Mathematical Methods 2

Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Module - 08
Partial Differential Equations
Lecture - 78
The heat equation: illustrative examples

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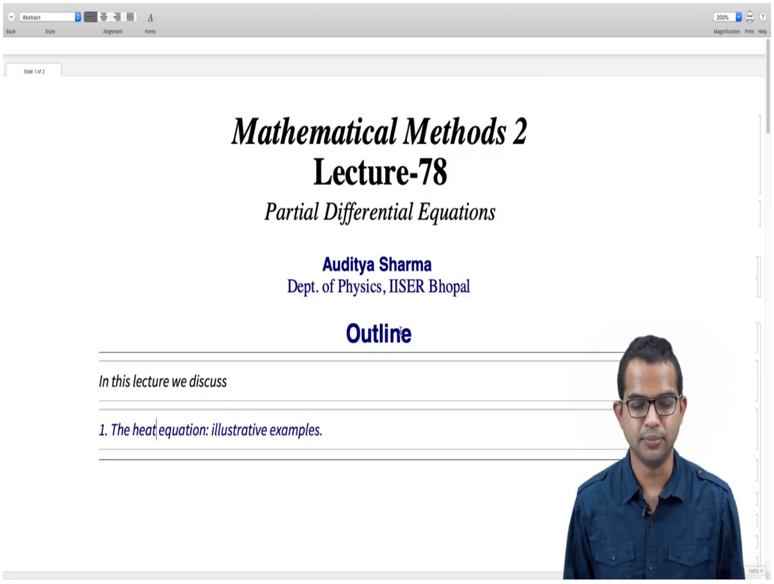

Hello everybody. So, in this lecture, we are going to look at some illustrative examples where we solve the heat equation using the method of separation of variables, but there are some tricks which come out in a nice way.

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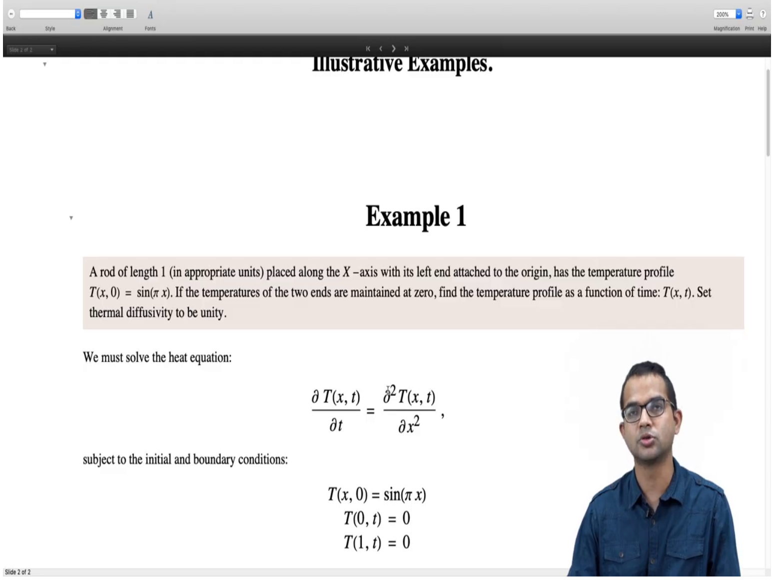

So, consider a rod which is of length l, length 1 in appropriate units, it is placed along the X-axis, the left end is attached to the origin, and the right end is of course at point 1. And let us say that it has this temperature profile sin of pi x at time t equal to 0. Now, if the two ends are maintained at zero temperature, and we allow the temperature to you know evolved as a function of time. So, initially it has this profile, but what happens to its profile as a function of time right? So, that is the question.

So, let us say that the thermal diffusivity is just unity, and we obtain the solution by solving the heat equation right. So, we have the heat equation dou capital T by dou t is equal to dou square capital T by dou x square.

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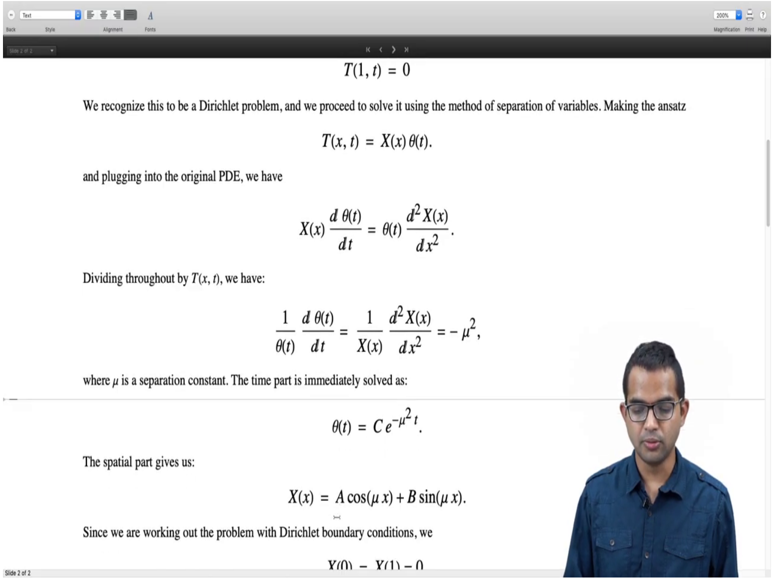

Now, subject to these initial and boundary conditions, the initial condition is that T of x comma 0 is just sin of pi x, and the boundary conditions are at x equal to 0 for all times it must be 0; and at x equal to 1 for all times, it must be 0. So, this is a Dirichlet problem right. And we proceed to solve it by the method of separation of variables: X of x times theta of t, then you plug it into the PDE. We have this differential equation and divide it throughout by capital T.

So, you get 1 by theta times d theta by d t is equal to 1 over X d squared X by dx squared. And both sides must necessarily be a constant which it is convenient to put it to be minus mu squared. So in these kinds of problems typically the temperature is going to decay as a function of time. So, it is appropriate to choose minus mu squared. And so the time part is immediately solved. And you get theta of t is equal C times e to the minus mu squared. All of these are standard steps we have seen.

Now, the special part can have a cosine solution or a sin solution, and arbitrary linear combination is possible in general.

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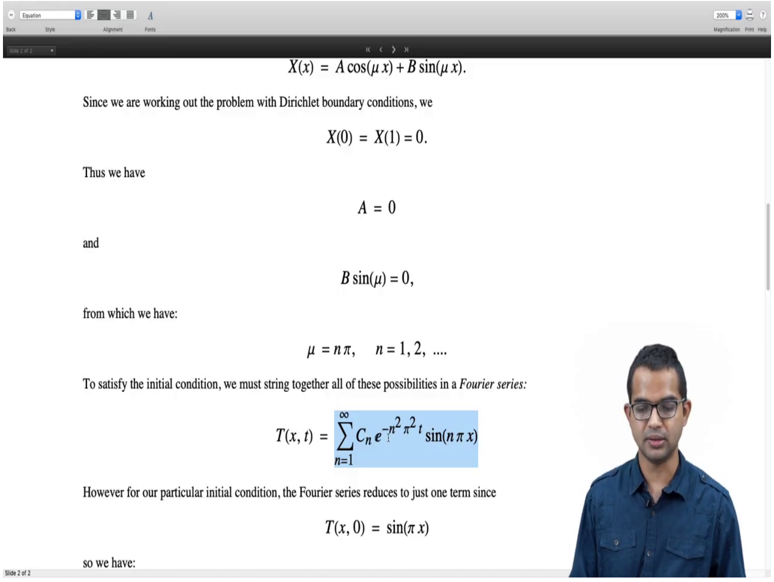

And to fit the boundary conditions here, you must have X of 0 equal to 0. So, X of 0 equal to 0, then A must be 0 right. So, cos will not work here. So, we have to work with sin. And also we have when X equal to 1, this must be equal to 0. So, B times sine of mu must be equal to 0. So, mu must be some integral multiple of pi. So, that is it. So, we have got the initial conditions, we have got the; we have got the boundary conditions. Now, we have to plug in this and put in the initial condition.

So, the full solution in general is going to be given by a Fourier series of this kind T of x comma t is a summation over e to the minus n squared pi squared t sin of n pi x. But in this very special case, we have an initial condition which is particularly convenient. So, the initial condition is t of x comma 0 is equal to sin of pi x. So, in fact, we are not going to get an infinite series, but we will get only one term in the series.

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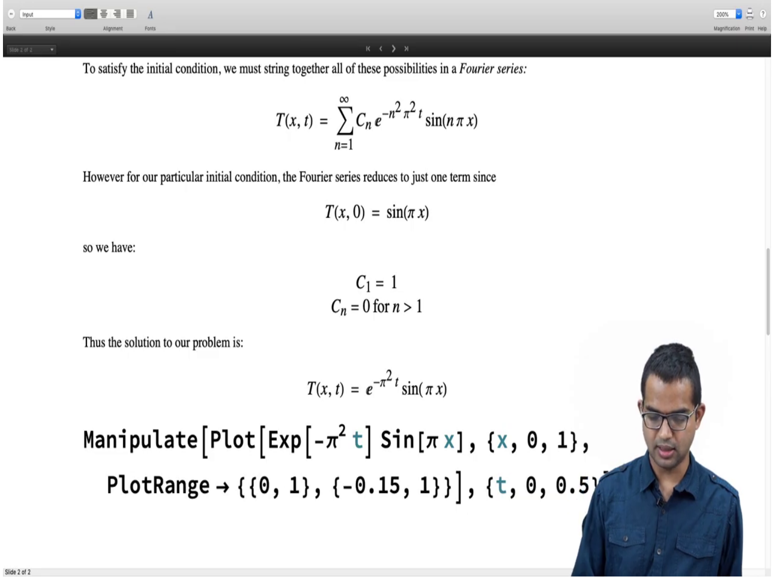

So, we immediately see that, in fact, all we need to do is put C 1 equal to 1, and all other C n's are 0 right. So, you can look at this a little more closely and convince yourself that indeed for this particular problem, it is very simple. And the solution is simply T of x comma t is equal to e to the minus pi square t sin of pi x. So, it is useful to plot this and trace the time evolution of this.

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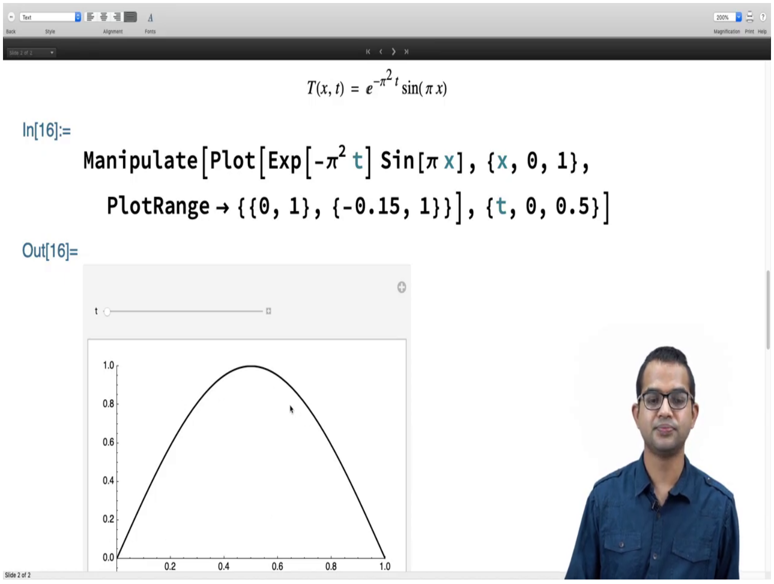

So, initially the profile is like this. So, it has a peak.

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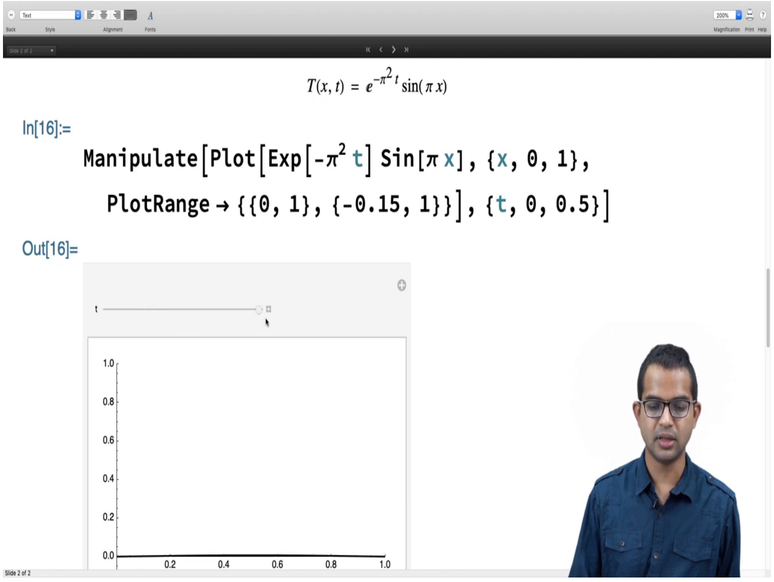

And then as a function of time, it keeps on going down; and eventually it goes to 0 right. So, one reason I included this solution is because if you remember from our first discussion of a heuristic discussion of the heat equation, we said that you know initially suppose there is some kind of curvature, we want to model come up with a PDE which is going to keep on killing this curvature, and eventually take it down to a steady state solution which will be of solution of the Laplace equation and that is exactly what is going on right.

We use this intuition to develop the PDE. And now we are solving this PDE and we are seeing that indeed that is exactly what it is doing right.

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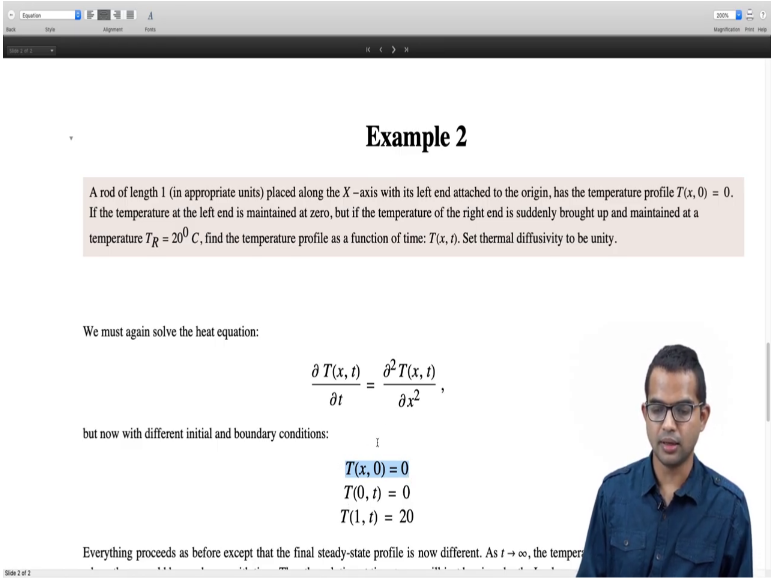

So, we look at one more example where a subtlety shows up which is a very close you know closely related problem to the current problem, but there is an additional subtlety which we will discuss in the next example. So, it is the same kind of a rod of length 1 which is along the X-axis. Now, the temperature profile is just 0. It is initially at 0 degree Celsius everywhere.

And now suppose you leave the left end at 0, but suddenly you heat up the right end and keep it at 20 degree Celsius. So, now, the question is what will happen to the temperature of this rod as a function of time and as a function of space right. Once again set thermal diffusivity to be unity. Now, I mean it is clear that we must solve the heat equation, we are interested in the dynamics.

And now the initial condition is a bit different, and the boundary condition is also different right. So, is a 0 throughout and. So, the initial condition here was sin pi sin of pi x. So, here it is just 0 right. So, there is no special variation at time equal to 0. But when you crank up this right end to 20 degree Celsius.

And so this boundary condition is going to make a difference as we will see. So, the solution itself is a little more complicated, but really it is the same method right. So, everything proceeds as before except that we must account for this final steady state profile right.

So, when let us think about it physically as t t goes to infinity, the temperature would settle down to a profile it will be a solution of the Laplace equation really right. So, in steady state this part is going to be 0. So, no change happens as a function of time that is what is meant by steady state. For when t becomes very large, dou T by dou t will be 0. So, it is going to be just dou square T by dou x square and that is equal to 0 which is the thing but the Laplace equation right.

So, at when t is infinity, if the right end is at 20 degree Celsius and left end is at 0 degree Celsius and if the temperature profile must be a solution of the Laplace equation, the way that can happen is if it is a linear growth from 0 to 20 degree Celsius at x equal to 1.

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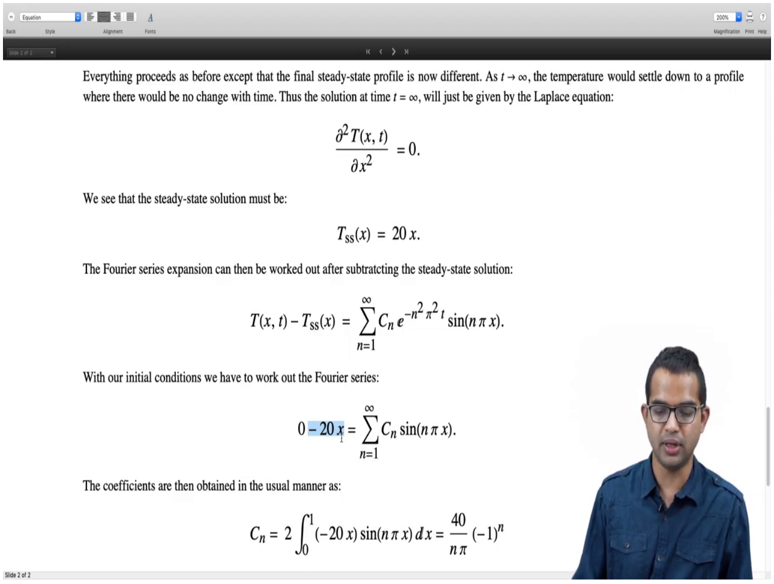

In other words, the steady state solution must be T ss of x is equal to 20 x. So, now, given that, so we know what the temperature profile is at t equal to 0, we know what the temperature profile is at t equal to infinity. So, in fact, the Fourier series must be worked out in this manner. So, you take this t of x comma t and then subtract out the steady state solution, and all the dynamics is really contained after you have subtracted out the steady state solution.

So, it is this quantity which is going to be this Fourier series. And in fact we will plug in the initial condition into this form now. When you do this, you are going to get, so I mean at t equal to 0, it is just 0 everywhere. So, 0 minus 20 x, so the steady state solution we have worked out using our physical intuition. And so you have minus 20 x is equal to this Fourier series.

Now, n goes from 1 to infinity, and now it is not going to be just one term because in fact all terms will leave. You can work out the coefficient: the straightforward exercise and computing coefficient of a Fourier series is just minus 20 x.

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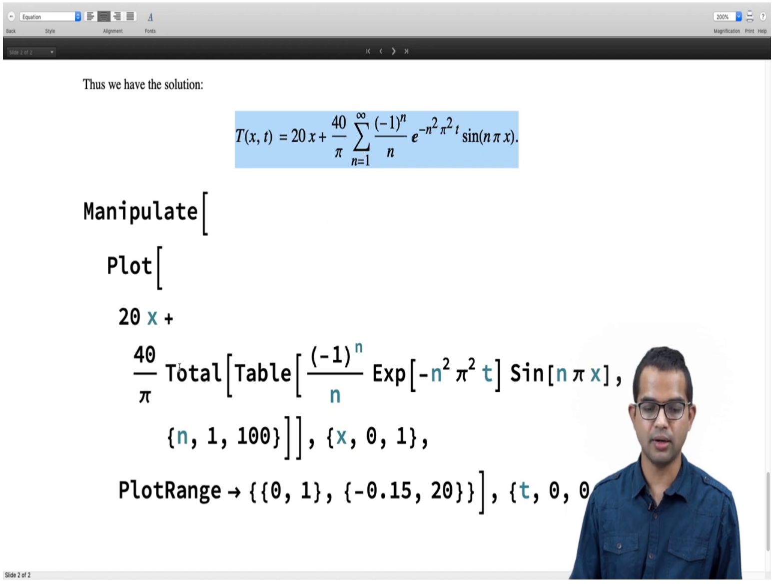

So, multiply by an appropriate sine function and integrate out between the limit 0 and 1, there will be this factor of 2 which comes over when do this integral; you should check this. And the answer is just 40 divided by n pi with you know signs alternating. So, minus plus and so on minus 1 to the power n you can write conveniently like this. So, the solution is this infinite series.

So, T of x comma t will be just 20 x plus 40 by pi summation over n minus 1 to the so it is going to be minus 1 to the n e to the minus n square pi square t divided by you know there is this 1 over this n and also sin of n pi pi x right. So, there is this is the final answer right which we can check by plotting it. I mean I have truncated this series to 100 right. So, n goes from 1 to 100, I have made a table, I have taken the sum, and then let me just plot it for you right. We do not have to go into the syntax of all this.

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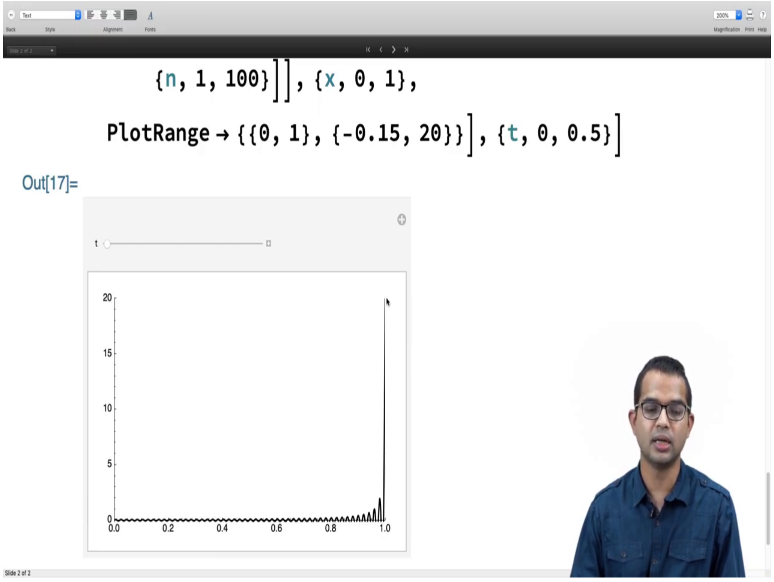

So, you see that at time t equal to 0, you know temperature is 20 only here because I have truncated it at after 100 terms, you see all this fluctuation, but actually it's all of this is exactly 0 this point alone is at 20.

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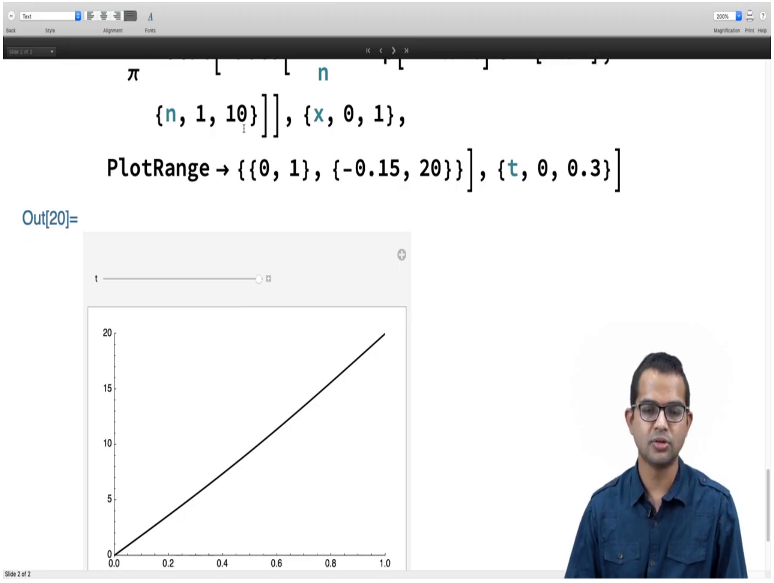

And then as time progresses, you see a smooth variation ok.

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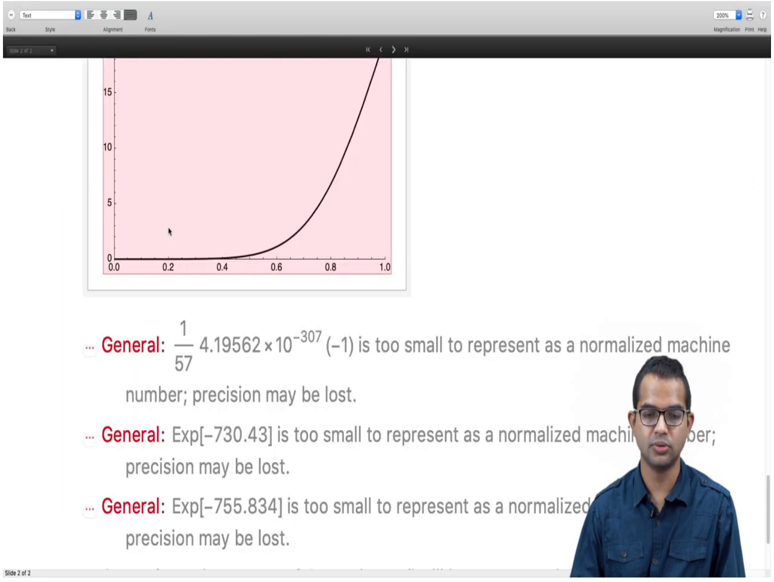

So, there are some difficulties with this plotting. So, we see that I mean. So, the problem is that we are using these exponentials of very very tiny I mean n going to 100 is going to make this a very very tiny number and that is running into difficulties with the precision, but the key point is that even with just 10, if I keep only 10 terms in this series expansion I already see that this is particular only problem is that the representation at time t equal to 0 is not so great, but it is not really a problem we understand this.

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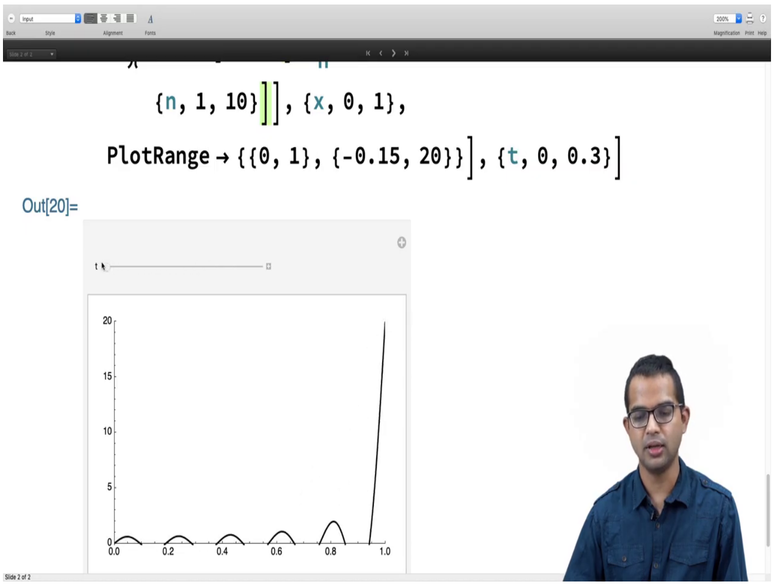

And so if you evolve this, the key point to observe here is this tendency to keep on destroying curvature right, and at some point it is going to become a solution of the Laplace equation, ok, alright.

Thank you, that is all for this lecture.