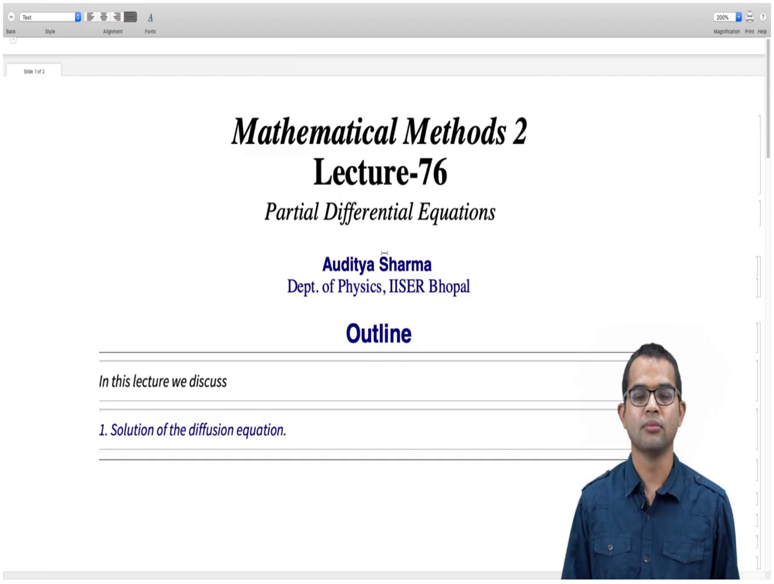
Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

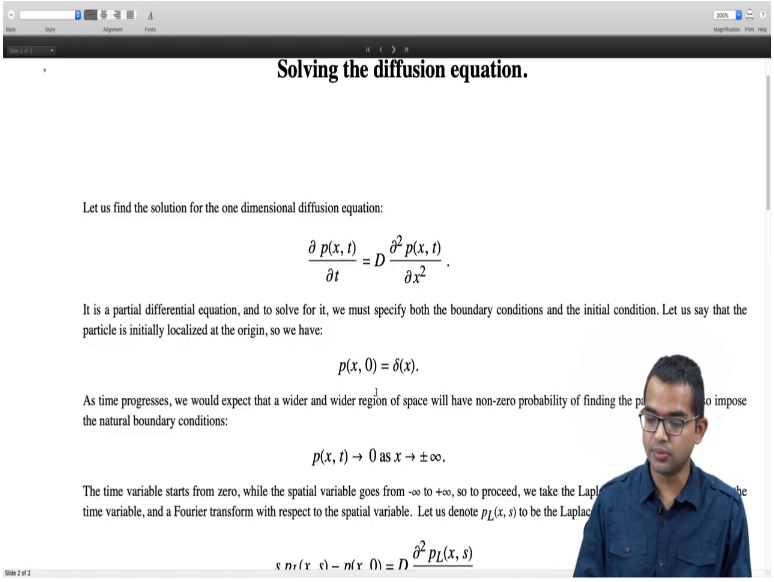
Module - 08
Partial Differential Equations
Lecture - 76
Solution of the diffusion equation

(Refer Slide Time: 00:19)



So, in this lecture, we will solve the diffusion equation in 1D using a very general approach.

(Refer Slide Time: 00:29)



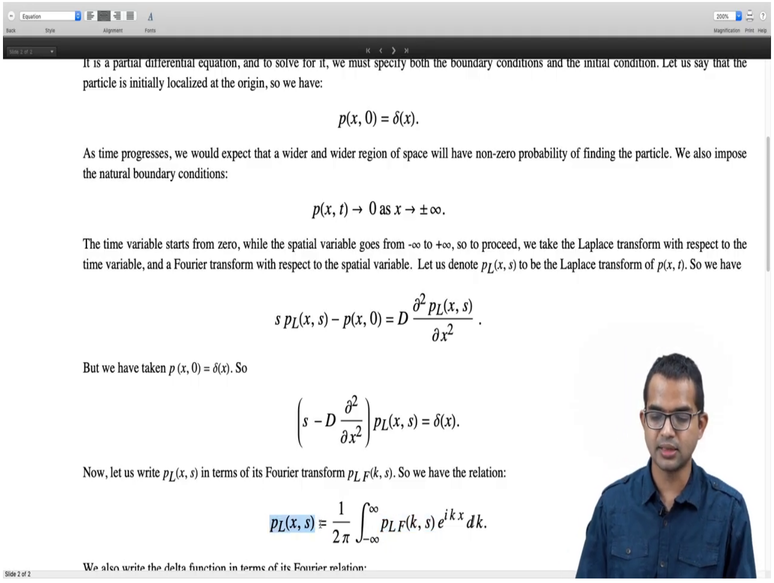
So, the diffusion equation in one dimension is just dou p by dou t is equal to D times dou square t by dou x square right. So, the initial condition that we will choose is such that it will actually allow us to consider other kinds of initial conditions as well. We will come to that later on. But let us say that our particle is localized at the origin at time t equal to 0 right. So, there is no chance for the particle to be found anywhere else at time t equal to 0, it is localized at the origin.

Now, we write this down as p of x comma 0 is equal to delta of x right. So, this is the density, probability density is just given by delta of x. Now, as time progresses, we would expect that a wider and wider region of space is going to be covered right, so there is going to be a non zero probability of finding the particle.

And so we also impose these natural boundary conditions which are very physical right. So, the probability of finding this particle you know goes to 0 as x tends to plus infinity or minus infinity right. So, as you go further and further away from the origin, the probability of finding this particle is going to be vanishingly small.

So, the time variable starts from 0, and the spatial variable also goes from minus goes from minus infinity to plus infinity. So, you solve this problem by taking a Laplace transform with respect to time which goes from where the variable t goes from 0 to infinity, so the Laplace transform is a suitable transform to take here. And on the other hand, with respect to the spatial variable which goes from minus infinity to plus infinity, we will take a Fourier transform right.

(Refer Slide Time: 02:17)



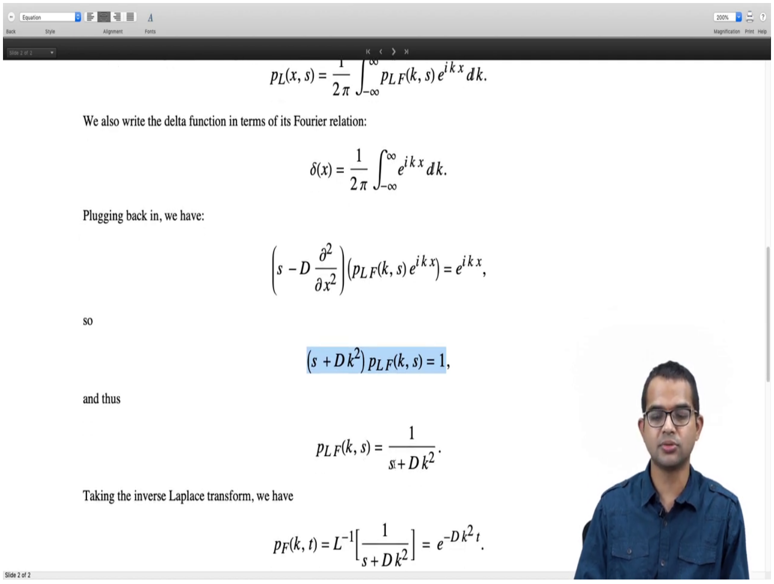
So, let us start with finding the Laplace transform. We will denote this as p L of x comma s instead of from p of x comma t. So, we take a look at this differential equation, and then when you take the Laplace transform. On the left hand side, it becomes s times p L of x comma s minus this p of x comma 0 you know as we know from Laplace transform theory.

And on the right hand side, it is just basically the same; in place of p of x comma t we write it as p L of x comma s right. It is just this function alone which is getting whose Laplace transform is being taken. So, it is just simply replaced by a different function here. And in place of t, we have s here.

Now, I mean p of x comma 0 is of course given to us to be just delta of x right. So, we can write this as s minus D times dou squared by dou x squared p L of x comma s is equal to the initial condition which is just delta of x. Now, we want to take the Fourier transform of you know this function as well. And so it is convenient to actually write p L of x comma s in terms of its Fourier transform.

So, right the relation that we are going to invoke here is actually the inverse Fourier transform. So, p L of x comma s can be written as 1 over 2 pi integral minus infinity to plus infinity p L F of k comma s so the e to the i k x d k right. So, p L of F is it has both where it is the function where both the Fourier and the Laplace transform has been carried out. So, p L of x comma s can be replaced by this expression.

(Refer Slide Time: 04:02)



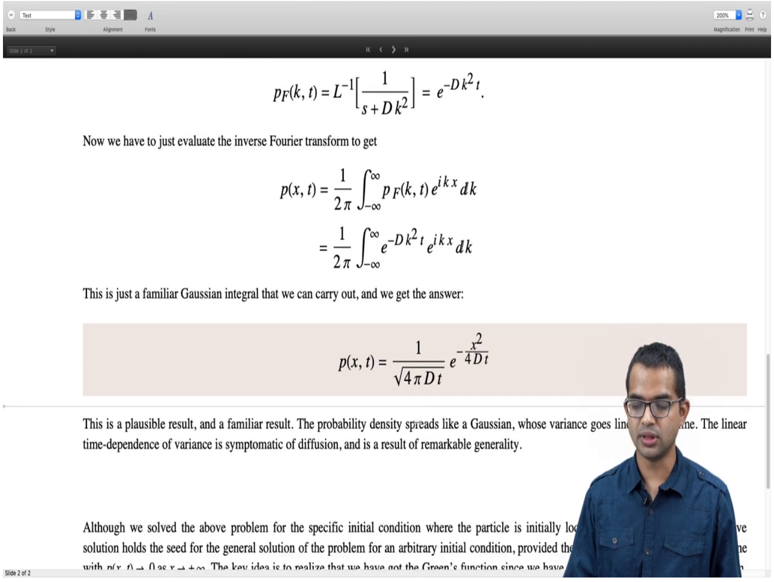
So, and then we also use this integral representation for the delta function which is simply given by this integral right. So, if you plug this stuff in, so we have on the left hand side s minus D times dou squared by dou x squared p L F of k comma s, then e to the i k x is the same as e to the i k x right. So, we have compared the integrands on both sides right.

So, this stuff has this integral, and then this stuff also has an integral. So, we can go ahead and write s plus D k squared p L F k s is equal to 1 right. So, this is a simplification which ensues, and then we can go ahead and solve for p L of a p L F of k comma s right. So, this is immediately seen to be 1 over s plus D times k square right.

So, all we have done is some simple jugglery involving you know taking a Fourier transform and a Laplace transform, and then simply arguing that the integrands are the same. And then some simplification and we immediately have s plus I mean you take this derivative, so you get a k squared which comes in and so s plus D k square times this is equal to you know the e to the i k x can be removed, and it is this quantity is seen to be 1. So, p L F of k comma s is equal to 1 over s plus D k squared.

Now, if you take the inverse Laplace transform of this function which is straightforward to do, we get e to the minus D k square t.

(Refer Slide Time: 05:51)



And then we have to take the inverse Fourier transform of this function to get back p of x comma t. If you do this, then you have this integral to be performed 1 over 2 pi integral from minus infinity to plus infinity e to the minus D k square t e to the i k x dk which returns to us this familiar Gaussian integral. So, this is a Gaussian integral, and then you get a Gaussian as the final answer.

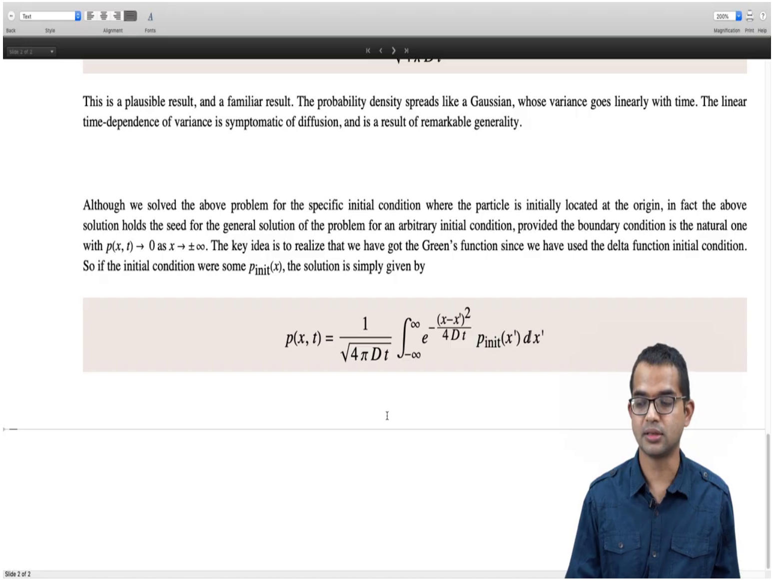
So, this p of x comma t is equal to 1 over square root of 4 pi dt and e to the minus x squared by 4 dt. So, this is a plausible result and something which is familiar right. So, we saw the discrete version which had this Gaussian appearance, and indeed the general solution also is Gaussian in nature right.

So, initially our particle was localized at the origin, and then we see that the probability density keeps on spreading. And in fact a key point to notice here is the width which is given by this variance. Variance goes as is linear in t, it is which is very its characteristic of diffusive motion right. So, what is called ballistic motion is characterized by you know the spread being proportional to time, whereas here it is the variance which is proportional to time right.

So, in diffusive motion, if you if a walker takes n steps you know he is typically covered and steps, whereas in ballistic motion it covers steps which is proportional to. And whereas in diffusive motion, n steps will probably give him only square root n of order of square root n, so that is symptomatic of diffusion; it holds in different dimensions, it holds in the continuous version, the discrete version and so on.

Now, we have solved this problem with some very specific kind of initial conditions, but this initial condition is such that it is amenable to generalization.

(Refer Slide Time: 07:58)



So, in fact, if we pause and think a little bit what we have found is really what a Green's function right. So, we worked out the solution for the delta function. So, if you have some other initial condition, if your initial probability density is some p in it of x, so we can just use the Green's function prescription and immediately write down a formal expression for the full solution p of x comma t is going to be 1 over this you know this stuff this constant from minus infinity to plus infinity.

Now, you have to shift this e to the minus x minus x prime of the whole square divided by 4 dt times this right. So, this comes about just from Green's function theory which we are familiar with. So, the advantage with the initial condition we have considered here is that we have worked out the full solution here.

And therefore, it automatically leads us to a more general solution assuming these natural boundary conditions, which is that this probability must die down to 0 at plus x equal to plus and minus infinity. We look at some other interesting boundary conditions. And we will solve the same PDE by a different method in the lectures coming up, but that is all for this lecture. Thank you.