Real Analysis II

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Lecture - 27.2
Approximation Theorems for Lebesgue Integrable Functions

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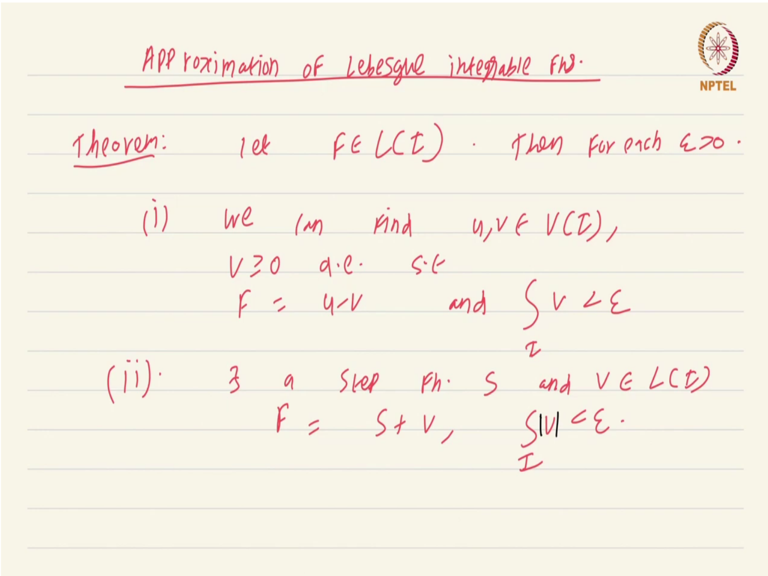

In this video we are going to study some approximation properties of the Lebesgue integrable functions. These Lebesgue integrable functions are in some sense almost a step function, they are also almost an upper function and the following theorem is going to make that precise.

Theorem; let F be a Lebesgue integrable function, then number one we can find we can find two upper functions u comma v in U of I with the property that v is greater than or equal to 0 almost everywhere. Such that F is equal to u minus v and integral of v over I is less than epsilon ok.

So, then for each epsilon greater than 0, this is possible. So, what this is saying is that any Lebesgue integrable function is almost an upper function except for a small non negative upper function; that is one approximation theorem. The second approximation theorem sort of says that this function is almost a step function. So, there exists a step function; step function S and V in L of I such that F is just equal to S plus V and integral of V is less than epsilon.

That is you just have to add a really small Lebesgue integrable function to the step function to recover the function F. So, these heuristic principles that F is almost a step function or it is almost an upper function is going to prove very useful later ok. Let us prove these the proofs are not hard.

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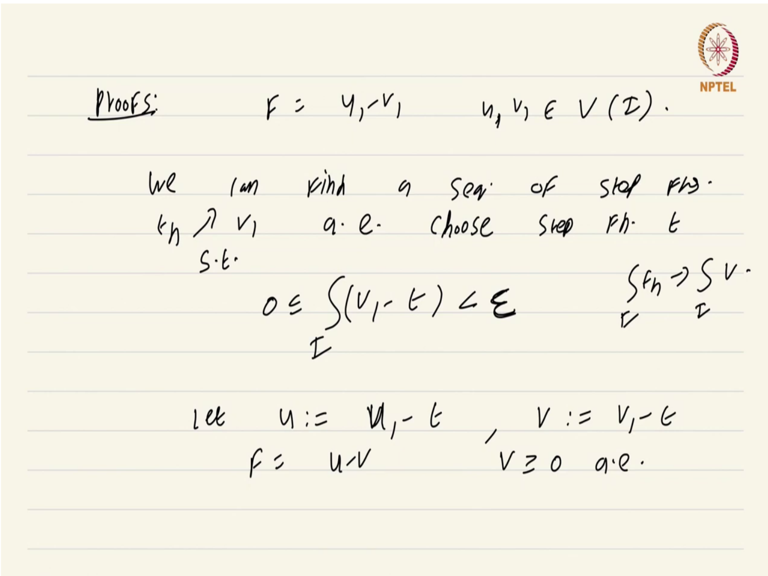

Proofs; well we already know that you can write F as u 1 minus v 1 where u 1 and v 1 are upper integrable functions upper functions essentially, they are that is just the definition. Now, by definition we can find a sequence of step functions t n that increase to v 1 almost everywhere ok.

Now, what we do is choose step function t step function t, such that such that 0 is less than or equal to integral v 1 minus t over I is less than epsilon ok. So, this just I mean you can do this because integral t n by definition converges to integral over I of v and integral of v 1 minus t 1 sorry v 1 minus t is just integral v 1 minus integral t. Note, this crucially uses the fact that t is an upper function sorry t is a step function not merely an upper function.

Since t is a step function I can break up the integral ok. So, because integral t n converges to integral v, I can find a t this t is just going to be some choice of t n where n is appropriately large such that this happens, I can do this ok.

Now, what I do is I let u to be just u 1 minus t and I let v be by definition v 1 minus t Then clearly F is u minus v and furthermore v is greater than or equal to 0 almost everywhere simply because t ns are increasing to v almost v 1 almost everywhere.

Furthermore by our choice of t we automatically have that integral of v is less than epsilon. So, part 1 is proved you just successively use the definitions and you got part 1. For part 2 which is about showing that you can find a step function which is very very close to the given Lebesgue integrable function we have to use part 1 and the way you use it is as follows.

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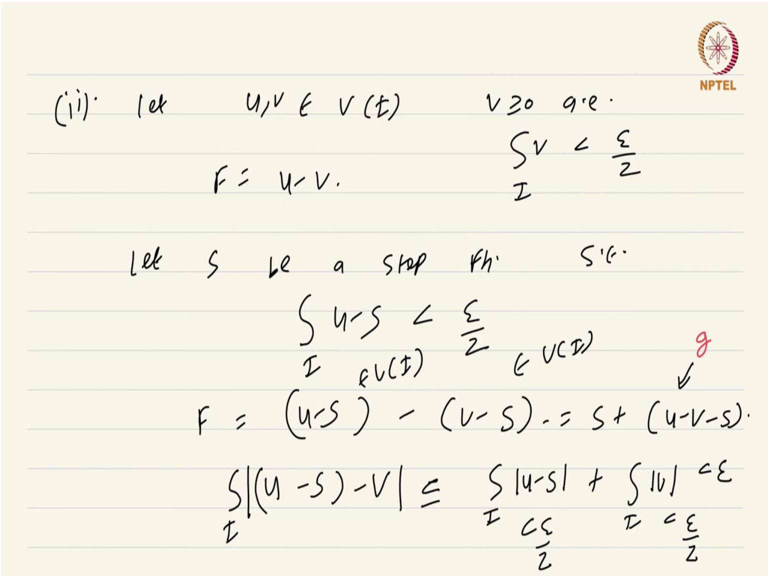

Let u comma v; so, this is the proof of part 2. Let u comma b v be upper functions such that v is greater than or equal to 0 almost everywhere and integral of v is less than epsilon by 2 ok. And for of course, F is u minus v this just follows from step 1.

Now, what you do is, let S be a step function such that integral of this u minus S over I is also less than epsilon by 2. The way you choose the step function is exactly the way you chose t in the previous step, for the same reason you can find a step function S such that integral over I u minus S is less than epsilon by 2.

Now, observe that F is again nothing but u minus S plus v minus sorry minus v minus S ok. And by our choice v minus s is an is an upper function u minus S is also an upper function. And integral of u minus S minus v, what is this integral going to be?

Well, certainly this integral one moment please let me just make a small change here, I actually claim something stronger I have integral over mod v itself is less than epsilon of course, integral over mod v will dominate integral over v.

So, I claim that even integral of the absolute value of v can be really small ok. So, coming back sorry about that error coming back. So, what I want to do is I want to estimate this because of course, this is just S plus u minus v minus S right. So, this part this part and for clarity let me just call this a new name.

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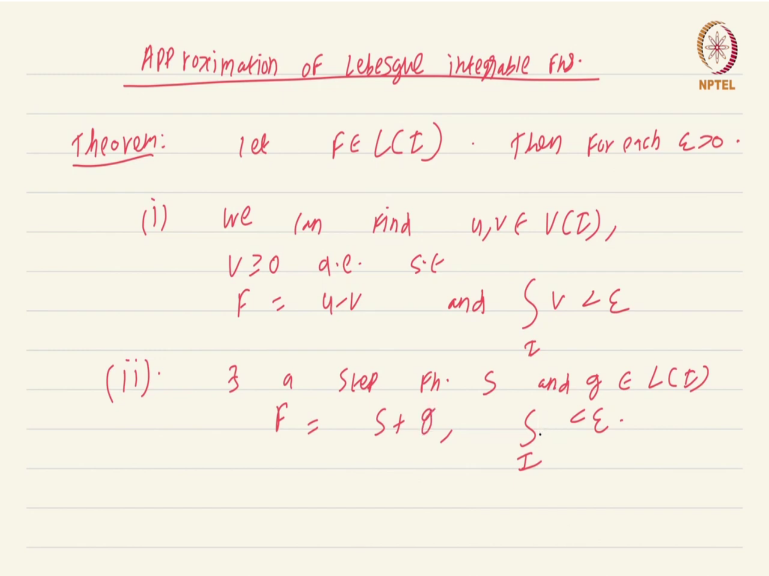

Let me just call this g. So, that there is no scope for confusion. So, coming back we will call this g we will call this g and note that this integral will certainly be less than or equal to integral over I mod u minus S plus integral over mod v ok. Now, because v is greater than or equal to 0 almost everywhere mod v and v are almost everywhere equal. So, this is less than epsilon by 2 because this is true for the function v and if two upper functions agree almost everywhere then obviously, they have the same integral ok.

Now, coming to this function integral of I u minus S, well this is also greater than or equal to 0 almost everywhere because S is less than u less than or equal to u almost everywhere by the way we chose S. So, this quantity is also less than epsilon by 2. So, the whole thing is less than epsilon ok. So, what have we managed to achieve we have written F as a combination of a step function and a Lebesgue integrable function whose integral of the absolute value is really small ok.

So, these two approximation theorems are useful heuristic principles that you can use to solve various problems. So, let us see let us see a some applications of this in the future, for the time being let us just analyze the behaviour of sets of measure 0. Recall that I had mentioned that if you have a Riemann integrable function if you just modify it on a countable set that is actually a set of measure 0, it is possible that you get a function that is not Riemann integrable at all.

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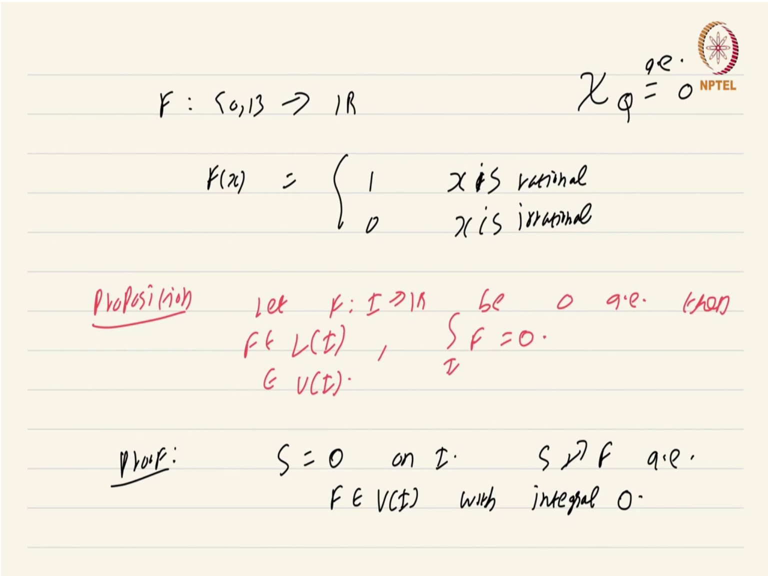

And you have no doubt seen this famous function F from close 0, 1 to R such that F of x is 1 if x is rational and 0 if x is irrational x is irrational. This is the characteristic function of the collection of rational numbers. This function is equal to 0 almost everywhere because Q is a countable set, but nevertheless this function is not integrable because it is not continuous at any point. So, the set of continuous set of um discontinuities will be a full measure set because of this this function is not going to be Riemann integrable by the Riemann Lebesgue criteria for integrability.

But we would like this function to have integral 0, its taking the value 1 on a really small set. No fear their Lebesgue integral will save the day because we have the following proposition. Let F from I to r be equal to 0 almost everywhere then F is Lebesgue integrable and integral of I F is 0 ok. In fact, we can make it stronger F is actually an upper function not just the Lebesgue integrable function it is something stronger proof well.

Let us just consider S equal to 0 on I look at this function then S increases to F almost everywhere this is just true trivially just by the fact that F is equal to 0 almost everywhere S increases to F almost everywhere which shows which shows that F is actually an upper function with integral 0 with integral 0 ok.

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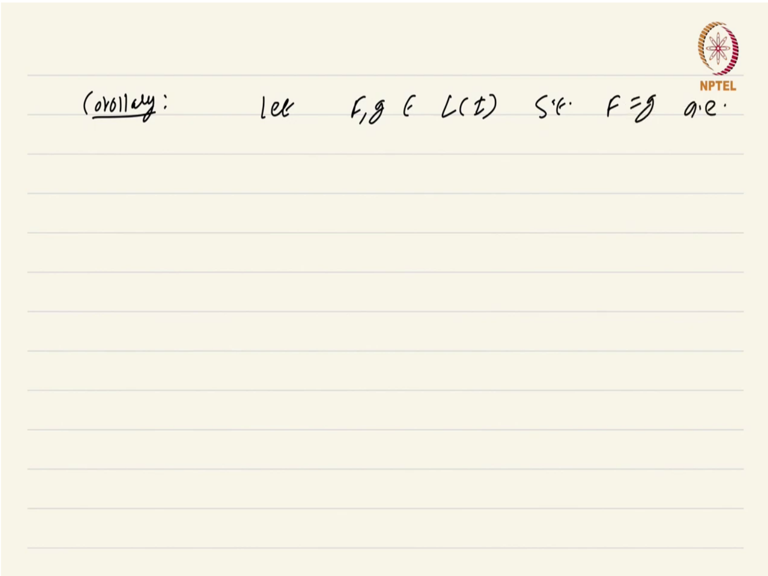

So, the this was rather trivial and a nice corollary of this is corollary is, let F and g be Lebesgue integrable functions such that F minus g or rather F equal to g almost everywhere ok.

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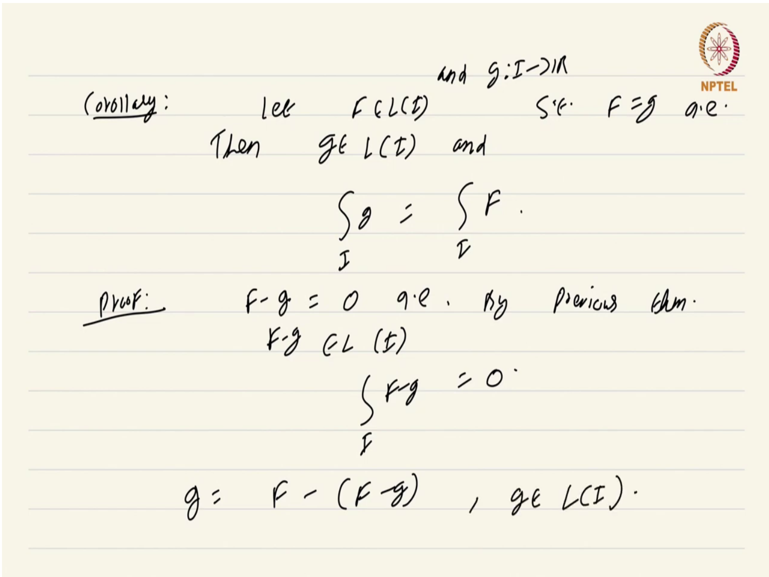

Let me make a slight change let F be Lebesgue integrable and g be just any old function from I to r such that F equal to g almost everywhere then g is Lebesgue integrable and integral over I of g is integral over I of F.

So, if you perturb a function that is Lebesgue integrable on a really small set on a set of measures 0, you are still going to land up with the function that us Lebesgue integrable and moreover both functions are going to have the same integral as expected. Proof; I mean most of the results are rather straightforward and just follow from the definitions that is one of the reasons why I said that this approach to the Lebesgue integral is really nice.

The alternative measure theoretic approach is more general and powerful, but at the same time it is an elaborate set theoretic, you will have to do a lot of set theoretic consideration dealing with algebras of sets which are not very illuminating. Whereas, this approach is straightforward function theoretic and elegant ok.

Look at F minus g, F minus g is 0 almost everywhere. So, by previous theorem by previous theorem F minus g is Lebesgue integrable and integral of F minus g I is just zero, but g g is F minus F minus g and because the Lebesgue integrable its collection of lebesgue integrable functions is a vector space this in fact shows that g isLebesgue integrable ok.

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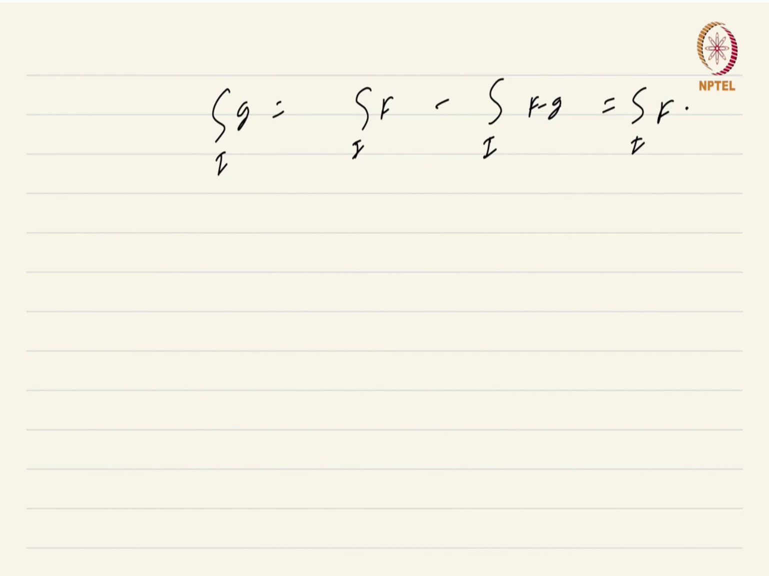

And moreover integral of g over I would be just integral of I F minus integral of I F minus g which is just integral of I F as claimed. So, this completes the proof that whenever you have a Lebesgue integrable function which agrees with some other function almost everywhere then that function is also Lebesgue integrable and has the same integral value ok.

So, this completes the basic properties of the Lebesgue integral. In the next set of videos we will start to study convergence theorems, which is essentially the reason why the Lebesgue integral exists in the first place. This is a course on real analysis and you have just watched the video approximation of Lebesgue integrable function.