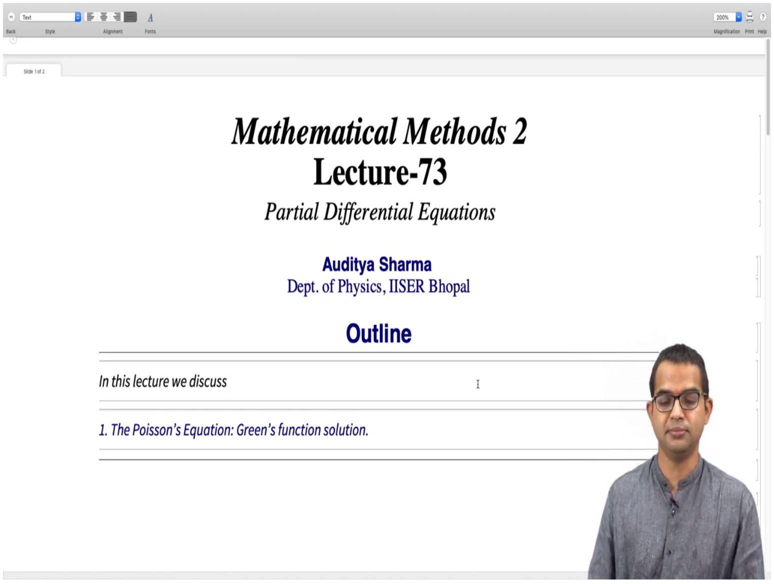
Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

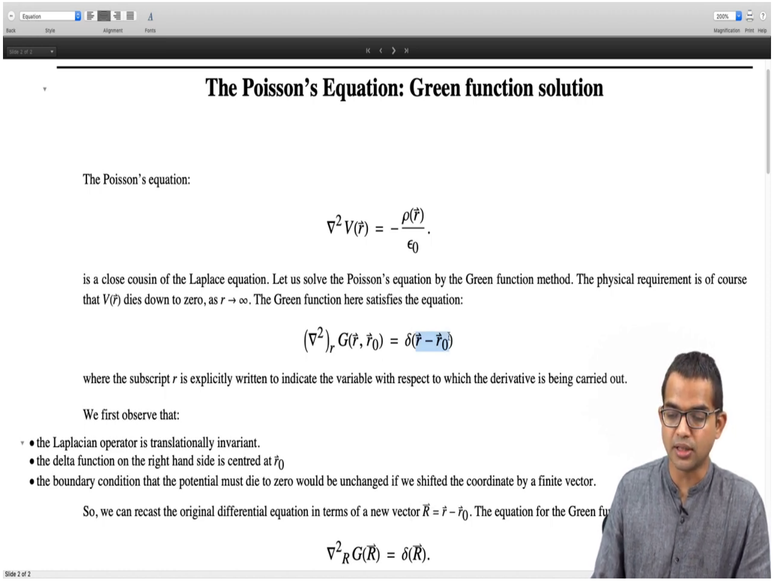
Module - 08
Partial Differential Equations
Lecture - 73
The Poisson's Equation: Green's function solution

(Refer Slide Time: 00:20)



So, we have looked at Laplace equation in quite some detail, we devoted several lectures to Laplace equation. In this one-off lecture, we discuss the Poisson equation which is a close cousin of the Laplace equation, before we move onto a different kind of PDE.

(Refer Slide Time: 00:41)



So, the Poisson equation is just you know something like the Laplace equation; but instead of zero on the right hand side, so there is a Laplacian operator acting on a scalar field and which is not zero, but it is some functions right. So, it's best to directly appeal to electrodynamics.

So, del squared V of r is equal to minus this charge distribution rho of r divided by epsilon naught right. So, this is something that we have seen. This is the Poisson equation. So, let us just work out the solution to this problem using the Green's function method right.

So, let us recall what the Green's function method was. Basically when you are given there is an in homogeneity that you introduce into the differential equation, so what you do is that in place of this function, you just put a delta function right.

So, and then try to work out the solution for the differential equation, if the right hand side is replaced by a delta function which so instead of have a having a complicated you know set of charges in this problem, you say that you assume that there is just a delta charge which is located at some particular point.

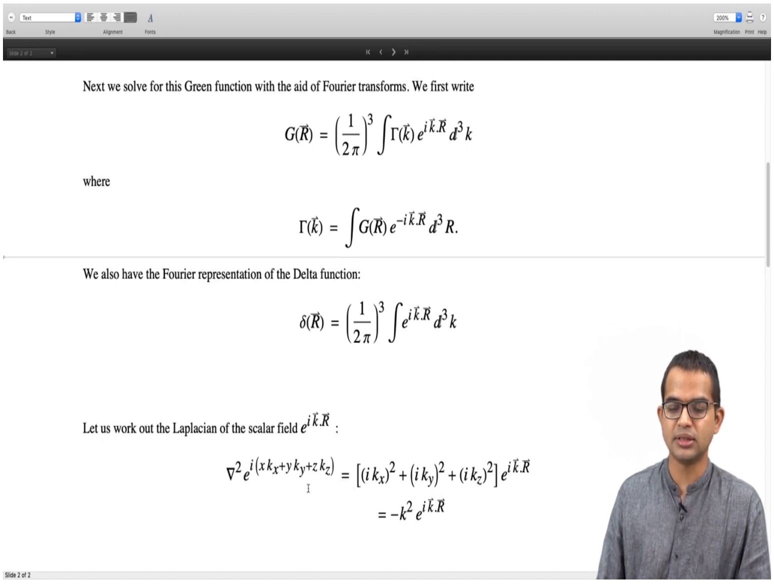
And then, you work out the effect of that and then, there is this way to combine the effect of lot of these in a entire distribution based on you know a kind of a superposition principle approach and then, you can stitch together the solutions from you know the Green's functions and then workout or write down at least a formal answer to an arbitrary function on the right hand side. So, that is the broad method of Green's function.

So, in this case, del squared del del squared G of r comma r naught is equal to delta of vector r minus vector r naught is the problem we must solve. So, in order to solve this, it is helpful to recast this equation in a slightly different form.

So, we make use of the transnational symmetry of this problem. So, we see that you know this r naught can be chosen to be anywhere. So, we might as well actually take that to be the origin, we shift it yeah this problem. So, as this r naught becomes the origin.

And so, this delta function is located about r naught and so, the boundary condition that the potential must die down to zero is going to be the same even if you make a shift right as long as it is a finite shift. Therefore, we can actually introduce this new vector capital R which is this r minus r naught and then just rewrite this as del squared with respect to this capital R variable; G of R is equal to delta of R right.

(Refer Slide Time: 03:29)



So, and then, it's a matter of solving for this differential equation right. So, in and to do this, we will make use of Fourier transforms right. So, this is you know this is also something familiar perhaps more familiar in 1D, but really it carries through to 3D as well.

So, G of vector R is written as 1 by 2 pi the whole cube of this integral. This integral is really a three-dimensional integral gamma of k is the Fourier transform right. So, this is like the equation involving; so it is like an inverse Fourier transform.

And where, the Fourier transform itself is given by this right, all this factor is chosen in such a way that you know these two are consistent with each other. You can distribute this factor in many different ways as you know various conventions are available. So, basically you choose one convention and stick to it. So, here, if you choose gamma of k to be this, then G of R is given by this expression as you can cross check this.

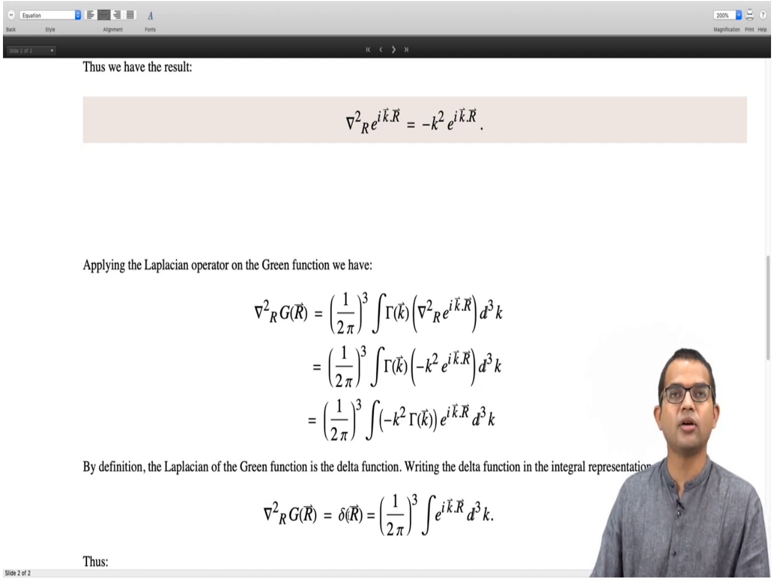
So, we also have this Fourier representation of the delta function which will come in handy. So, delta of R is given by this 1 by 2 pi the whole cube integral e to the i k dot R you know d cube k, integrated over all these you know three degrees k x, k y and k z.

So, it will be useful to work out the Laplacian of this field e to the i k dot R right. So, del squared of e to the i k dot R is nothing but x k x plus y k y plus z k z and then, you can check that in fact, you will just get i k x the whole squared.

So, you know do it once and the second time and then, basically you will get i k x the whole squared. I mean you write down this del squared operator in Cartesian coordinates. Because we have expanded this in Cartesian coordinates and then, you get this expansion and then this remains as it is right.

So, basically, you will leave this as it is. You can check this and then, but you then you immediately see that this i squared is minus 1 and then k x squared plus k y squared plus k z squared will add up to k squared and then, finally, the answer can be written again in a form which is independent of the coordinate system that you are using right. So, to go from you know from here to here, we made use of the Cartesian coordinate system which is useful.

But really the final answer is real. Del squared of e to the i k dot R is equal to minus k squared e to the i k dot R which is this result itself is independent of the coordinate system. (Refer Slide Time: 16:14)



So, this result will come in handy in a moment as we will see. So, let us go back and you know rewrite whatever we already have. So, which is I mean we have just introduced the Fourier transform of this Green function and then, now we are going to take this Laplacian operator and operate it upon the Green's function. So, that is why we wanted to calculate this. You will see that this will come in handy, when we compute the Laplacian of the Green's function.

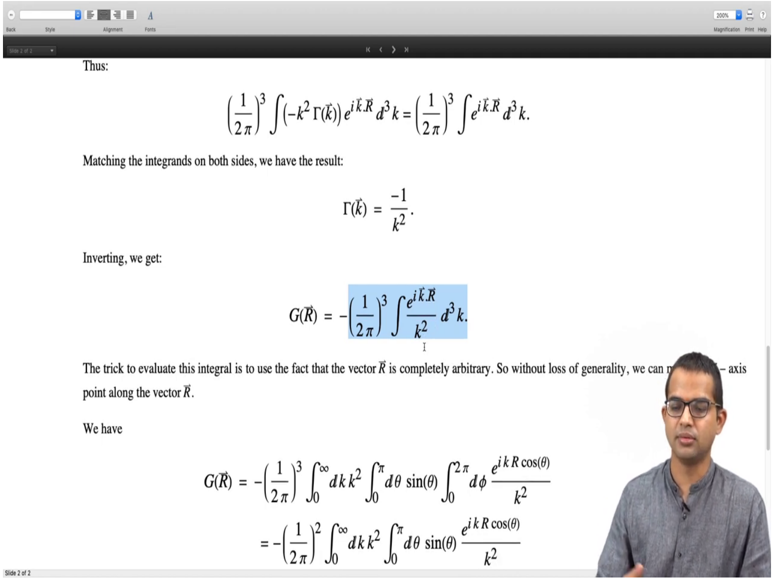
So, when you do Laplacian of the Green's function, you write G of R as this inverse Fourier transform relation and then, you notice that it is only this part which lies within the integral which has any dependence on R.

So, we can directly take this del del R squared in to its integral and then, operate on this part of this integrand and then just replace this part of the integrand by minus k squared times e to the i k dot R. And then, so this it is useful to rewrite this as this minus k squared times gamma of k then e to the i k dot R.

So, we have an expression for the Laplacian of the Green's function, but then what is the Laplacian of the Green's function. By definition, the Green's function is a function which you know whose Laplacian is the delta function. This is how we defined the Green's function.

So, in fact, we have another you know answer for this which is del squared the Laplacian of G of R is actually delta of R; but we also have an integral representation of delta R, which we already yeah you know stated sometime ago. So, if we write that out explicitly, then we see that this quantity and this quantity are both the same.

(Refer Slide Time: 08:00)



And so, writing it out explicitly and comparing the integrands, so we can immediately read off this delta of gamma of k. So, gamma of k is the Fourier transform of the Green's function is actually nothing but 1 minus 1 over k squared. So, this comes you know just directly as a consequence of this equality.

So, once we have this, what we have managed to do is the Green's function is somewhat tricky. So, we have worked out using these arguments that you know the answer for the Fourier transform of the Green's function. Once we have the Fourier transform of the Green's function, we can get the Green's function by taking the inverse Fourier transform of this quantity.

So, which is just given by you know 1 by 2 pi the whole cube you know integral e to the i k dot R divided by times this function d cube i k, like we wrote down formally earlier: we are just substituting the functional form for gamma of k. Now, this integral, it turns out, is something which we can work out analytically.

So, the way to do this is to immediately see that this angle of this vector R is actually completely arbitrary right and so, without loss of generality, we can actually choose that to be the z axis and once we do that, it becomes straightforward. You can write k dot r as you know k R cos theta in this coordinate system.

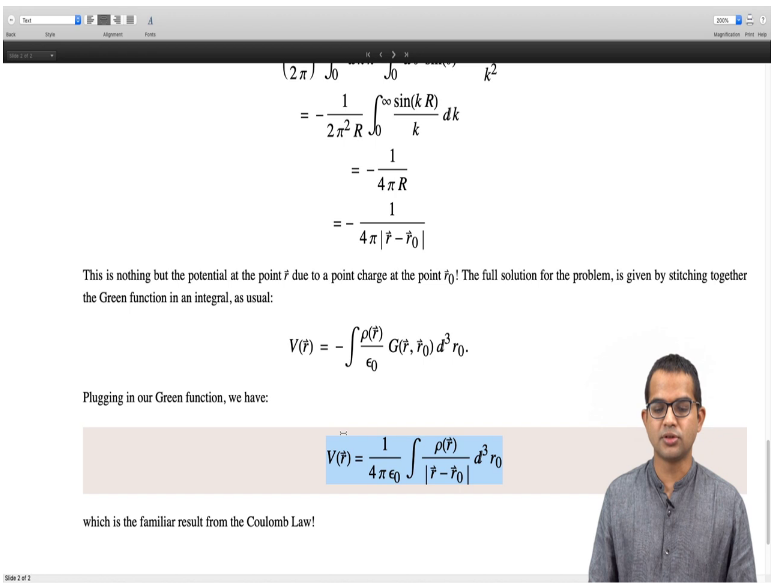
So, we have taken R to be along the z axis and then, we write it in terms of spherical coordinates and then, you have d k k squared integral 0 to pi d theta sin, sin of theta and then integral 0 to 2 pi. So, this is like I mean rewriting d cube k in spherical coordinates.

So, you get this sin theta and you also get this k squared. But also the integrand itself becomes this quantity and now, it is actually straightforward to integrate this. First, we integrate with respect to phi which will just give us a 2 pi. So, this will become minus 1 by 2 pi the whole squared. Then, the next step is to integrate this part with respect to theta which is straightforward to do because you can just simply put cos theta is equal to x.

So, this is d theta sin theta will become d x and the sin which you have to be careful about and you can check that you know this integral will just simply give you a you know e to the i k i k R and minus e to the minus i k R, then these are 2 i sitting in the denominator, you can arrange all the coefficients appropriately. So, you will basically get sin of k R.

So, this k squared will cancel with this k squared and so, there is this you know another factor of k which comes in because of this integral. So, you can check that you know this integral will basically be this whole stuff, you know the factors included will just boil down to this integral 0 to infinity sin of k R by k d k.

(Refer Slide Time: 10:57)



But this is also a standard inti and then, there is this factor outside by the minus sin. Now, this integral itself is a standard integral, you know this is like doing integral sin x by x 0 to infinity d x which is a standard integral you get a pi by 2. So, if you do this carefully, the final answer is just minus 1 by 4 pi R.

But R is nothing but mode of r minus r naught and so, this Green’s function, if you pause for a moment is actually something that you know which we should have perhaps expected it is just the potential at the point r due to a charge at the point r naught right. So, you have just a unit charge which is being placed at r naught and then, we work out the potential due to it.

And the full solution of this problem is actually given by stitching together the Green’s function in an integral as usual right. So, we take the Green’s function and then you have minus rho of r divided by epsilon naught you know like here right. So, this is given to us.

So, we have to put in this stuff here and then, multiply it by yeah the Green’s function and then, do this integral right. So, if you plug this in, then immediately you get this final answer which is actually a familiar result which is nothing but the Coulomb law right.

I mean the Coulomb law is really you know a way of working out the potential due to many charges at a point right. You work out the potential due to a particular charge and then you go to another charge, you work out the potential at a point due to that charge and a third charge and so on; but if you have a distribution of charges, then it becomes a function rho of r like this.

Then, you have to work with an integral right. If you had a discrete set of charges, then you would get a sum; but in this case, you write it as a distribution of charges and so, this is a familiar result from basic electrostatics. So, anyway, we worked out a familiar result using a somewhat you know heavy machinery in some sense. But it is an illustrative way of getting some more experience with working with Green’s function right.

We have looked at Green’s functions in the context of ODEs, but you know something similar also plays out in PDEs. We looked at one you know rather simple example, where it's actually possible to write down G the Green’s function analytically and also, work out the full solution in a formal way and it also makes a connection to a result that we are already aware of from Coulomb’s law ok.

Thank you.