Mathematical Methods 2
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Module - 08
Partial Differential Equations
Lecture - 74
The heat equation: a heuristic discussion

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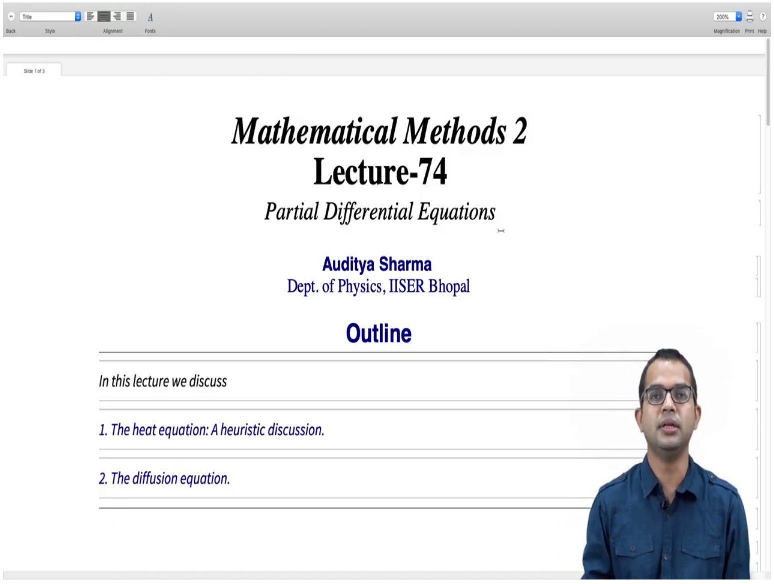

Starting with this lecture, we are going to discuss a different kind of PDE namely the heat equation or equivalently the diffusion equation. So, in this lecture, we will motivate the heat equation with the aid of a heuristic discussion. And we will also look at how the diffusion equation comes about, and how essentially the two are really the same ok.

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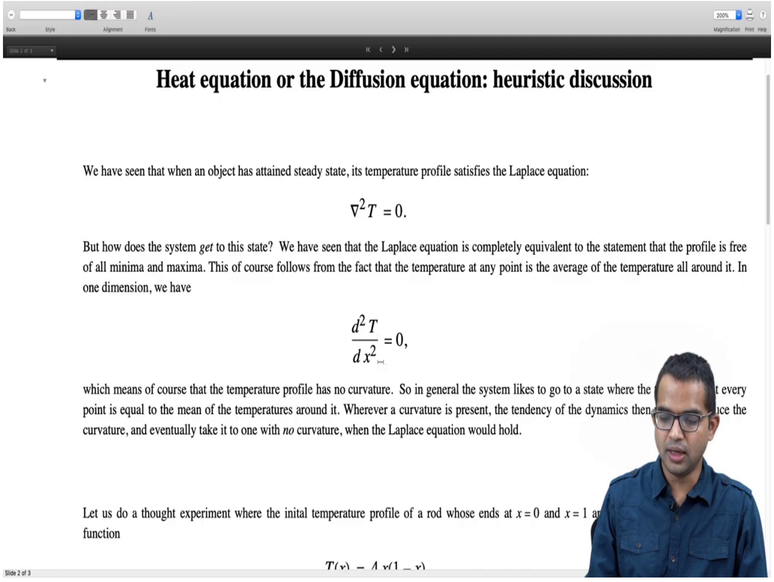

So, we have seen that the Laplace equation is del square T equal to 0 when a heating rod for example has reached steady state, then this is the differential equation which it obeys from which you can compute its temperature profile for instance. But how does this system get to such a steady state right?

So, what is the dynamics which would drive your system towards the Laplace equation right? So, in order to answer this question, we can recollect how the Laplace equation is essentially the same as saying that there are no maxima or minima in the profile right. So, which comes about from the fact that you know a solution to the Laplace equation is such that the value of the function at any point is equal to the average of the values that the function takes in its neighborhood right.

So, if you know if any point is a maximum or a minimum a local minimum or a local maximum, then for sure you cannot get the value of that function by taking the mean of all neighboring points right. If it is a maximum, then the mean is necessarily going to be less than that value. And likewise you know if you are at a minimum it is going to be the mean is going to be greater than that right.

So, therefore, the Laplace equation basically roots out all these you know all curvature essentially right. So, in 1D of course, this is obvious because it is just d squared T by dx squared equal to 0 which means really literally it means that there is no curvature right. So, the basic property of the Laplace equation is that it tends to root out curvature right.

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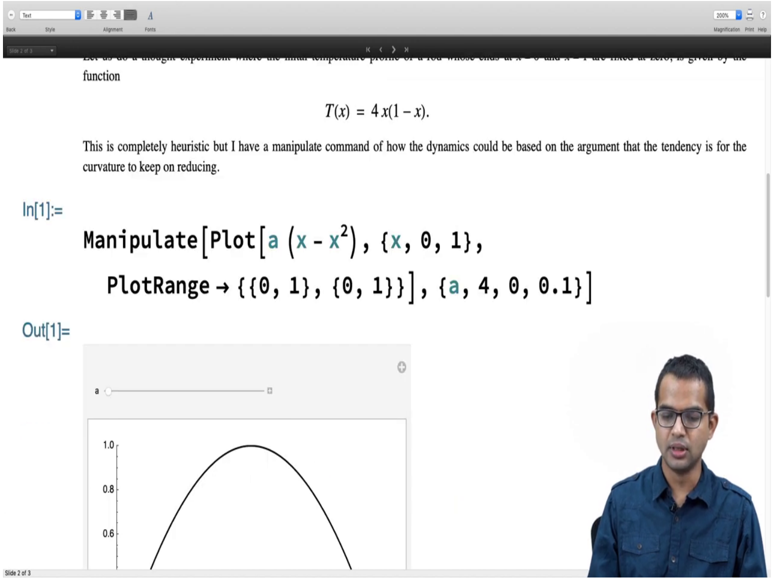

So, we can do a thought experiment I mean in a heuristic way about what would happen if suppose your system is not in such an equilibrium right. So, the tendency will be to go towards this kind of a steady state. And so in order to do this, let us look at the temperature profile of some rod right.

So, I have fixed the ends x equal to 0 and x equal to 1 of this rod. And suppose this rod has this temperature profile T of x is equal to 4 times x times 1 minus x I just made up some function right. And let me plot this function and in fact, I have in this plot I have made it T of x is equal to a times x times 1 minus x I am plotting this, and I allow this to take different values right.

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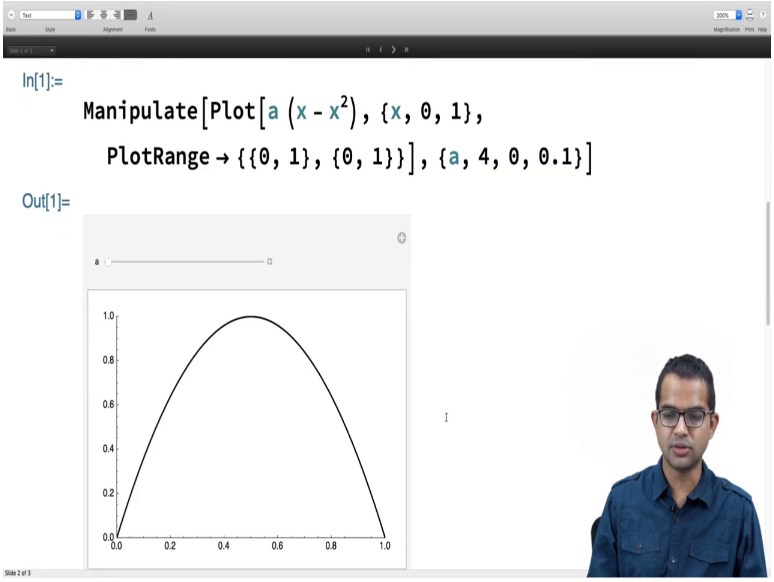

So, I am imagining a scenario where I mean I will start with this profile. But as you know time evolves right, so we see that I mean this end and this end they are pegged at this temperature 0 right. If you want to move towards this function becoming a solution of the Laplace equation basically then we will want every one of these points to tend towards the average of its neighboring point. So, let us look at this top point.

So, there is a maximum. So, clearly this function cannot be a solution of the Laplace equation. So, if you go to this point, then if you replace the value of this function here by the mean of its neighboring points, both of them are slightly lower. So, its value is going to tend to decrease. So, and likewise the value here is going to try to decrease and so on. So, in general there will be a tendency for this to keep on decreasing right.

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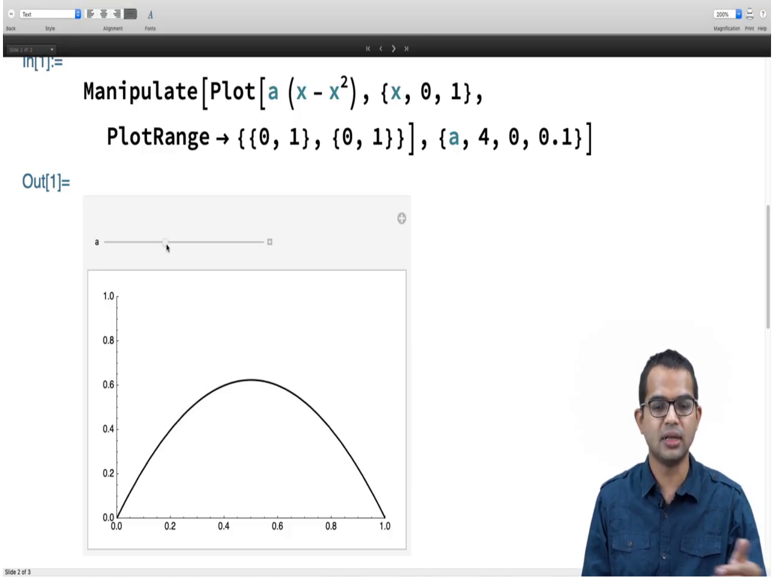

So, this is some kind of cooked up dynamics I made right, I do not, I am not claiming that this is precisely the rate and all these things are not being taken care of.

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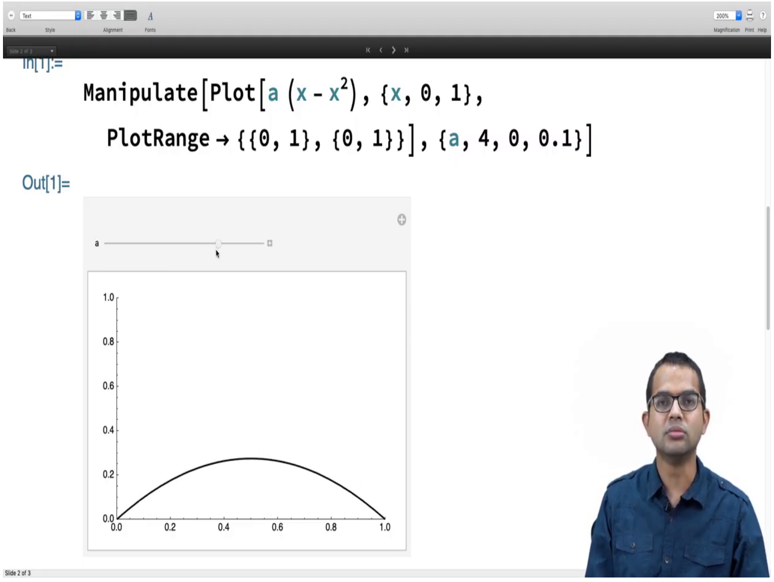

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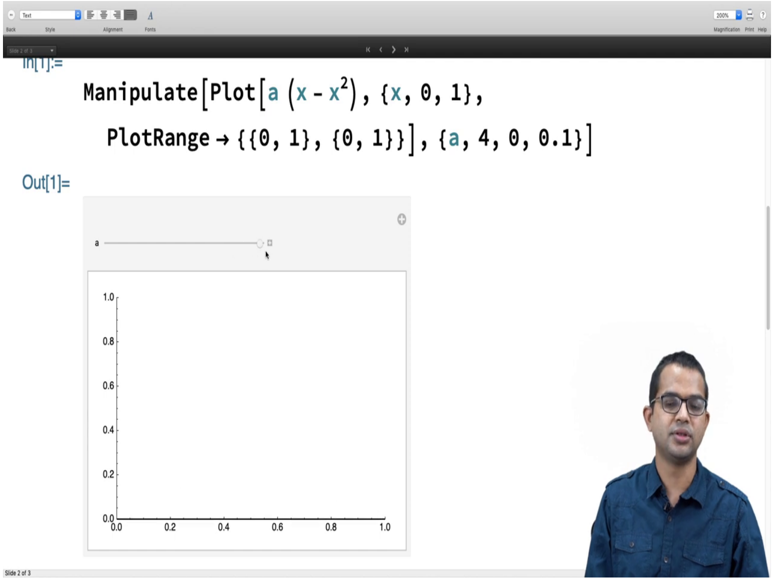

But qualitatively so the tendency will be for it to root out this curvature and eventually it is going to go to steady state. And in this case it is kind of a trivial steady state where T of x is just 0 right.

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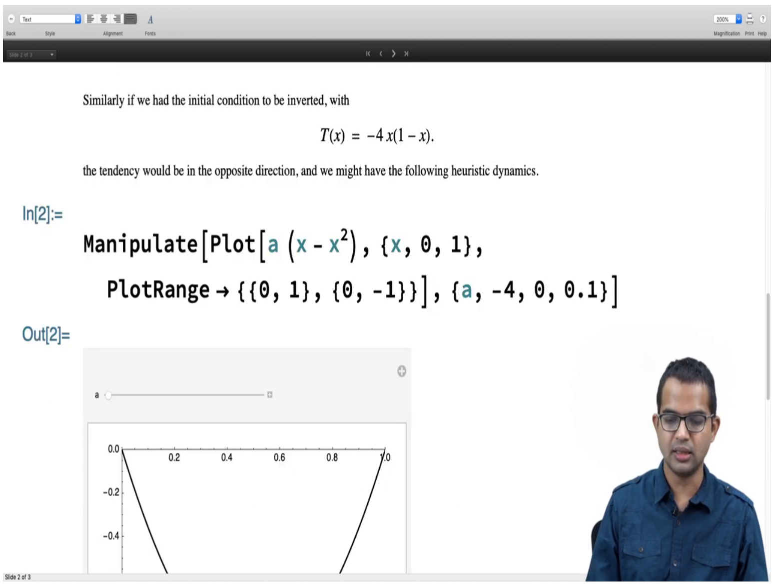

So, if this something similar would happen if we had inverted the initial conditions, if you had started with minus 4 x times 1 minus x, and then it might be something like this.

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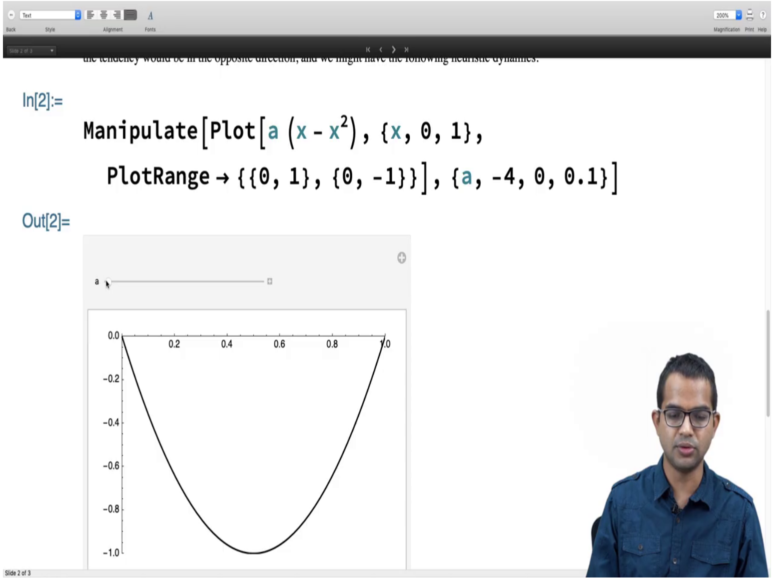

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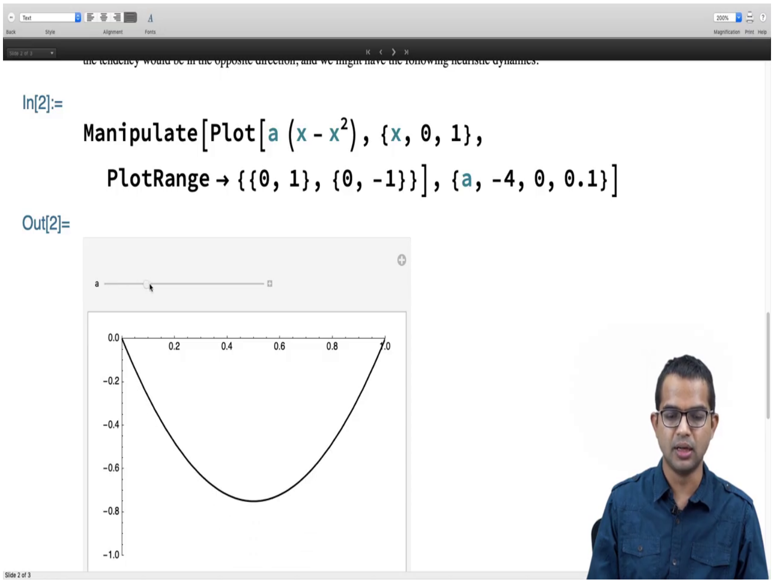

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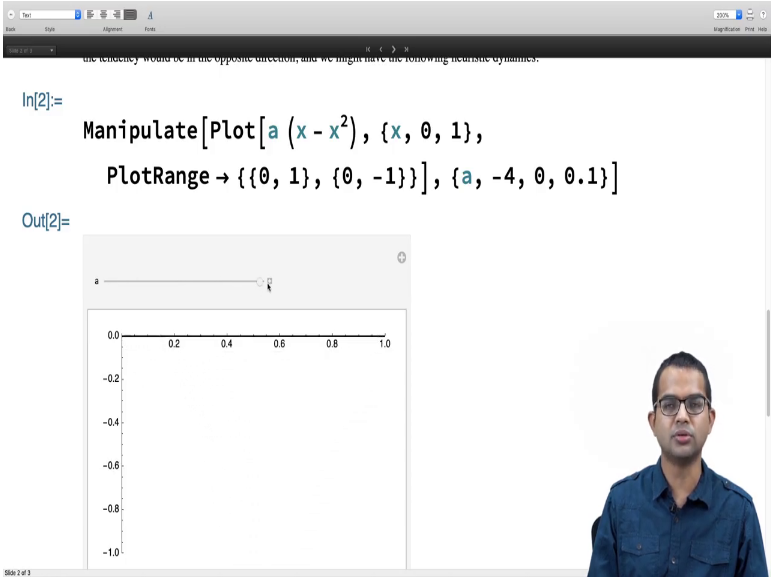

The time evolution might take it in the other direction, and eventually it is going to go to 0.

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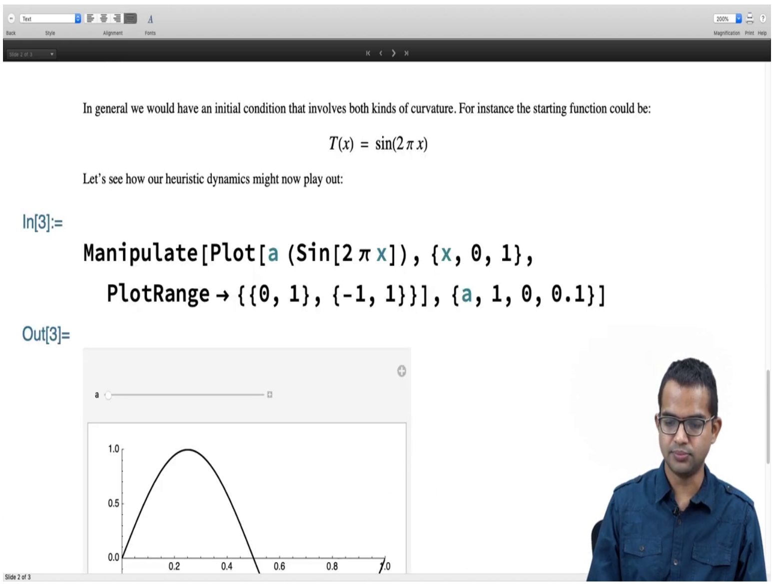

Yeah, just to complete discussion I also have another initial condition where I cooked up this function where you have both kinds of features.

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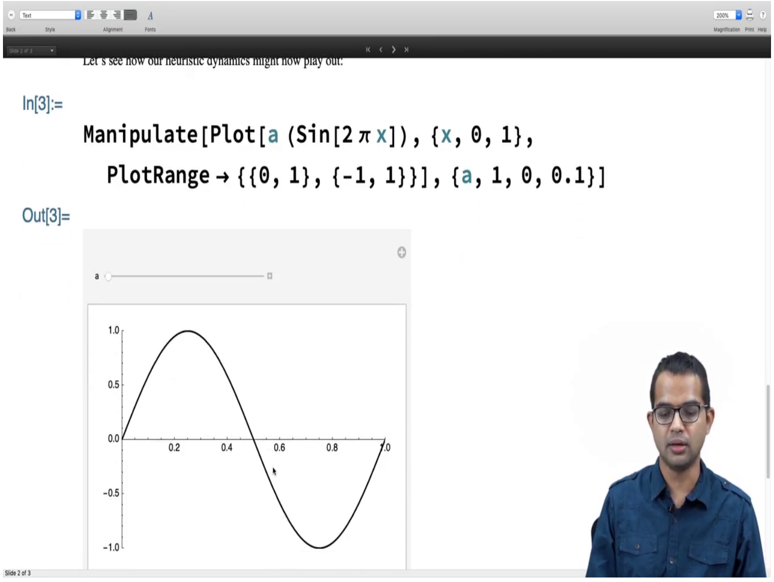

So, there is a; there is a peak and a you know top positive peak and a negative peak. And so the tendency will be to root out you know curvature of every kind. So, as time progresses, we might end up with a situation like this right.

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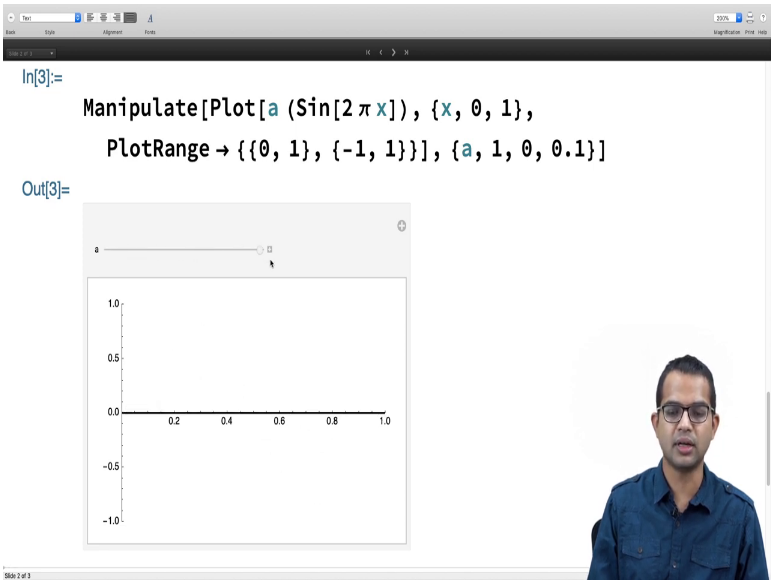

So, and in general, there is no need that you know the ends have to be at 0 you might have a more complicated scenario like this.

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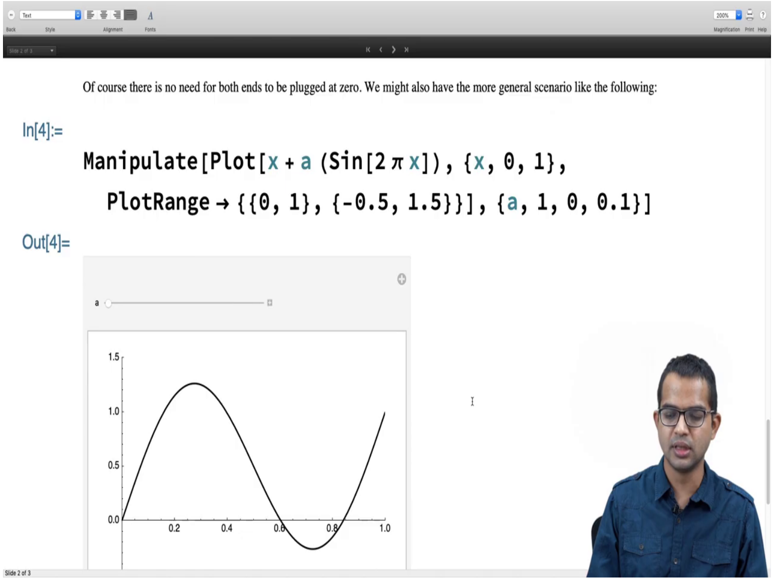

Where the right hand is pegged at some other temperature.

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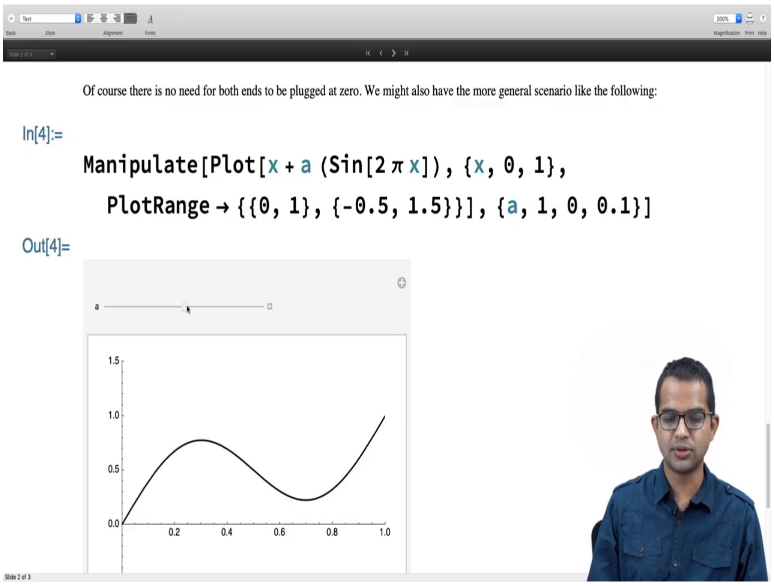

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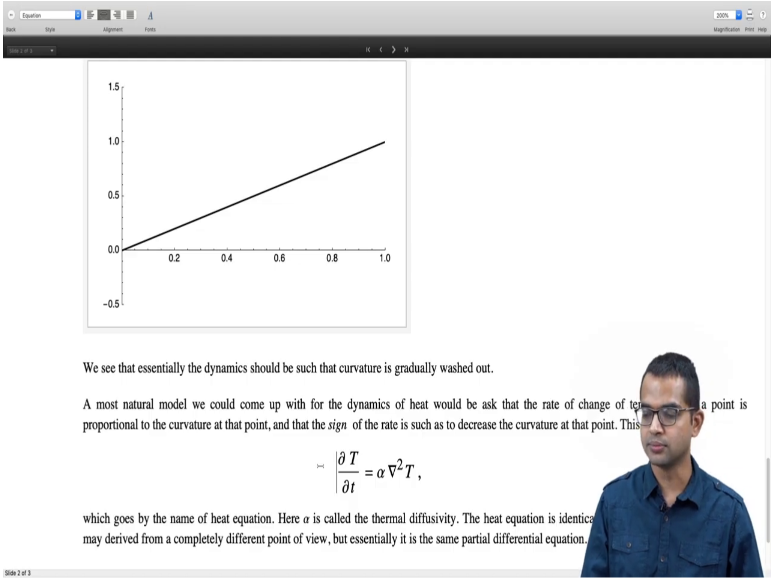

And so eventually some dynamics you would expect it to do this, basically it is going to root out all this curvature right. So, what is the simplest way to model such you know destruction of curvature and so that is really that it will get us to the heat equation?

So, a natural model to come up with is to say that whenever there is curvature you know the temperature is going to change by an amount which is proportional to the curvature itself, and in a direction such that the curvature is you know reduced and eventually destroyed right.

So, that is given by just this very simple equation dou T by dou t, so that is like the change in the temperature it should is proportional to the curvature del square T Laplacian of this field. And so there is this constant alpha which is a positive constant. So, you can check that you know this equation will with a positive value of alpha is going to keep on decreasing the curvature.

And alpha goes by the name of thermal diffusivity. So, this heat equation, this is the heat equation. And so we have argued in one d, but essentially the idea is the same even in higher dimensions. So, the tendency will be to root out curvature right. So, we will see now that in fact the heat equation is the same as what is also called the diffusion equation. And the diffusion equation is derived using the following arguments right.

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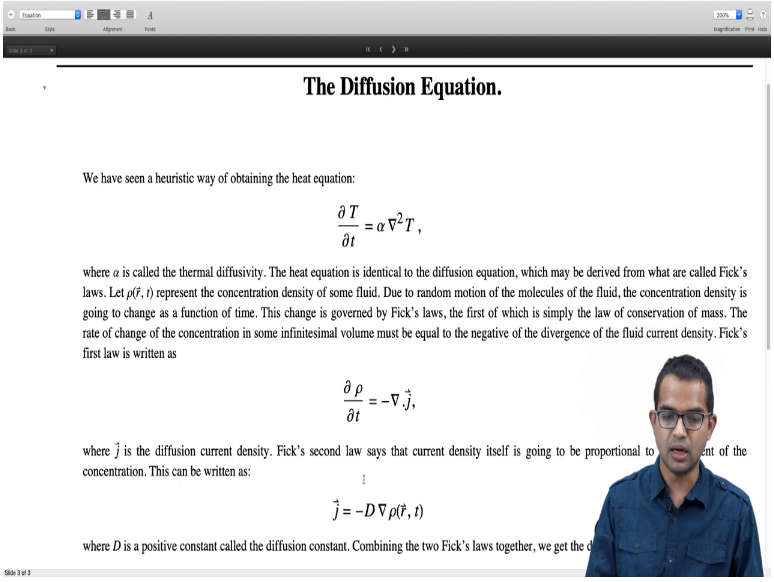

So, let us look at what are called Fick's laws. So, these empirical laws which presumably were discovered by Fick. These are a couple of laws which are really you know statements of conservation right. So, let us say that you are considering a concentration density of some, so there is a fluid which is concentrated in some region right, it could be a perfume which is you know which is initially localized in some region, and then you might ask how does this perfume spread as a function of time.

So, concentration density of rho of r comma t, and due to random jiggling motion of the molecules you would expect that there would be a tendency for this to keep on expanding right. So, for the purpose of this discussion, we assume that the only dynamics comes from this kind of random motion of the particles. You know we ignore gravity and other kinds of phenomena like convection and so on.

Ignoring all of that, and thinking of this as purely a result of random motion of molecules, we have these two laws right which are called Fick's laws. So, first law first Fick's law is really a statement of conservation of mass right. So, it says that if you consider some you know some infinitesimal region and find out what is the rate of change of the concentration density in that region the dou rho by dou T must be equal to the negative of the divergence of the current right.

So, the current is one way of keeping track of how many particles are exiting some small region, and then this concentration density is another way of keeping track of it. And so both of these you know there is a bookkeeping exercise which says that there is no loss of there is no destruction of mass right. So, therefore, dou rho by dou T must be equal to minus divergence of the current density.

And Fick's second law says, so this is sort of like the you know assumption we made when in our heuristic discussion of while deriving the heat equation which is that you know the curvature there is this tendency to destroy curvature, and we just took it to be proportional rate of change to be just proportional to del squared t.

So, something like that is being said here which is that the current density itself right. I mean, we can imagine how particles which are concentrated in a region will tend to go from a region of higher concentration to a region of lower concentration. So, the direction of current is of course from a region of higher concentration to a region of lower concentration.

But so the assumption here which is within the realm of what is called linear response theory which is that this current is actually just proportional to this gradient right, the sign is fixed in such a manner that you know particles move from a region of higher concentration to a lower concentration.

So, if you invoke this argument, then you have this current is given by minus some constant times the gradient of this concentration density of this fluid. And so this D has the name of it goes by the name of diffusion constant. If you combine these two Fick's laws you know you plug in place of the current density j you plug in this minus D times gradient of rho.

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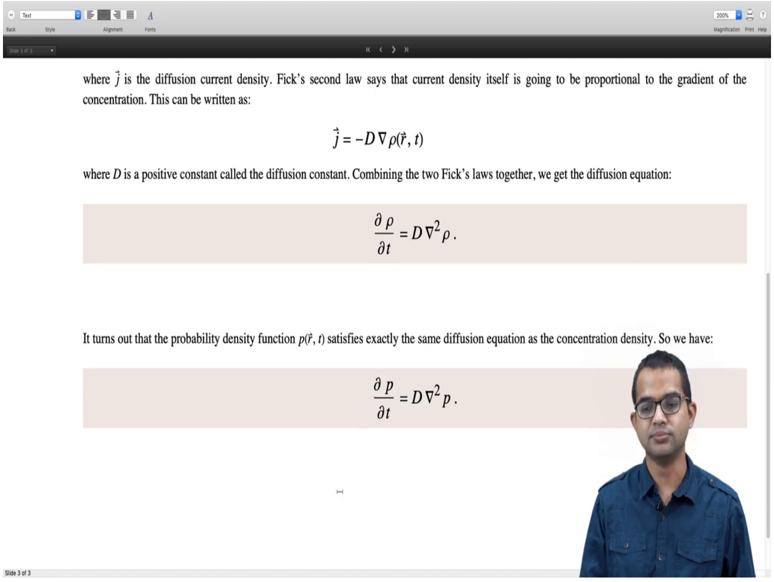

Together, we immediately get the diffusion equation that is dou rho by dou T is equal to D times del squared rho which is really the same as the heat equation that we saw from heuristic point of view. Now, it turns out that often it is convenient to work with a probability density function rather than a concentration density. So, basically probability density is like you ask what is the probability that a given particle is in some region, whereas concentration is about asking how many particles are in the region.

So, they are very closely related, and in some sense they are really the same. So, often we work with the probability density function, and then you know the partial differential equation satisfied is simply given by this equation dou p by dou T is equal to D times del squared p which goes by the name of the diffusion equation, ok.

Thank you.