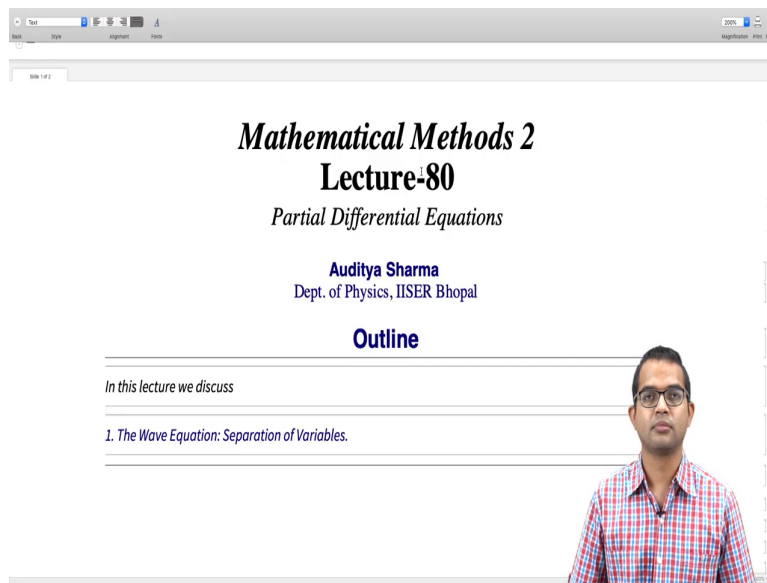


Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Module - 08
Partial Differential Equations
Lecture - 80
The Wave Equation: Separation of Variables

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Mathematical Methods 2
Lecture-80
Partial Differential Equations

Auditya Sharma
Dept. of Physics, IISER Bhopal

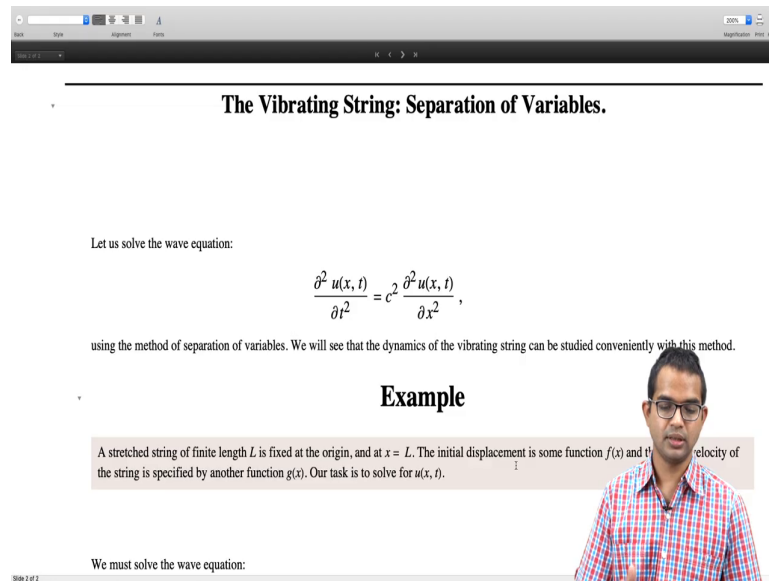
Outline

In this lecture we discuss

1. The Wave Equation: Separation of Variables.

Ok. So we saw how the wave equation can be solved from a very general perspective. So, in this lecture, we will solve the Wave Equation again using Separation of Variables and we will see how in particular this method is suitable for the vibrating string problem, ok.

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The Vibrating String: Separation of Variables.

Let us solve the wave equation:

$$\frac{\partial^2 u(x, t)}{\partial t^2} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2},$$

using the method of separation of variables. We will see that the dynamics of the vibrating string can be studied conveniently with this method.

Example

A stretched string of finite length L is fixed at the origin, and at $x = L$. The initial displacement is some function $f(x)$ and the initial velocity of the string is specified by another function $g(x)$. Our task is to solve for $u(x, t)$.

We must solve the wave equation:

So, the wave equation is like here. So, it is a second derivative with respect to time, second derivative with respect to position and this c squared is here, where c is representative of the speed of the wave right. So, let us solve this problem using the method of separation of variables you know applying it specifically to the problem of the vibrating string.

So, the stretch string like we had before in the earlier lecture is of length L . So, it is fixed at the origin and at x equal to L . So, the initial displacement is some function f of x , right. So, f of x is some function such that it must be 0 at x equal to 0 and it must be 0 at x equal to L and the initial velocity of the string is given by some other function. So, at every point x , there is a certain velocity associated with the motion of the string. So, our task is to solve for u of x comma t right.

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$$\frac{\partial^2 u(x, t)}{\partial t^2} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} u(0, t) &= u(L, t) = 0 \\ u(x, 0) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= g(x) \end{aligned}$$

As usual, we begin with the ansatz:

$$u(x, t) = X(x)T(t).$$

Plugging this into the original PDE and separating variables, we have:

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = k$$

where k is a separation constant. If k is positive, we would get exponentials in both space and time, which would be incompatible with our boundary and initial conditions. $k = 0$ leads to a trivial solution. So we take $k = -\lambda^2$. This leads to two separate ODEs:

$$\frac{d^2 X}{dx^2} = -\lambda^2 X$$

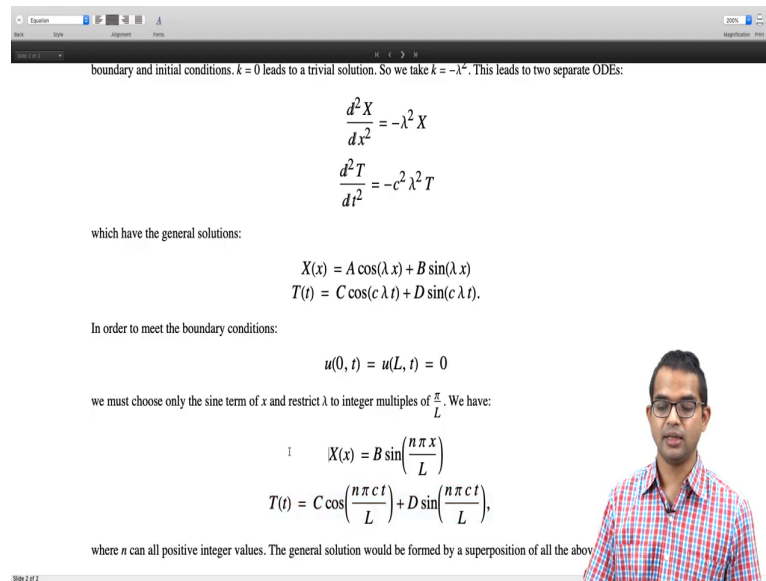
Slide 2 of 2

So, we have this wave function which needs to be solved and it is always helpful to write down explicitly all the boundary conditions and initial conditions. Here the boundary conditions are you know at x equal to 0 and at x equal to L . This function must be 0 and at time t equal to 0, u of x comma 0 is f of x and $\frac{\partial u}{\partial t}$ of x at time t equal to 0 is another function g of x which is some arbitrary function which is specified.

Now, we begin with this ansatz which is to separate these variables u of x comma t as X of x times T of t . When we plug this solution or this ansatz into the differential equation and we separate variables, we have $\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2}$ and since we have you know a function of X must be equal to some other function of t for all X and T , this both of these must be equal to some constant let us call it k .

Now, if k is positive, we would get exponentials in both space and time which would be incompatible with our boundary conditions and k equal to 0 will give you a trivial solution as you can check. So, k is a negative number that is relevant for our boundary conditions. So, we take k to be minus lambda square.

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boundary and initial conditions. $k = 0$ leads to a trivial solution. So we take $k = -\lambda^2$. This leads to two separate ODEs:

$$\frac{d^2 X}{dx^2} = -\lambda^2 X$$
$$\frac{d^2 T}{dt^2} = -c^2 \lambda^2 T$$

which have the general solutions:

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x)$$
$$T(t) = C \cos(c \lambda t) + D \sin(c \lambda t).$$

In order to meet the boundary conditions:

$$u(0, t) = u(L, t) = 0$$

we must choose only the sine term of x and restrict λ to integer multiples of $\frac{\pi}{L}$. We have:

$$X(x) = B \sin\left(\frac{n \pi x}{L}\right)$$
$$T(t) = C \cos\left(\frac{n \pi c t}{L}\right) + D \sin\left(\frac{n \pi c t}{L}\right),$$

where n can all positive integer values. The general solution would be formed by a superposition of all the above.

And this leads to two separate ODEs. So, it is lambda squared and minus c squared lambda squared T. So, this will give us A cos of lambda x plus B sin lambda x for capital X of x and T of t is going to be some other constant times cosine of c lambda t plus D times sin of c lambda t, right. So, these are basically like the classical harmonic oscillator problem, very familiar territory.

So, in order to meet these boundary conditions right when at x equal to 0 and x equal to L, you must have this function going to 0. So, X of x can only have the sine term. You cannot have the cosine term and also this lambda is constrained to be a multiple of pi by L, right. So, that at x equal to L, you must get 0.

So, X of x once you are constrained to take this, it also constrains T of t specifically it forces lambda to be an integral multiple of pi by L. So, we can write X of x like here and T of t is C cosine of n pi c t by L plus D sin of n pi c t by L, right.

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where n can all positive integer values. The general solution would be formed by a superposition of all the above solutions:

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[C_n \cos\left(\frac{n\pi c t}{L}\right) + D_n \sin\left(\frac{n\pi c t}{L}\right) \right].$$

The initial conditions

$$u(x, 0) = f(x)$$

lead to Fourier sine series:

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

The coefficients can be immediately worked out using the Fourier trick:

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Again the condition:

$$\frac{\partial u}{\partial t}(x, 0) = g(x)$$

leads to another Fourier sine series:

So, the general solution would be formed by a superposition of all these solutions, right. You can take this with this and then n can take all positive integer values. So, in fact we should consider a sum like this u of x comma t . So, we do not need to put this constant B here because it is going to get absorbed in this coefficient C_n and D_n need to be determined.

So, the initial conditions we will make use of these initial conditions to determine these coefficients. So, u of x , 0 equal to f of x . So, which basically means this sum right. So, if you put t equal to 0 , this is going to vanish and this is going to just go to 1 .

So, then you are just left with the summation $C_n \sin$ of $n \pi x$ by L is equal to f of x . So, this is a Fourier sine series and we can extract these coefficients using the standard Fourier trick. So, c_n you can immediately write down to be $\frac{2}{L}$ integral from 0 to L f of x \sin of $n \pi x$ by L . And again the velocity condition you know relates this to g of x .

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leads to another Fourier sine series:

$$g(x) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) \frac{n\pi c}{L}.$$

Once again, we solve for the coefficients using the Fourier trick:

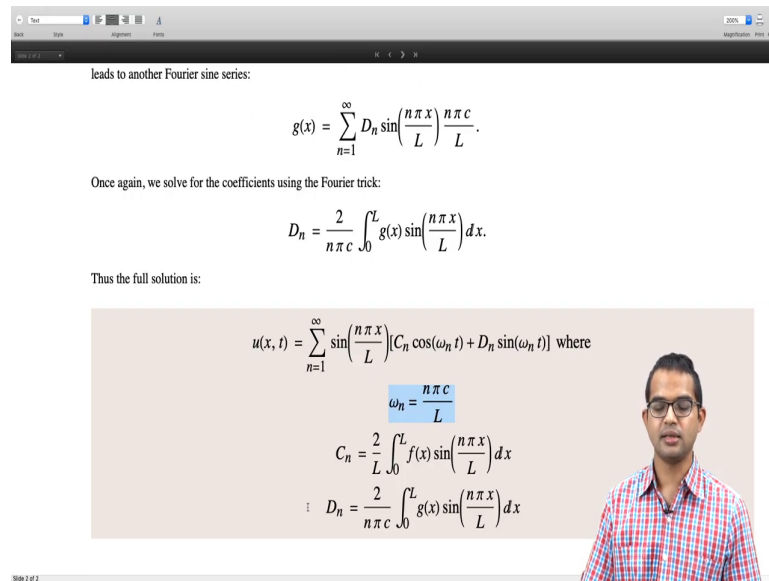
$$D_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Thus the full solution is:

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) [C_n \cos(\omega_n t) + D_n \sin(\omega_n t)] \text{ where}$$

$$\omega_n = \frac{n\pi c}{L}$$

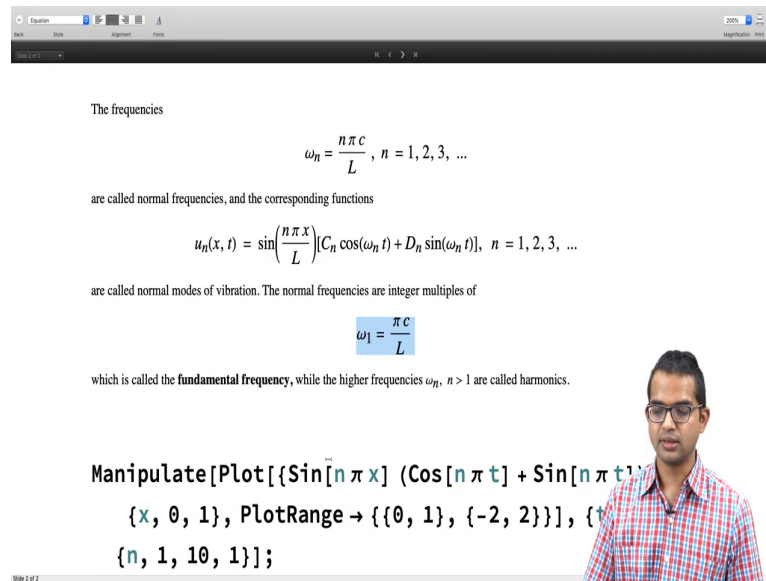
$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$D_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$


And that also will give us another Fourier sine series, right. So, the Fourier sine series where g of x is expanded in this manner, so you have to also take care that you have these other you know stuff here along with D_n which form the coefficient. So, if you take care you will get D_n is equal to 2 divided by $n\pi c$ integral 0 to L of $g(x) \sin$ of $n\pi x$ divided by L dx .

So, the full solution is like here we can write it as you know this in this summation over n \sin of $n\pi x$ by L , then this $C_n \cos$ of $\omega_n t$, it helps to write as $\omega_n t$ plus $D_n \sin$ of $\omega_n t$ where ω_n is an integral multiple of πc by L and C_n and D_n we just worked out.

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The frequencies

$$\omega_n = \frac{n\pi c}{L}, n = 1, 2, 3, \dots$$

are called normal frequencies, and the corresponding functions

$$u_n(x, t) = \sin\left(\frac{n\pi x}{L}\right) [C_n \cos(\omega_n t) + D_n \sin(\omega_n t)], n = 1, 2, 3, \dots$$

are called normal modes of vibration. The normal frequencies are integer multiples of

$$\omega_1 = \frac{\pi c}{L}$$

which is called the **fundamental frequency**, while the higher frequencies $\omega_n, n > 1$ are called harmonics.

`Manipulate[Plot[{Sin[n π x] (Cos[n π t] + Sin[n π t])`
`{x, 0, 1}, PlotRange -> {{0, 1}, {-2, 2}}, {t`
`{n, 1, 10, 1}];`

Now, these frequencies ω_n equal to $n\pi c$ by L are called normal frequencies and the corresponding functions, right. So, this entire function corresponding to an integer n , these are called the normal modes of vibration of your stretched string. Perhaps we have encountered a description of this in an elementary discussion, maybe even going back to high school, but we did not think of this as a solution of a PDE at that point.

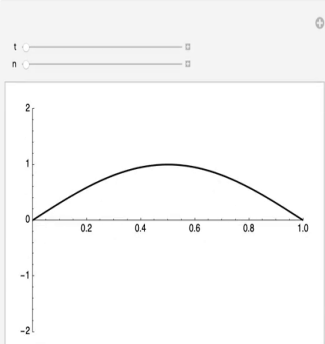
So, now you see the story in a more sophisticated manner and now you see that you know these normal modes which we probably saw pictorially at some point, you know they have this nice mathematical way of arriving at this same result. And these are called the normal modes of vibration.

The normal frequencies are in integer multiples of what is called the Fundamental Frequency and the higher frequencies are referred to as Harmonics. So, you know a description in this language is important when we are studying the physics of you know musical instruments for example.

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Manipulate[Plot[{Sin[n π x] (Cos[n π t] + Sin[n π t])},  
  {x, 0, 1}, PlotRange -> {{0, 1}, {-2, 2}}, {t, 0, 10},  
  {n, 1, 10, 1}]
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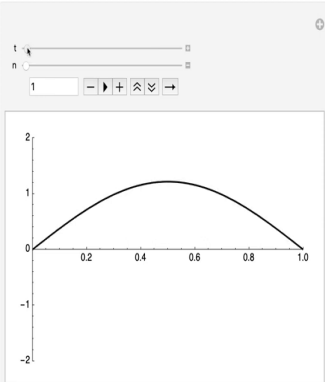
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So, let us quickly plot this. It is instructive to look at what happens to these normal modes as we change n.

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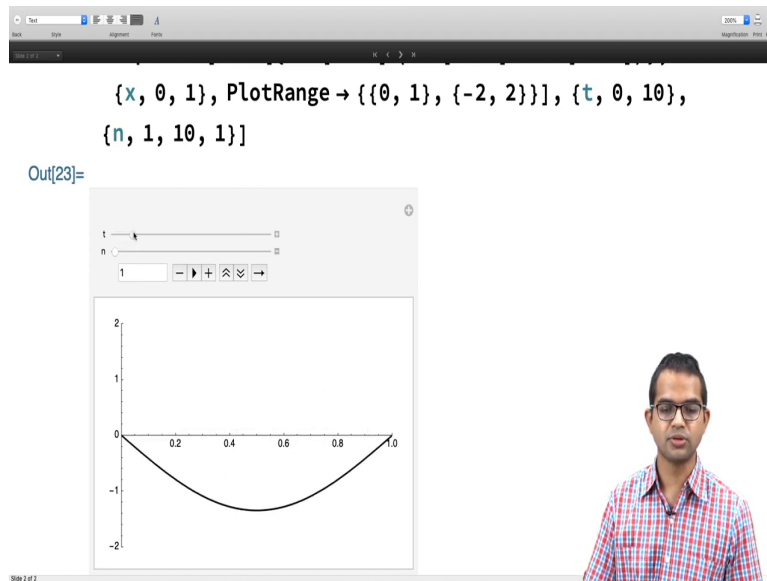
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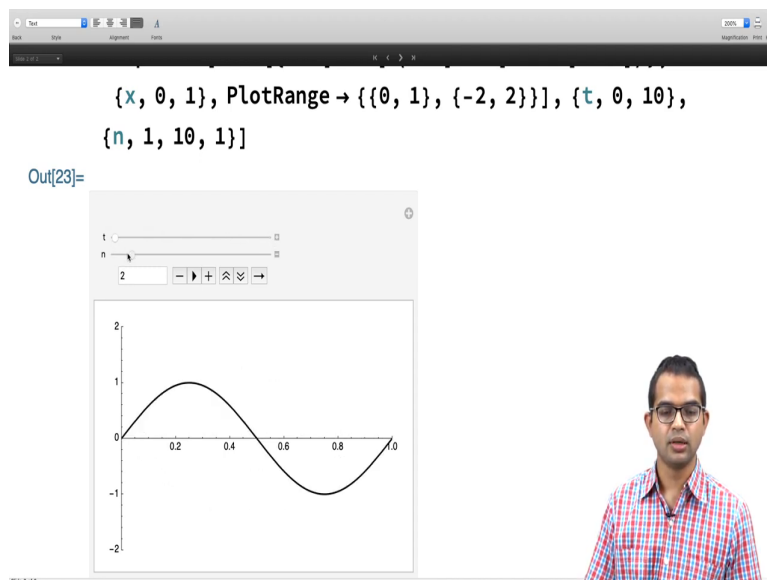
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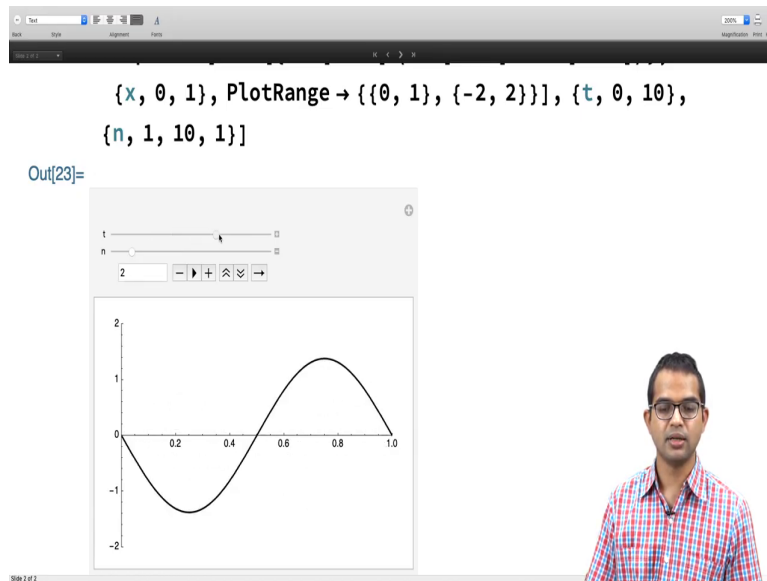
So, when I fix n to be 1 and then look at you know the normal mode as a function of time, so this is what the fundamental frequency does, right. So, this is how an instrument you know produces sound as well. So, the fundamental keeps doing this.

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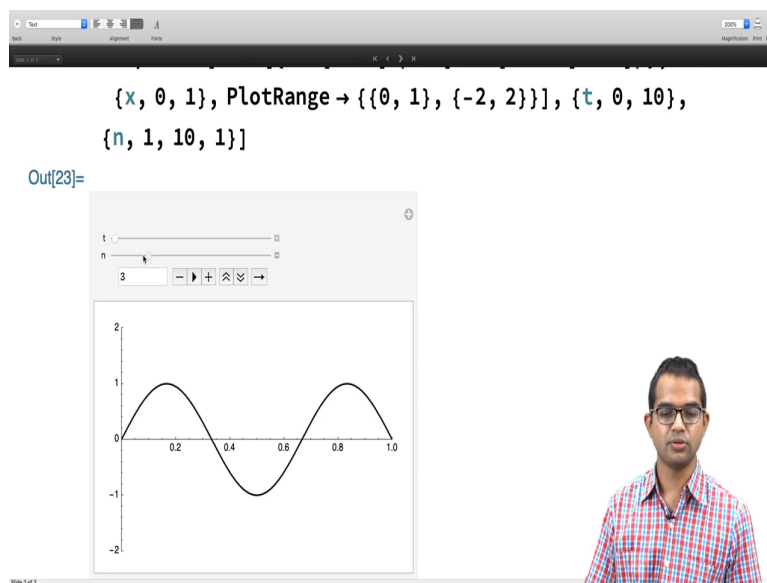
And then if you go to n equal to 2, then you see that it has not only nodes at the ends, but there is also one more node.

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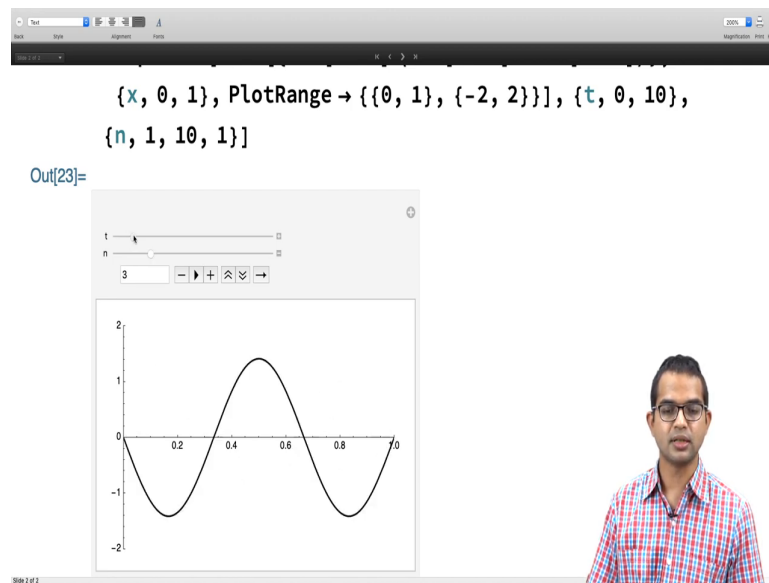


So, as a function of time the node remains unchanged, but the dynamics happens elsewhere and then I make n equal to 3.

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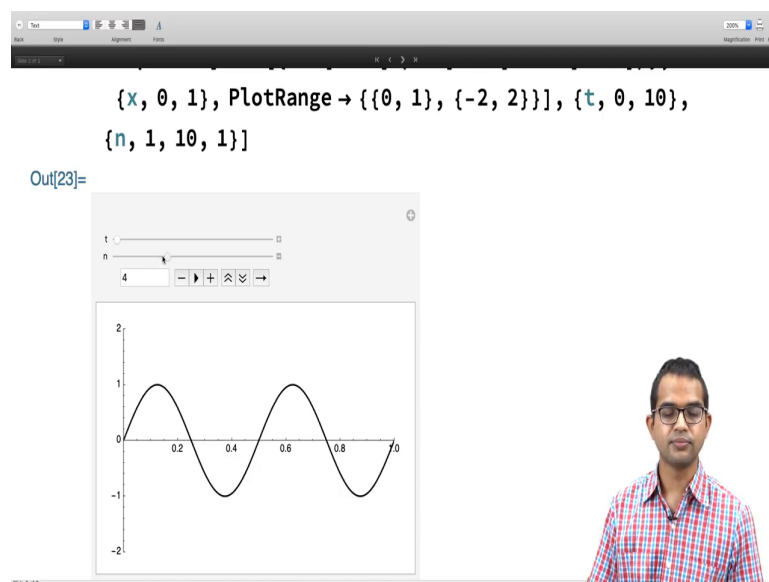


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So, there are going to be two nodes in the center in addition to the edges and so, they keep oscillating. So, it is interesting that we get these what are called standing waves also from the wave equation.

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And which we have seen can be thought of as superpositions of traveling waves, right.

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{x, 0, 1}, PlotRange -> {{0, 1}, {-2, 2}}, {t, 0, 10},  
{n, 1, 10, 1}]
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Out[23]=

The image shows a Mathematica notebook interface. At the top, there is a code cell containing the command: $\{x, 0, 1\}, \text{PlotRange} \rightarrow \{\{0, 1\}, \{-2, 2\}\}, \{t, 0, 10\}, \{n, 1, 10, 1\}$. Below the code, the output is labeled "Out[23]=". The output is a plot of a sine wave. The x-axis ranges from 0 to 1 with major ticks at 0.2, 0.4, 0.6, 0.8, and 1.0. The y-axis ranges from -2 to 2 with major ticks at -2, -1, 0, 1, and 2. The sine wave starts at (0,0), reaches a peak at approximately x=0.25, crosses the x-axis at x=0.5, reaches a trough at approximately x=0.75, and ends at (1,0). There are 4 nodes marked on the x-axis. Above the plot, there are sliders for 't' and 'n', and a numeric input field for 'n' which is set to 4. A man in a red and white checkered shirt is visible in the bottom right corner of the notebook window.

So, you get more and more nodes.

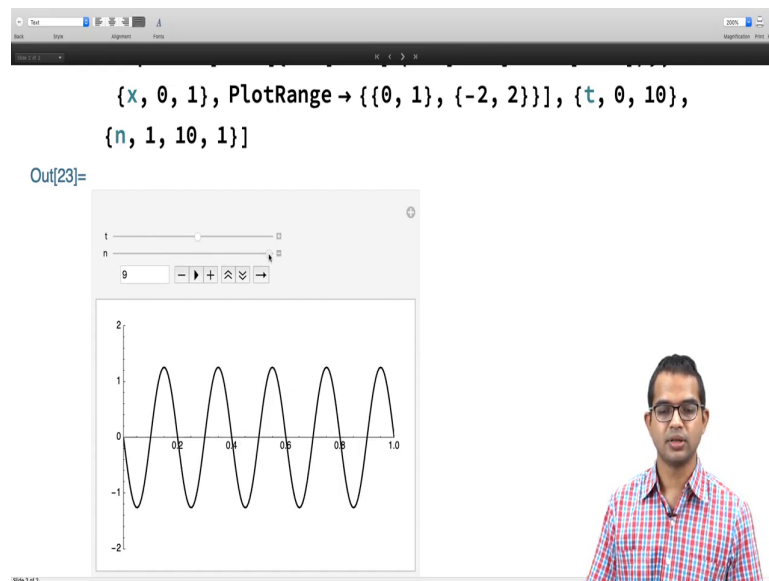
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{n, 1, 10, 1}]
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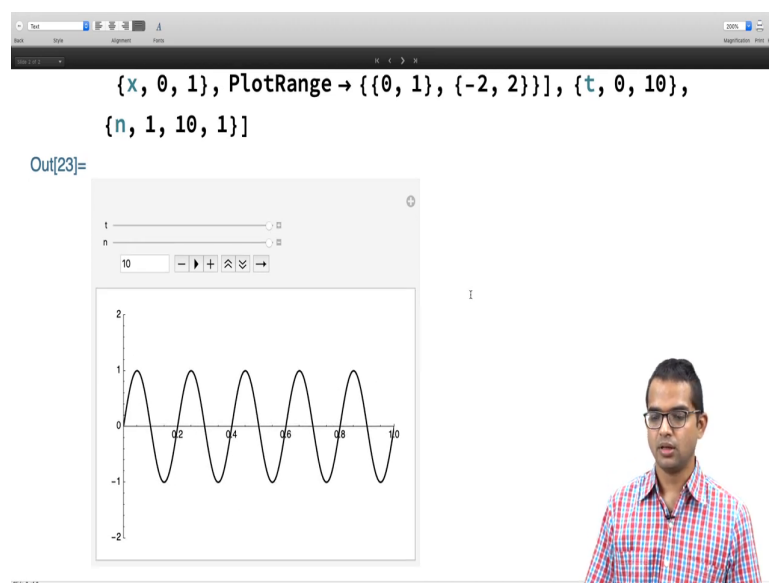
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As you increase n , you can check that even for large you can make n very large and you will get a large number of nodes and so, you get these kinds of patterns, ok. Thank you. So, that brings us to an end of this discussion of partial differential equations.

Thank you.