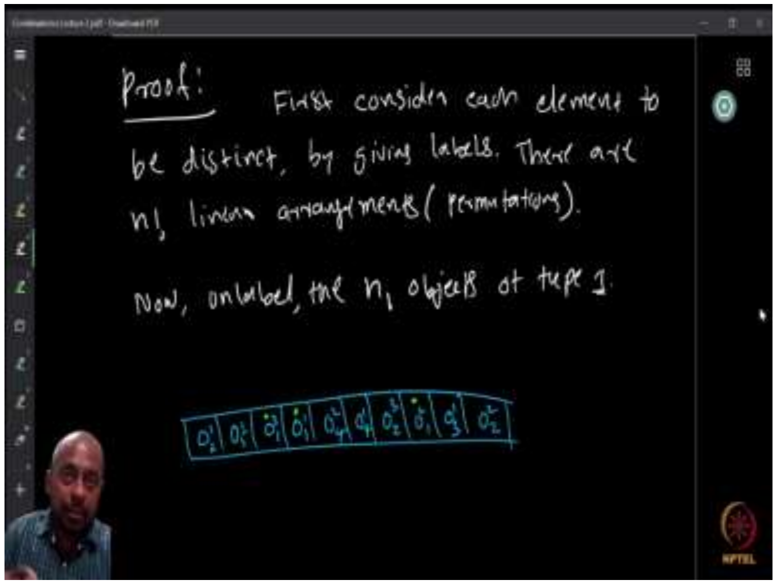

So, here is the proof. So, first consider each element to be distinct by giving labels. So, we start with the multiset, but now, the multiset has same element appearing many times. So what I am going to do is that I will make, O_1 to be O_1^1, O_1^2, O_1^3 et cetera $O_1^{n_1}$.

Similarly O_i will be O_i^1, O_i^2, O_i^3 et cetera $O_i^{n_i}$. I am putting more labels to make them all distinct labeled objects. So now, since I have n objects in total, once I do the labeling, I get exactly n different objects. So now, I look at the permutations of them. So, how many are there? $n!$ permutations are there. After this, I do the unlabeling. So, what I do is that I unlabel the n_1 objects of type 1.

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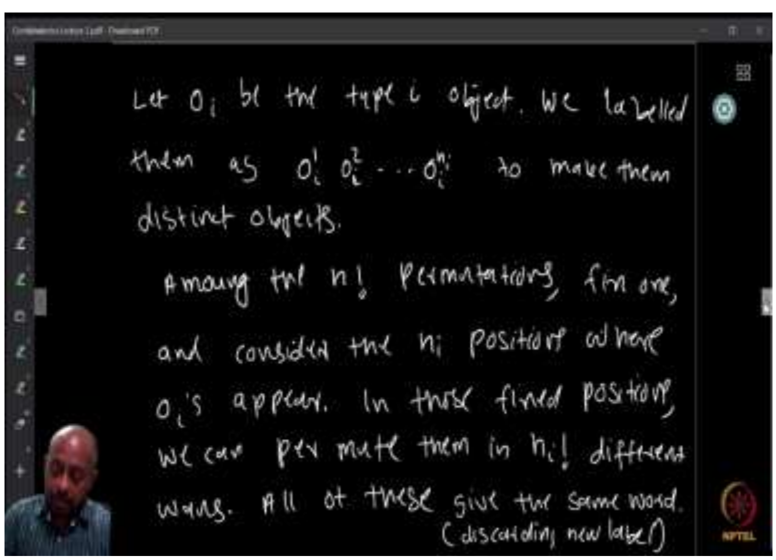

So, once I do this, what happens? In general, what happens is that, if I unlabel, so here O_1 is 3 since O_1^1, O_1^2 and O_1^3 are there. So, there are three objects of type O_1. So, what I do is that when I unlabel these three, what happens is that, the different things will become the same. So, there were three of these objects.

So, 3! different ways I can permute and each of them will give the different objects in the labeled fashion. But once you unlabel, each of these 3 factorial will be corresponding to the same set we were looking at. So therefore, that is over counting. So, similarly, O_i be the type i object. We have labeled them as O_i^1, O_i^2, O_i^3 et cetera $O_i^{n_i}$, to make them distinct. Now, among the $n!$ permutations, let us fix one of the permutations.

Once you fix the permutation, consider the n_i positions where the O_i's appear. And then between these fixed positions, I will not change the positions, but I can permute the copies of O_i,'s , they were O_i^1, O_i^2, O_i^3 et cetera $O_i^{n_i}$. These guys, I can permute them between themselves. They will all lead to the same ordering that we were looking at because, when we were looking at the unlabeled ordering.

So, n_i factorial permutations were giving the same thing. So therefore, the over counting they permuted n , and a multiple of n_i factorial. So therefore, I have to discount this by, using the division principle by dividing n_i factorial.

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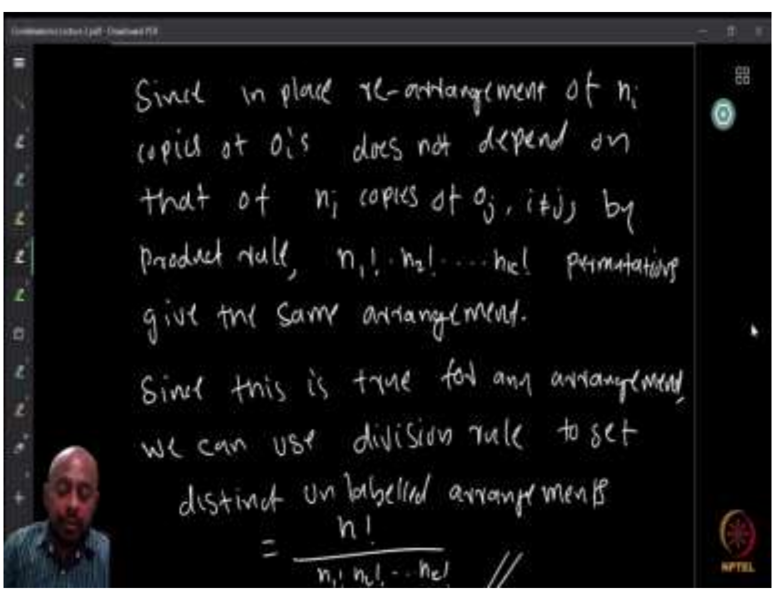

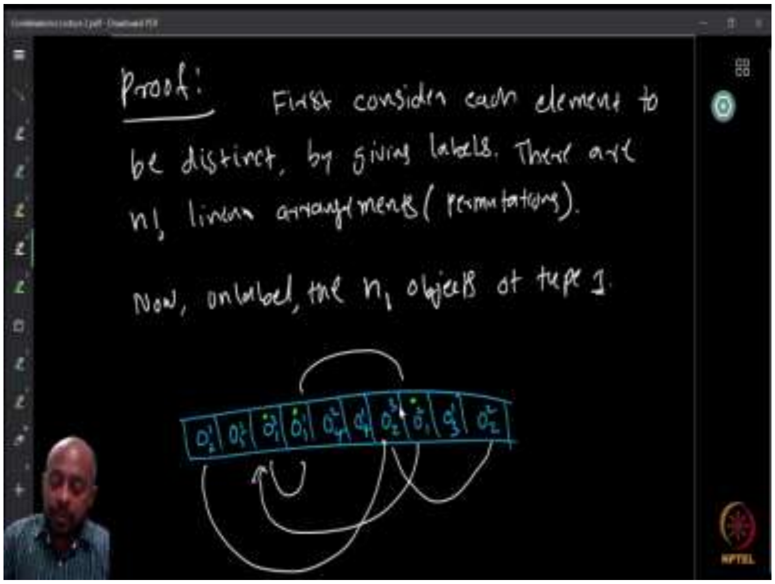

So, since the, in place re-arrangements of n_i copies of O_i's does not depend on that of the n_j copies of O_j. So, I look at the n_i positions, re-arrange them. I look at the n_j positions of the other objects and then re-arrange them there. That does not affect here. For example. When we were looking at this, we were looking at copies of O_1. So, even though I permute them between themselves, no matter what, I will say that this goes here now, and then this guy, who was here will go here and then this guy who was here comes back here.

I can re-arrange O_i and O_j independently because their positions are different. So therefore, these arrangements are independent. So therefore, these permutations, n_i factorial permutations are all independent.

So, the total number of over counting is n_1 factorial for the first object, n_2 factorial for the second object and n_k factorial for the last object. And they are independent so I can multiply using the product rule. So, product of n_1 factorial, n_2 factorial, …, n_k factorial is the total number of over countings.

So therefore, now I can say that since I have over counted every object these many times, I can divide by the division rule. So therefore, I get $\frac{n!}{n_1!n_2!\ldots n_k!}$. So, that is how we prove this. So, this is the multinomial co-efficient.