Mathematical Methods 2
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Module - 08
Partial Differential Equations
Lecture - 72
The Laplace Equation in Spherical Coordinates: illustrative examples

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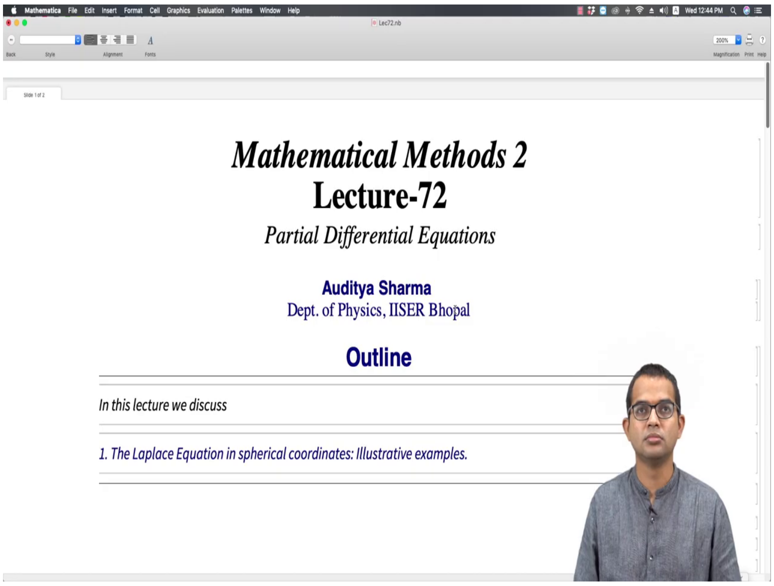

So, in this lecture we are going to look at a few examples where we apply the techniques which we developed to solve the Laplace Equation in Spherical Coordinates. So, when problems which you know have spherical symmetry in-built into the boundary conditions, then the method that we develop is going to be applicable. So, we are going to solve some examples in this lecture ok.

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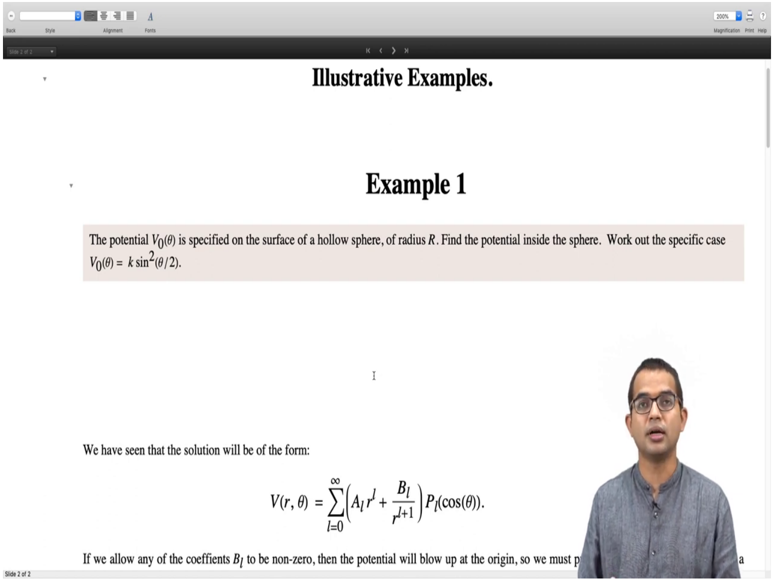

These examples are taken from David Griffiths' electrodynamics textbook, there are many interesting problems you know around this topic. So, interested readers can consult this book and other books also have similar examples as well ok. So, let us start with this problem where you are given a hollow sphere, of radius R and the potential on the surface of this sphere is specified V naught of theta.

And we will work out one specific case of this V naught of theta which is k times sin squared of theta by 2 after we have solved this problem. Our goal is to solve for the potential inside the sphere. So, it is really a Dirichlet problem: the boundary conditions on the surface are given to us and we must find the potential within the sphere.

And since the potential is dependent only on theta. So, our solution is also going to be independent of the angle phi as we have seen. So, the solution is going to be of this form. There is going to be this radial part which has you know terms of the kind r to the l and 1 over r to the l plus 1 and P l of cosine theta comes from the angular part. We should combine these two together and allow all possible values of l, l goes from 0 all the way up to infinity.

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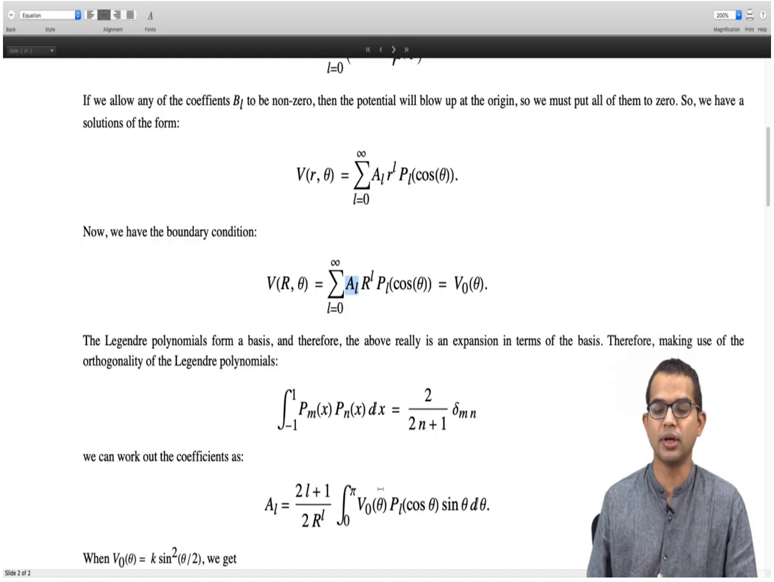

So, first of all we argue that in this particular problem for when you are looking for a solution within the interior of this sphere B ls all of them have to go to 0. Because otherwise at r equal to 0 at the origin the potential is going to blow up which would be unphysical. So, right away we can remove all these coefficients in B l and we are left with this expansion V of r comma theta is equal to summation over l, A l r to the l b P l of cosine theta and our job is to compute this coefficients A l.

Now, we are given the boundary condition which is that when you put small r is equal to capital R, this infinite series expansion must go to V naught of theta for all values of theta right. So, a little thought here reveals that in fact, what is happening is we are expanding this function V naught of theta in terms of Legendre polynomials right.

So, the Legendre of polynomials form a basis and so, this is a legitimate expansion and. So, in order to compute this coefficients A l we use a trick which is like the Fourier trick and so, here I mean really what is going on is we are making use of the orthogonality property of Legendre polynomials which you will recall has this form right.

So, if you take any two Legendre polynomials of different degrees multiply them and integrate from minus 1 to 1, if their degrees are different then you are going to get 0 basically they are orthogonal to each other and if they are the same I mean polynomials which are multiplied here then you are going to get 2 divided by 2 n plus 1 which we have worked out from first principles in an earlier discussion.

Now, we will simply use this result and so, we can write e A l in this form right. So, we are really looking at a Legendre polynomial of cosine of theta. So, here we have this sine theta d theta and the integral is from 0 to pi, but really if you just put cosine of theta is equal to x you can rewrite this integral in terms of x as we will do now.

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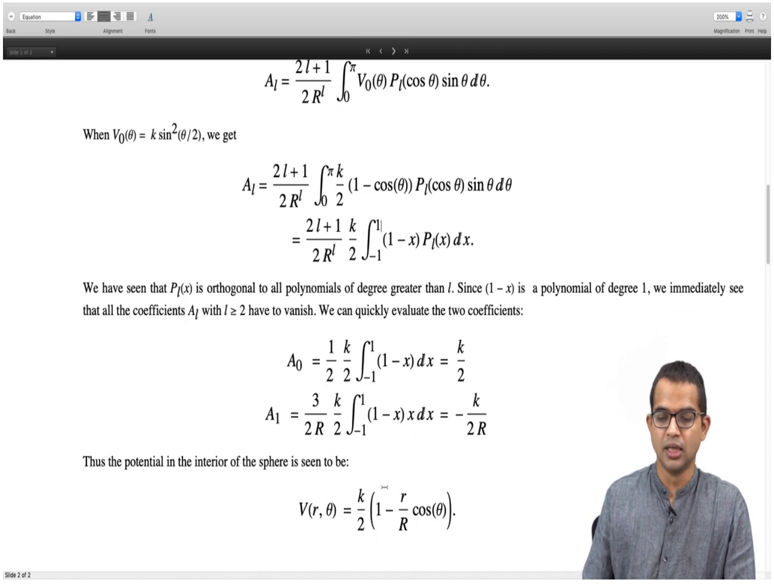

So, for the specific problem of interest when you put V naught of theta is equal to k times sin square theta by 2, you know it's useful to write this sin square theta by 2 as 1 minus cos theta and then we have 1 minus. So, when you put cos theta is equal to x. So, you get 1 minus x times P l of x and then the sin theta will go by sin theta d theta will give you d x and there is a minus sign which you get absorbed.

You can check this and over all the integral will be just minus 1 to 1 and this stuff. And so, now, we argue that, in fact, since 1 minus x is a polynomial of degree 1. So, whenever l is greater than or equal to 2 right. So, this integral is going to vanish right. So, this comes from a basic property of these orthogonal polynomials as we have discussed right.

So, a polynomial of whatever degree can be written in terms of, you know, Legendre polynomials of that degree and below and then Legendre polynomials of different degrees are anyway orthogonal to each other, right. So, this is the argument which you can go back and look up again and so, immediately we see that A l is going to be 0 for l greater than or equal to 2.

So, there is only A 0 and A 1 which we must evaluate which is very easy to do because we just plug in here and so, you can check that A naught is just k by 2 and A 1 is minus k by 2 R right. So, check these integrals and then we are able to immediately write down the final answer which is that V of r comma theta is equal to k by 2 times 1 minus r by R cosine of theta. You can try out other kinds of potentials V naught of theta and see what kind of answers you get.

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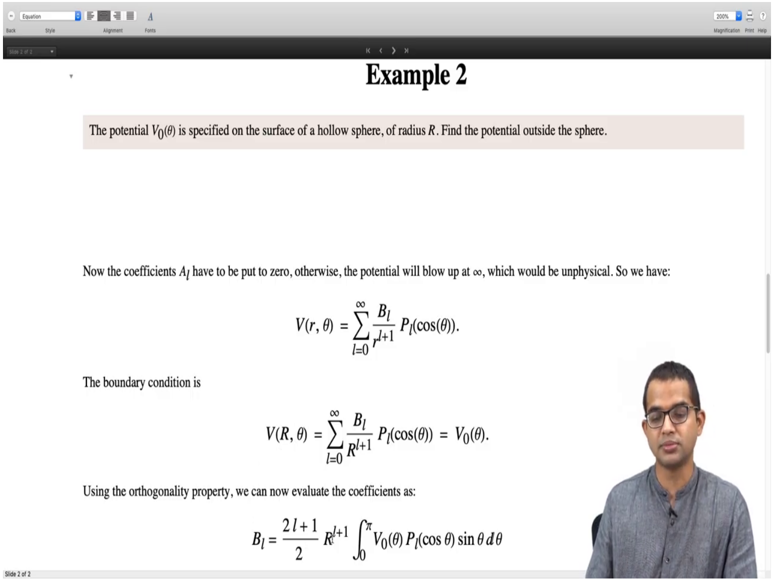

So, now, let us look at example 2 which is basically the same problem, but trying to understand what happens to the potential outside of this hollow sphere. So, you can think of the boundary conditions on the surface as one boundary condition which is given to us.

But also now we should ensure that the potential for very large r must go to 0 right. So, that is the physics which ensures which we have to impose onto this problem right. So, there is no; there are no charges which are distributed far away and so, therefore, indeed it would be unphysical for this potential to be known 0 far away.

So, we are you know really that is also another boundary condition, one boundary condition is you know the boundary condition which has been specified on the surface and then V of r must go to 0 for large R. So, if that must happen then all these coefficients A l have to vanish in our expansion. So, we are now going to look at a solution of this form where all these B l's are potentially there but none of the A l's you know are allowed to be there.

And so, the boundary condition we must fit this to is that where small r is equal to capital R once again this expansion must go to V naught of theta. So, now, again we invoke the orthogonality property, now the coefficients B l are given by 2 l plus 1 over 2 times R to the l plus 1 and then an integral which is similar integral as we had earlier will come up here now 0 to pi V naught of theta P l of cos theta sin theta d theta.

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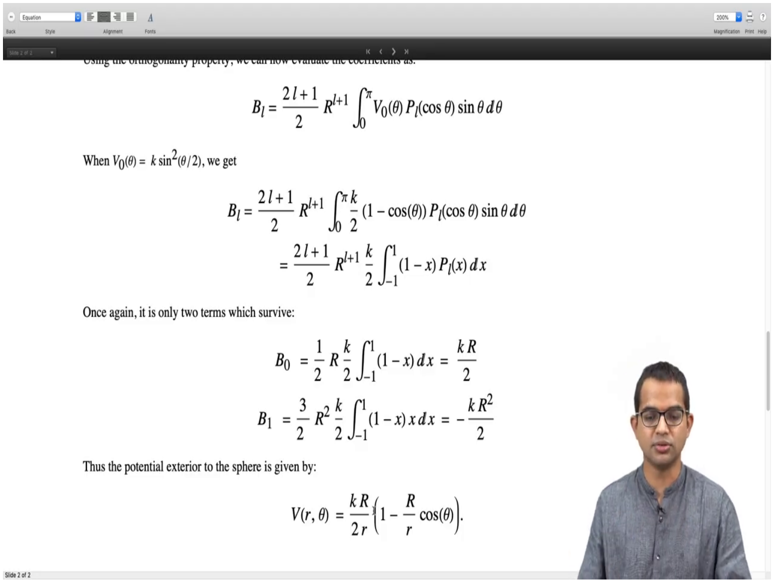

So, for the specific case k times sin square theta by 2 here we will get you know this stuff outside is a bit different, but really inside it's all pretty much the same and now you know you have to take care of these outside factors and coefficients if you do this carefully you will see that again only B naught and B 1 are the only two coefficients which will survive.

And they are given by k R by 2 and minus k R squared by 2 and you should check this and convince yourself that indeed the potential V of r comma theta outside of this sphere is going to be given by this expression. So, you see that when r becomes very large it's going to go to 0 and when r is equal to capital R the boundary condition holds and all you have to do is check that V of r comma theta satisfies Laplace equation.

And so, basically this uniqueness theorem tells us that if you can find a solution which fits the boundary conditions Dirichlet boundary conditions, then for sure that is the solution. So, there is a way to quickly directly check this. Once you have the final answer you must always check the boundary conditions and that it satisfies Laplace equation and then for sure you have got the correct answer ok.

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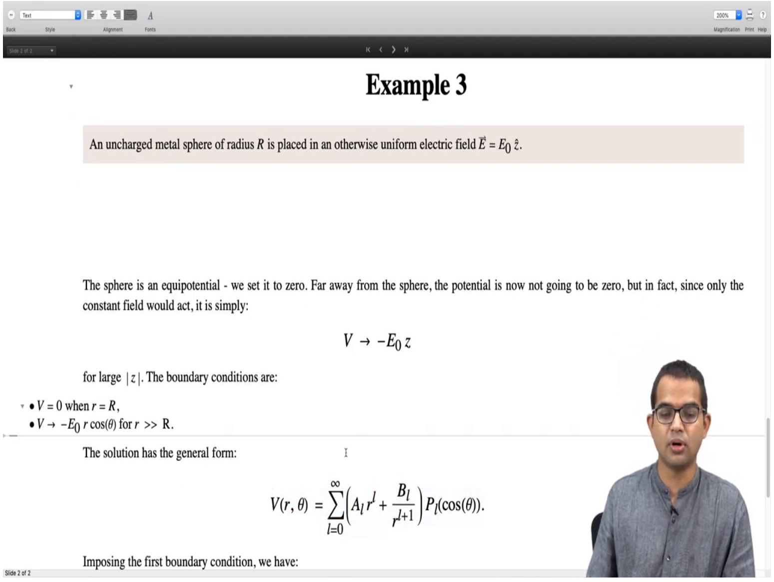

Then we look at one more example. So, here again it's as you know a problem with spherical symmetry. So, you are given a you know metal sphere of radius R and its placed in an otherwise uniform electric field which is a constant electric field which is a constant electric field whose direction we can take without loss of generality be along z.

And our job is to find the potential you know outside of the sphere right. We will come to what happens inside in a moment. So, basically the surface of the square is an equivalent potential. So, we can just put that potential to be 0 here right. So, far away from the sphere I mean you can put it to be some constant, but you might as well take that constant to be 0 right.

So, you fix the 0 of the potential and then the potential everywhere else is fixed with respect to that. So, we take that potential constant potential here on the surface to be 0 far away from the sphere the potential is now not going to be 0 because you have an electric field which is going away all the way to infinity, so in fact, you would have a potential which goes as minus E naught z very far away from the origin.

So, these are in fact, the two boundary conditions. So, V equal to 0 at small r equal to capital R and V must go to minus E naught times r cos theta which is what z is for r much greater than R.

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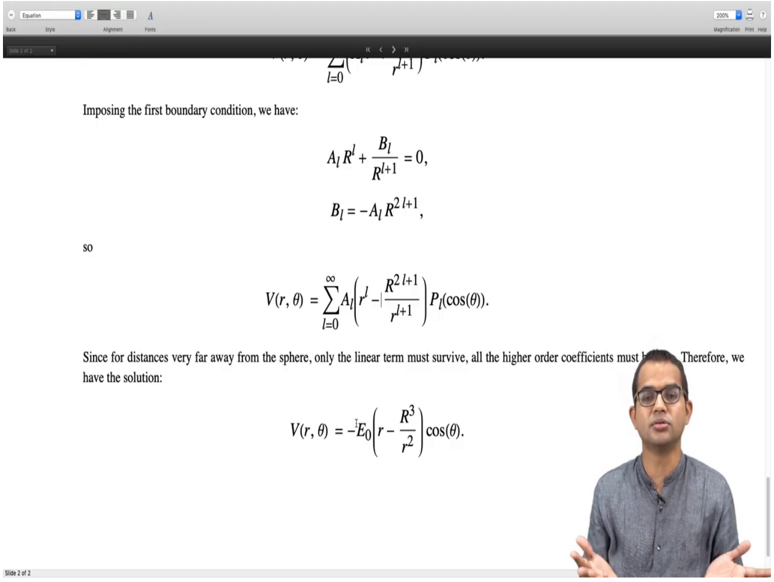

So, the solution we know which is of this general form, in this form if you put the first boundary condition it implies that when you put small r equal to capital R this must be equal to 0. Therefore, B l is equal to minus A l times R to the 2 l plus 1. So, V of r comma theta is you know this is the expansion we have. So, A l times r to the l you know minus capital R to the 2 l plus 1 divided by r to the l plus 1 then there is P l of cosine of theta. Now we have imposed one of the two boundary conditions.

So, the other boundary condition is a rather strong one you will see that when its straightforward to apply this and in fact, the key point is that when r is very large you can only have something which goes as r cos theta so; that means, that only l equal to 1 is the only term which is allowable every if even one other term is present then you are not going to get this, you will go you are going to get other kinds of terms which is not acceptable for this.

And therefore, immediately it forces A l to be 0 for all l other than l equal to 1. So, the only linear term must survive and that we know that it should also go to this particular form. So, you can immediately write down the answer V of r comma theta is equal to minus E naught times small r minus R cube divided by r square a whole times cos theta. Again you can check that another boundary conditions hold and that it satisfies the Laplace equation and so, this is the solution.

So, we can ask what happens to the potential inside the sphere in this case. So, you know one answer which we can immediately guess is you know there is an entire surface which is at 0 if the potential is 0 everywhere inside that is for sure its satisfying Laplace equation and it satisfies the boundary conditions also.

And if you can find one solution for a Dirichlet problem then indeed that is the solution right. So, even by just looking at the symmetry of the problem you can argue that if there is going to be accumulation of positive charge on one side of the sphere and negative charge on the other side of the sphere everywhere inside the sphere the potential is just going to be 0. That is something you can argue just on physical grounds, but also from this sort of Dirichlet problem and uniqueness theorem angle as well ok.

Thank you.