Mathematical Methods 2
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Module - 08
Partial Differential Equations
Lecture - 80
The Wave Equation: Separation of Variables

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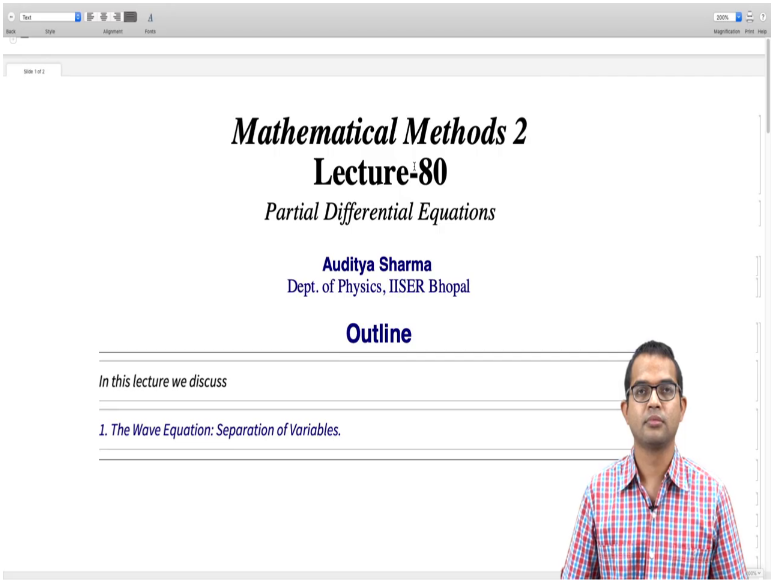

Ok. So we saw how the wave equation can be solved from a very general perspective. So, in this lecture, we will solve the Wave Equation again using Separation of Variables and we will see how in particular this method is suitable for the vibrating string problem, ok.

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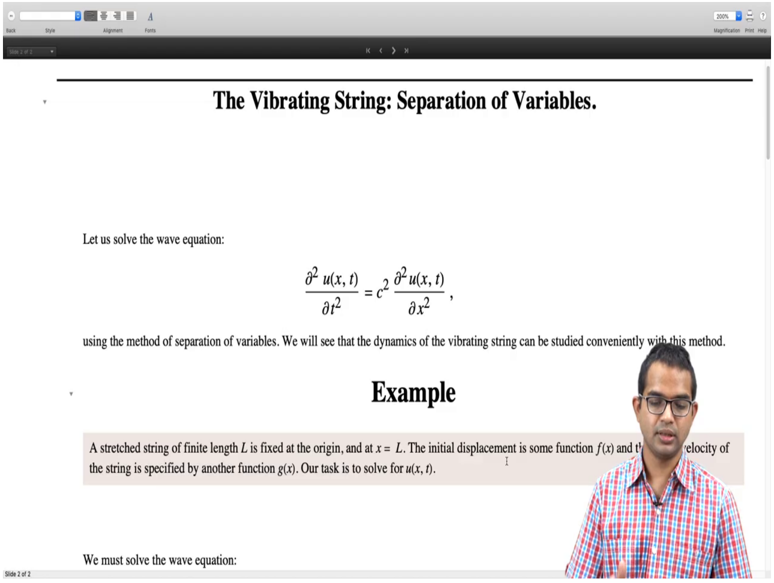

So, the wave equation is like here. So, it is a second derivative with respect to time, second derivative with respect to position and this c squared is here, where c is representative of the speed of the wave right. So, let us solve this problem using the method of separation of variables you know applying it specifically to the problem of the vibrating string.

So, the stretch string like we had before in the earlier lecture is of length L. So, it is fixed at the origin and at x equal to L. So, the initial displacement is some function f of x, right. So, f of x is some function such that it must be 0 at x equal to 0 and it must be 0 at x equal to L and the initial velocity of the string is given by some other function. So, at every point x, there is a certain velocity associated with the motion of the string. So, our task is to solve for u of x comma t right.

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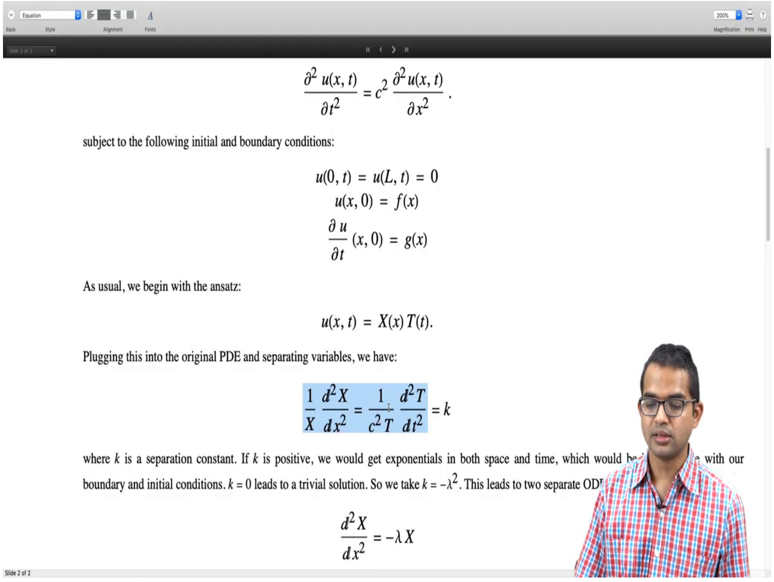

So, we have this wave function which needs to be solved and it is always helpful to write down explicitly all the boundary conditions and initial conditions. Here the boundary conditions are you know at x equal to 0 and at x equal to L. This function must be 0 and at time t equal to 0, u of x x comma 0 is f of x and dou u by dou t of x at time t equal to 0 is another function g of x which is some arbitrary function which is specified.

Now, we begin with this ansatz which is to separate these variables u of x comma t as X of x times T of t. When we plug this solution or this ansatz into the differential equation and we separate variables, we have 1 over X d squared X by d x squared is equal to 1 over c squared T times d squared T by d t squared and since we have you know a function of X must be equal to some other function of t for all X and T, this both of these must be equal to some constant let us call it k.

Now, if k is positive, we would get exponentials in both space and time which would be incompatible with our boundary conditions and k equal to 0 will give you a trivial solution as you can check. So, k is a negative number that is relevant for our boundary conditions. So, we take k to be minus lambda square.

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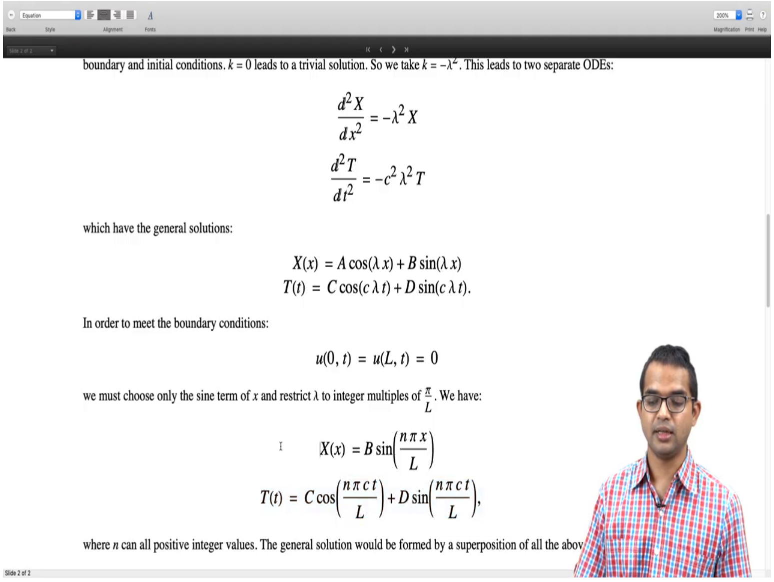

And this leads to two separate ODEs. So, it is lambda squared and minus c squared lambda squared T. So, this will give us A cos of lambda x plus B sin lambda x for capital X of x and T of t is going to be some other constant times cosine of c lambda t plus D times sin of c lambda t, right. So, these are basically like the classical harmonic oscillator problem, very familiar territory.

So, in order to meet these boundary conditions right when at x equal to 0 and x equal to L, you must have this function going to 0. So, X of x can only have the sine term. You cannot have the cosine term and also this lambda is constrained to be a multiple of pi by L, right. So, that at x equal to L, you must get 0.

So, X of x once you are constrained to take this, it also constrains T of t specifically it forces lambda to be an integral multiple of pi by L. So, we can write X of x like here and T of t is C cosine of n pi c t by L plus D sin of n pi c t by L, right.

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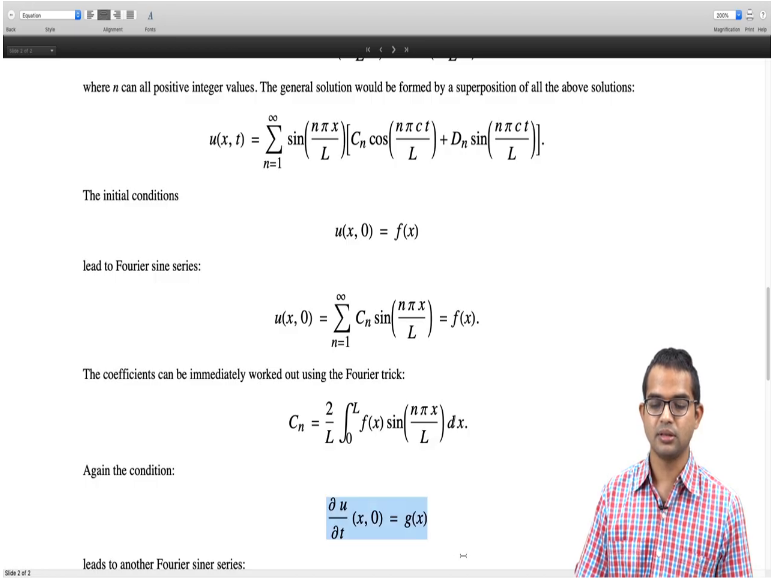

So, the general solution would be formed by a superposition of all these solutions, right. You can take this with this and then n can take all positive integer values. So, in fact we should consider a sum like this u of x comma t. So, we do not need to put this constant B here because it is going to get absorbed in this coefficient C n and D n need to be determined.

So, the initial conditions we will make use of these initial conditions to determine these coefficients. So, u of x, 0 equal to f of x. So, which basically means this sum right. So, if you put t equal to 0, this is going to vanish and this is going to just go to 1.

So, then you are just left with the summation C n sin of n pi x by L is equal to f of x. So, this is a Fourier sine series and we can extract these coefficients using the standard Fourier trick. So, c n you can immediately write down to be 2 by L integral from 0 to L f of x sin of n pi x by L. And again the velocity condition you know relates this to g of x.

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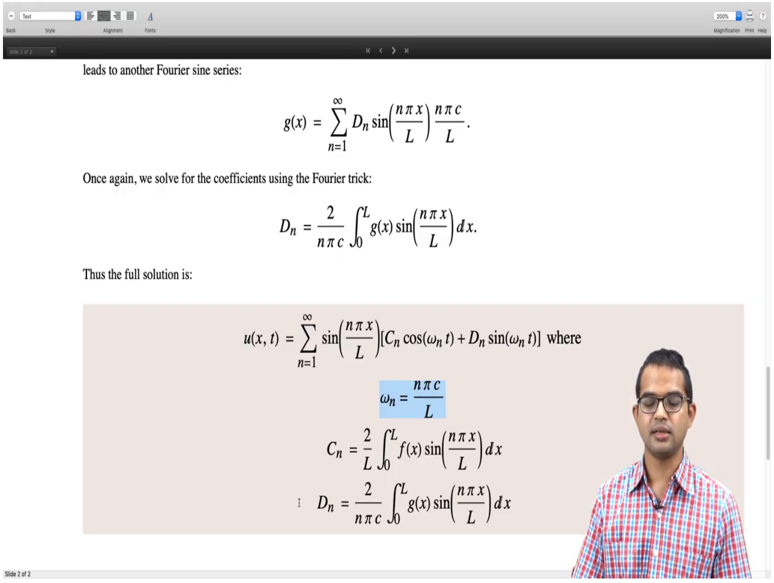

And that also will give us another Fourier sine series, right. So, the Fourier sine series where g of x is expanded in this manner, so you have to also take care that you have these other you know stuff here along with D n which form the coefficient. So, if you take care you will get dn is equal to 2 divided by n pi c integral 0 to l g of x sin of n pi x divided by l dx.

So, the full solution is like here we can write it as you know this in this summation over n sin of n pi x by L, then this C n cosine of omega n, it helps to write as omega n t plus D n sin omega n t where omega n is an integral multiple of pi c by L and C n and D n we just worked out.

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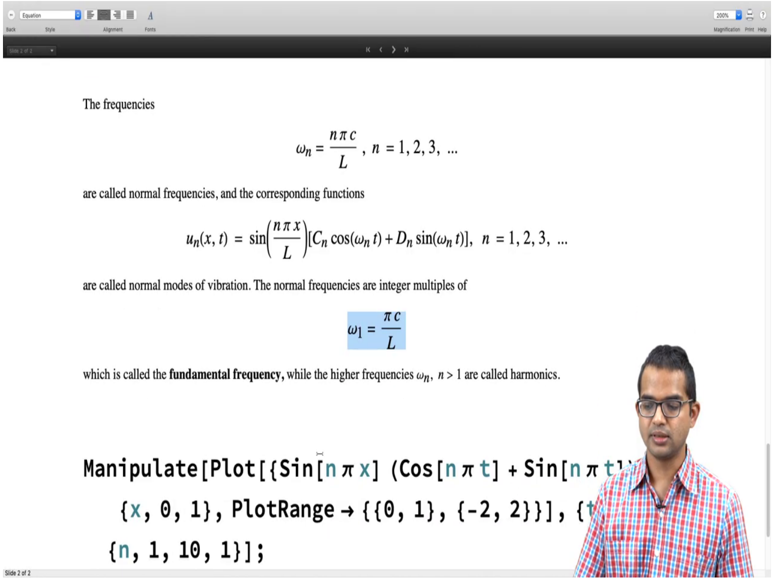

Now, these frequencies omega n equal to n pi c by L are called normal frequencies and the corresponding functions, right. So, this entire function corresponding to an integer n, these are called the normal modes of vibration of your stretched string. Perhaps we have encountered a description of this in an elementary discussion, maybe even going back to high school, but we did not think of this as a solution of a PDE at that point.

So, now you see the story in a more sophisticated manner and now you see that you know these normal modes which we probably saw pictorially at some point, you know they have this nice mathematical way of arriving at this same result. And these are called the normal modes of vibration.

The normal frequencies are in integer multiples of what is called the Fundamental Frequency and the higher frequencies are referred to as Harmonics. So, you know a description in this language is important when we are studying the physics of you know musical instruments for example.

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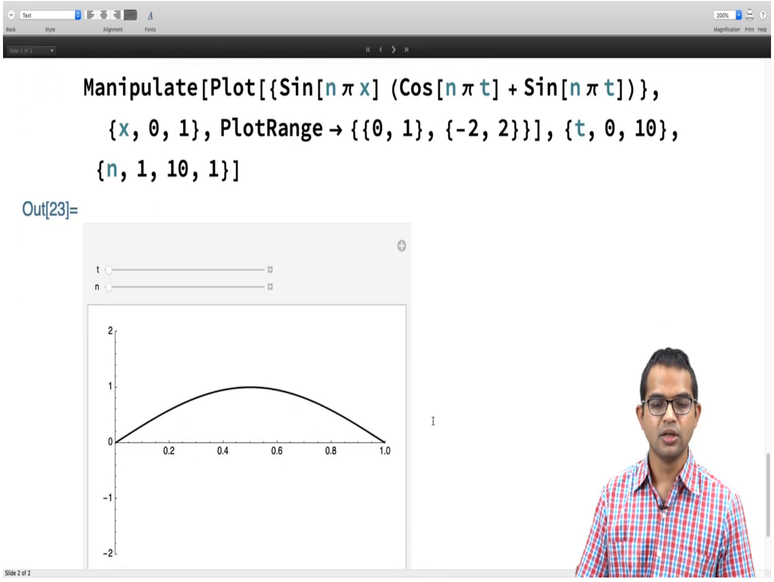

So, let us quickly plot this. It is instructive to look at what happens to these normal modes as we change n.

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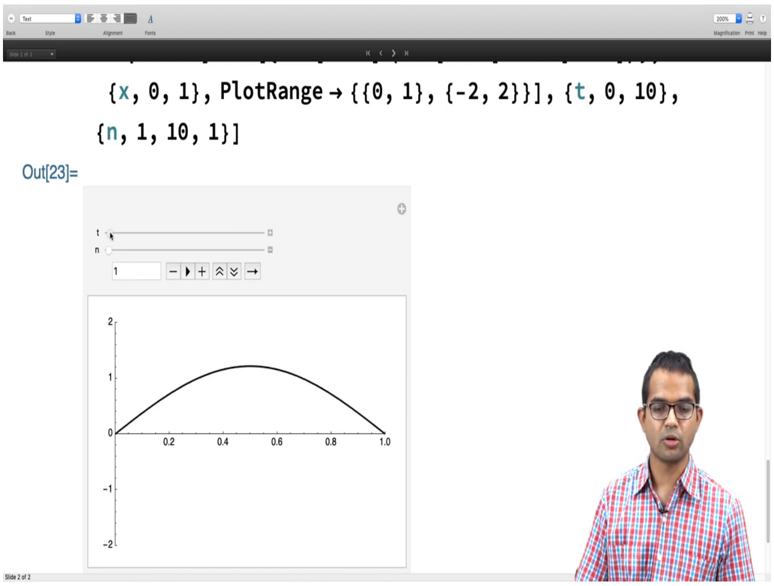

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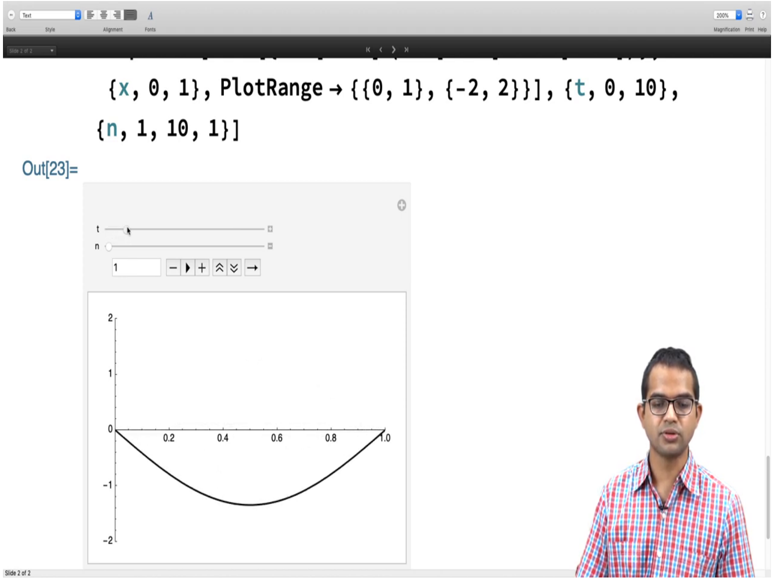

So, when I fix n to be 1 and then look at you know the normal mode as a function of time, so this is what the fundamental frequency does, right. So, this is how an instrument you know produces sound as well. So, the fundamental keeps doing this.

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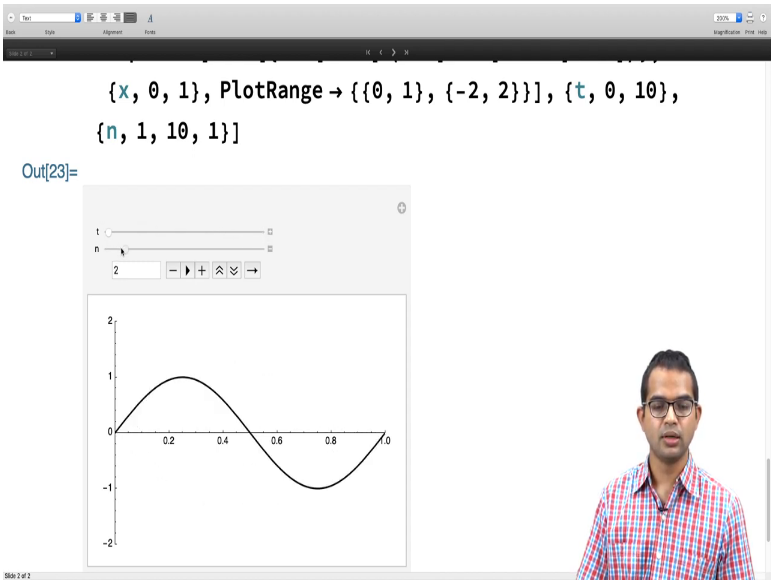

And then if you go to n equal to 2, then you see that it has not only nodes at the ends, but there is also one more node.

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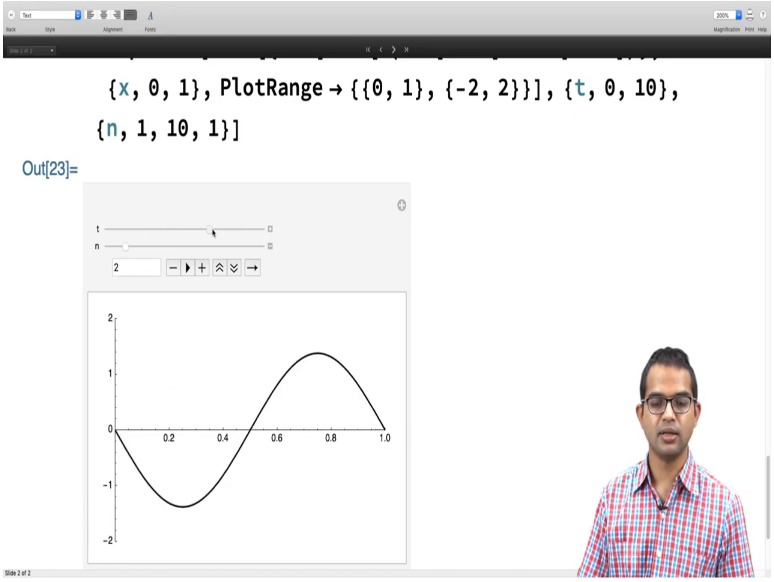

So, as a function of time the node remains unchanged, but the dynamics happens elsewhere and then I make n equal to 3.

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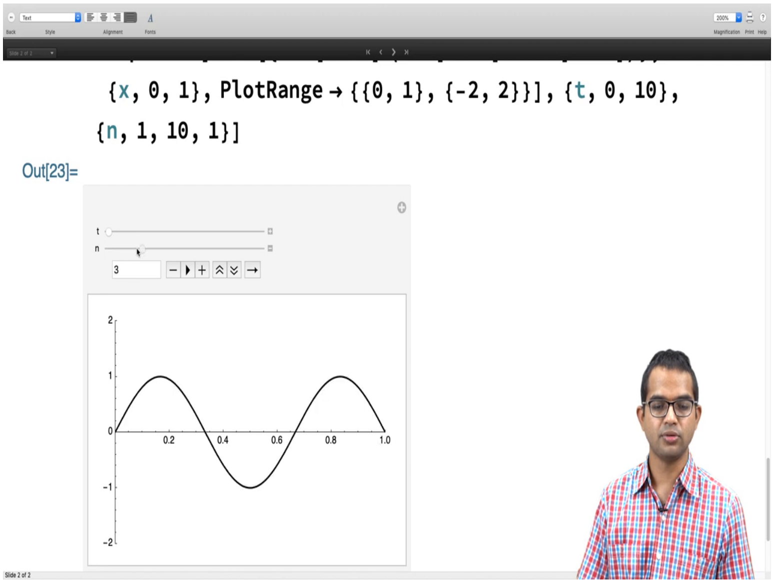

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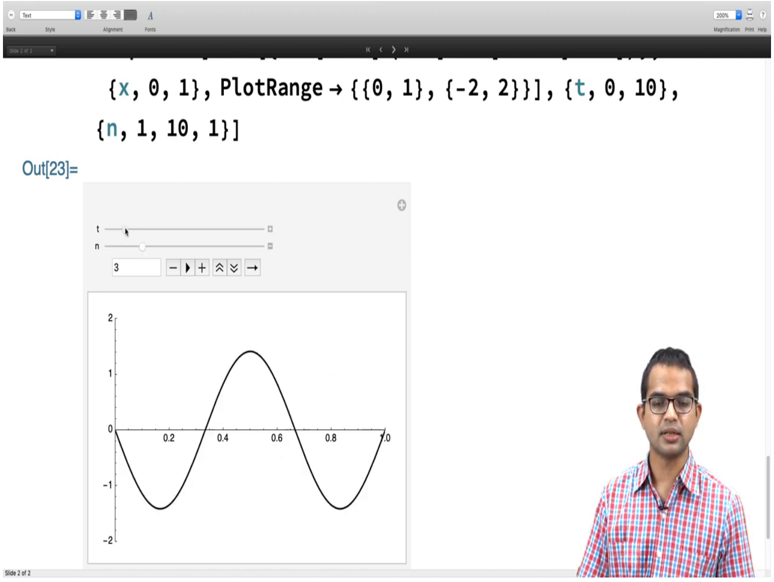

So, there are going to be two nodes in the center in addition to the edges and so, they keep oscillating. So, it is interesting that we get these what are called standing waves also from the wave equation.

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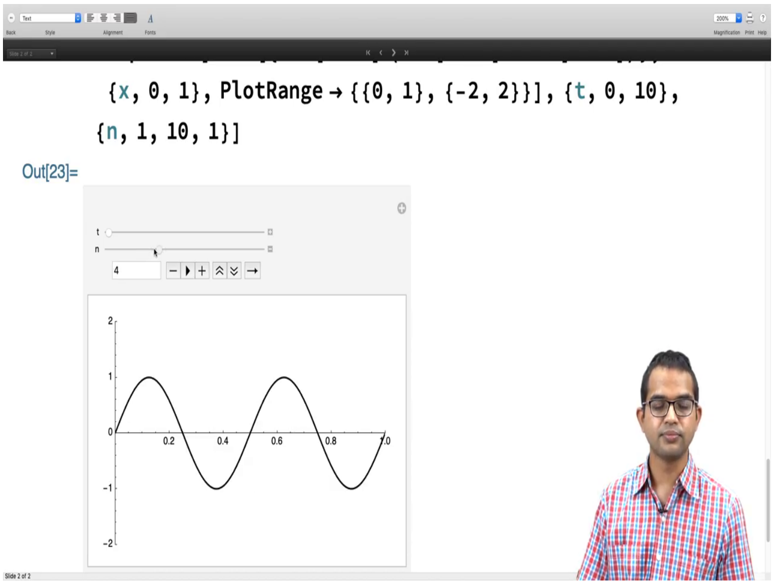

And which we have seen can be thought of as superpositions of traveling waves, right.

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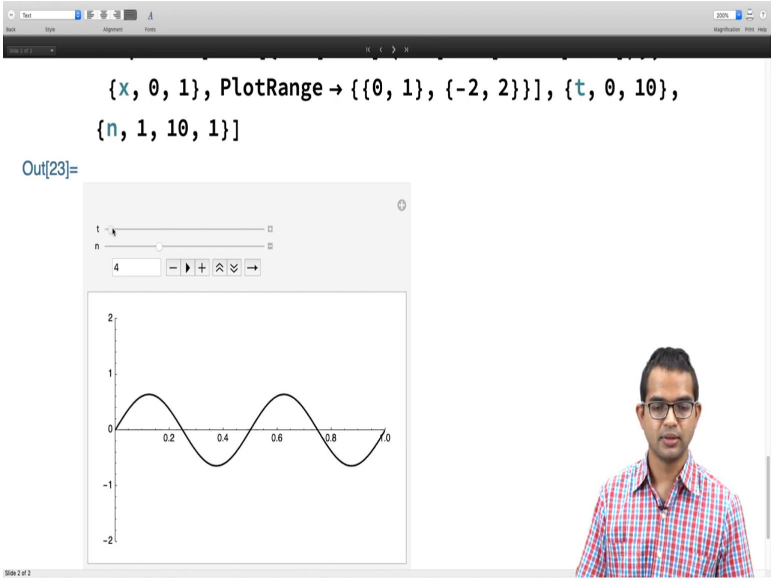

So, you get more and more nodes.

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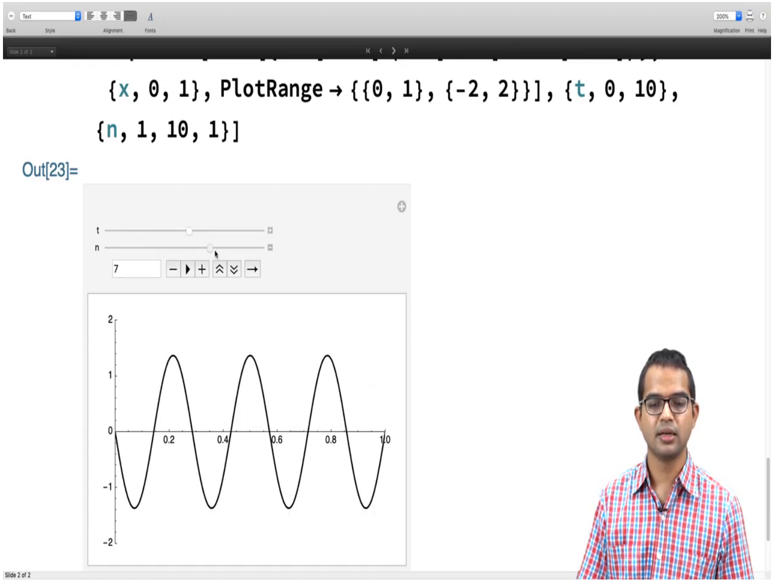

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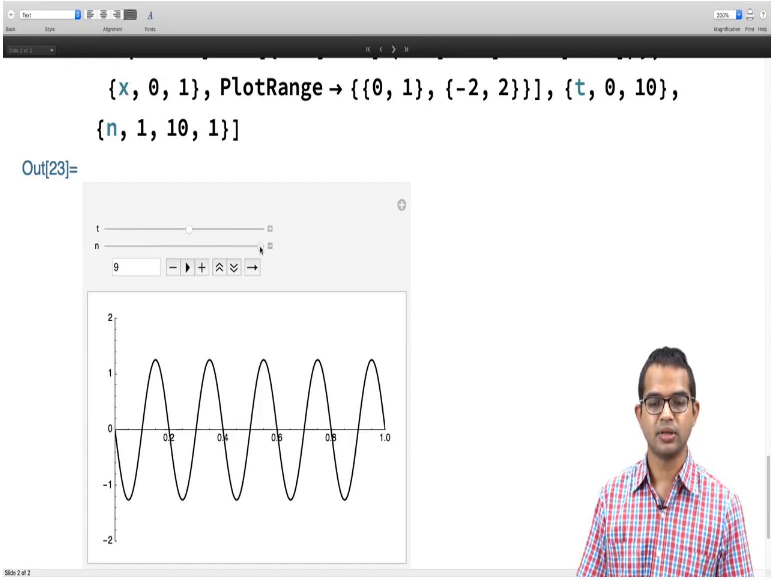

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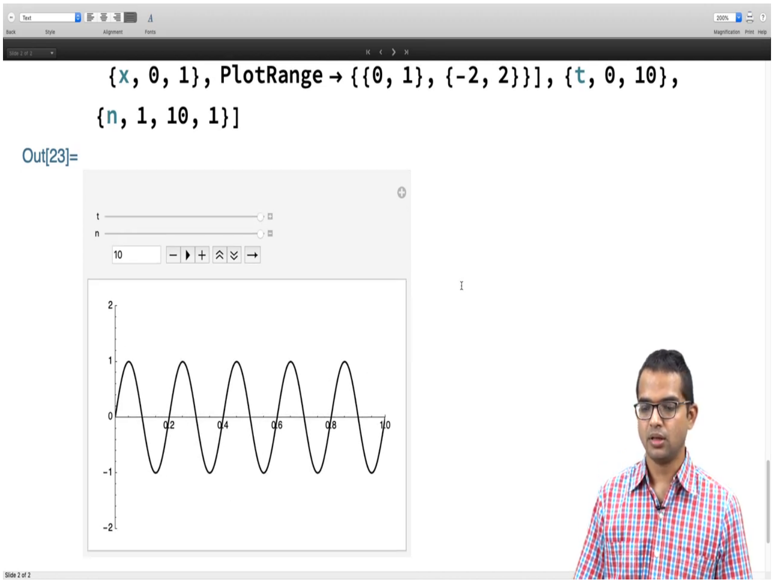

As you increase n, you can check that even for large you can make n very large and you will get a large number of nodes and so, you get these kinds of patterns, ok. Thank you. So, that brings us to an end of this discussion of partial differential equations.

Thank you.