Mathematical Methods 2
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Complex Variables
Lecture - 07
Differentiation rules for a complex function

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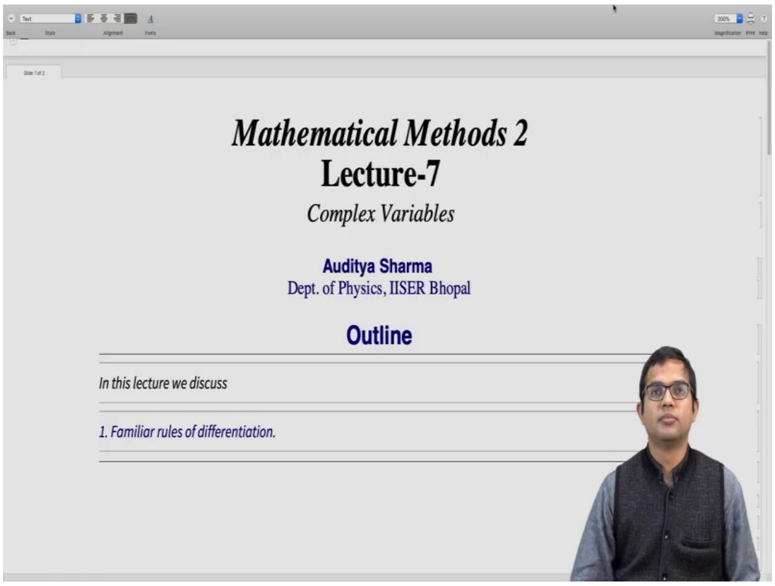

So we have seen the notion of continuity for a function of a complex variable. We have also seen the notion of a derivative for a function of a complex variable, how it is a natural generalization of the idea of a derivative for a function of a real variable.

But we have also seen how differentiability is a rather strong condition and how you can have functions which are continuous, but not differentiable. Differentiation is a rather involved idea, when you are working with the function of a complex variable.

So, in this lecture, we will look at some familiar rules of differentiation which we take for granted when working with functions of a real variable; how do they extend when we go to the world of functions of a complex variable?

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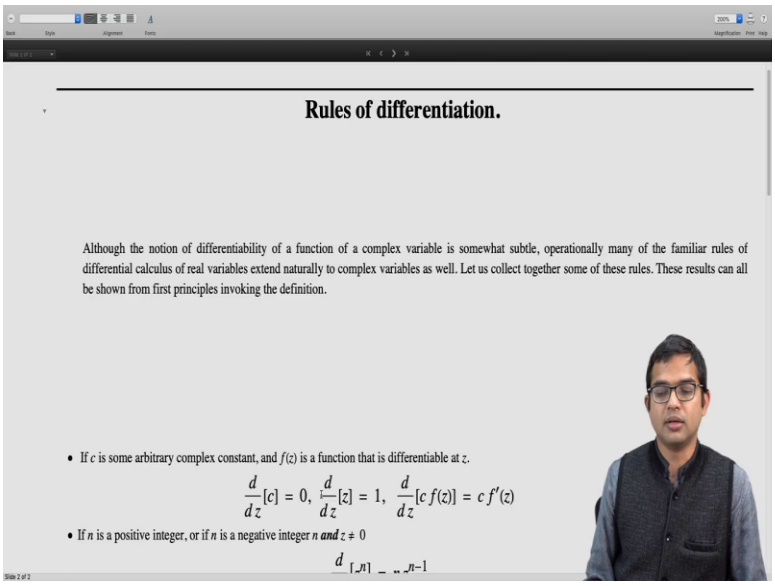

So, we look at some of these familiar rules of differentiation. So, basically they extend almost directly from differential calculus of real variables; we have to be a little careful with how we apply them.

So, if c is some arbitrary complex constant and f of z is a function, that is differentiable at z; then the derivative of the constant is 0, just like with a function of a real variable, so the derivative of the function z is 1. And if you want to take the derivative of some constant times f of z, then the derivative of the function itself must be defined. So, that is very important.

So, then you have c times f prime of z. So, this f prime of z we are guaranteed has meaning, because we are looking at a function which is differentiable at z.

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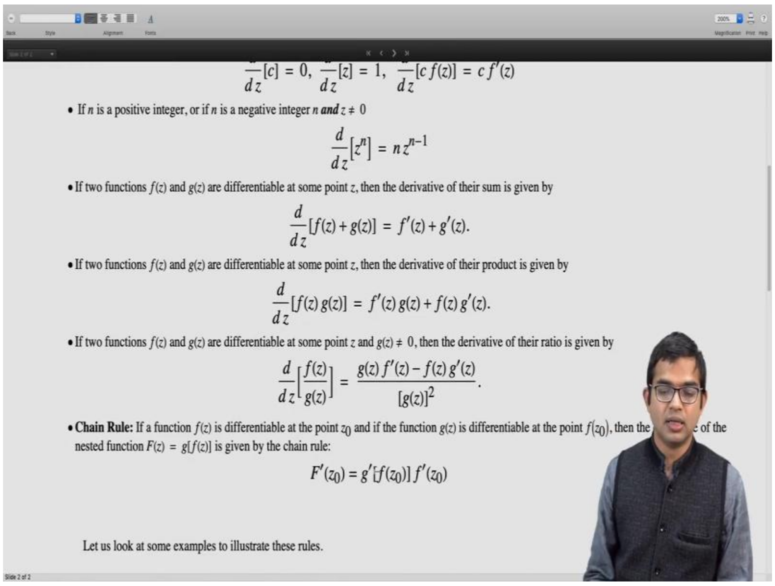

Now, if n is a positive integer, then the derivative of z to the n is just n times z to the n minus 1, right. So, we already worked this out from first principles for z squared, but you can do it for z cubed, z to the 4; and any arbitrary z to the n.

And in fact, this result will also extend to negative integers n, provided z is not equal to 0, right. So, you are going to get in the denominator a 1 by z power something. So, that is why we have to, we have this extra restriction when n is a negative integer; but the rule that the derivative of z to the n is equal to n times z to the n minus 1, will hold for both positive and negative integers and for negative integers provided z is nonzero.

Now, if there are two functions f of z and g of z, both are differentiable at some point z; then the derivative of that sum is just given by some of the derivatives. This is not a surprise; it is something that we take for granted and it also extends here. So, again this is the product rule; we have two functions f of z and g of z, both of them are known to be differentiable at a certain point z.

Then if you want to find the derivative of their product, then it is simply given by f prime of z times g of z plus f of z times g prime of z, so this is also a familiar rule. And again the quotient rule, if you want to take the derivative the ratio of two functions f of z divided by g of z; it is meaningful provided g of z is not 0. If you are at a point where g of z is not 0, then it is like taking the derivative of the product of two functions.

I mean you are taking the derivative of the product of the function f of z and 1 over g of z. So, slightly different, but effectively you can work it out based on the next rule that also we will point out, so that is the chain rule. So, but anyway the point is that, it is f of z times 1 over g of z; so it will come out to be just g of z times f prime of z minus f of z times g prime of z divided by g of z the whole squared.

So, then we have the chain rule which is that, if a function f of z is differentiable at some point z0 and this function g of z is differentiable at the point f of z0, correct. So, it is in this sense that, you can think of you know f of z times 1 over g of z and so if you know that g of z is differentiable; then you can also work out the derivative of 1 over g of z and use this kind of a chain rule.

So, the idea of the chain rule is g of z is differentiable at some point f of z0; then the derivative of this nested function capital F of z which is equal to g of f of z, it is just given by this chain rule which we are very, which we are familiar with. And so, all of these rules are of course best understood by applying them right. We have applied these rules many times and the same kind of rules that we used for functions of a real variable, we just follow through.

Provided we take this small extra care to ensure that, these functions are all differentiable; if g of z is differentiable at the point f of z0, then f prime of z0 will be g prime of f of z0, then of course you have to multiply by this derivative of prime itself of z0, right.

So, basically all the rules that we are familiar with will naturally extend to functions of a complex variable; provided you know differentiability of different pieces of the function are all already given.

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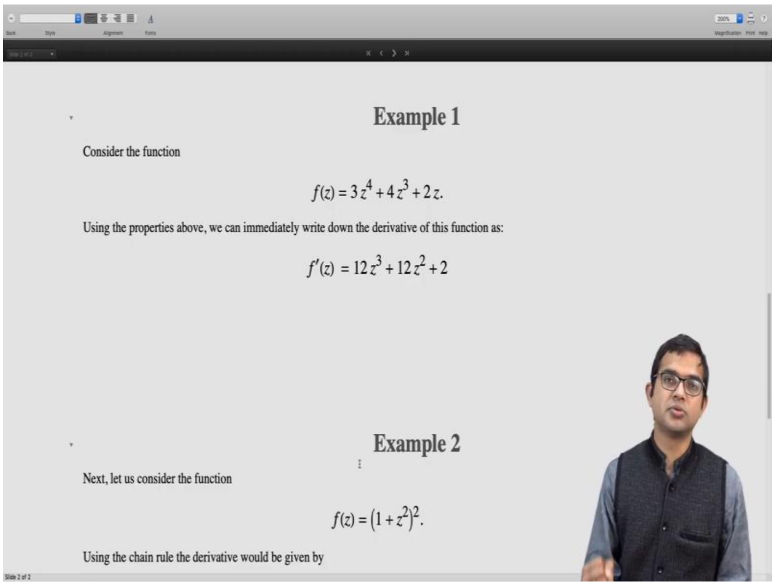

So, let us look at a few examples and this is best understood with many examples, by playing with examples. So, I have cooked up some function f of z that is equal to 3 times z to the 4 plus 4 times z cube plus 2 z. So, simply invoking these properties above, we should be able to write down the derivative of this function.

So, in general at any point, z is a very nice smooth function; because it is a polynomial, all very nice properties are available for this. So, it is just going to be 12 times z cube plus 12 times z squared plus 2. If you wish, you can try to derive this result directly from first principles; start with the definition, take the limit and convince yourself that the limit holds no matter in which direction you approach the point of interest.

Let us look at another function, 1 plus z squared the whole square. So, now, I can treat this 1 plus z squared this entire thing as some w and I can call it. So, it is like w squared, so that is where the chain rule applies.

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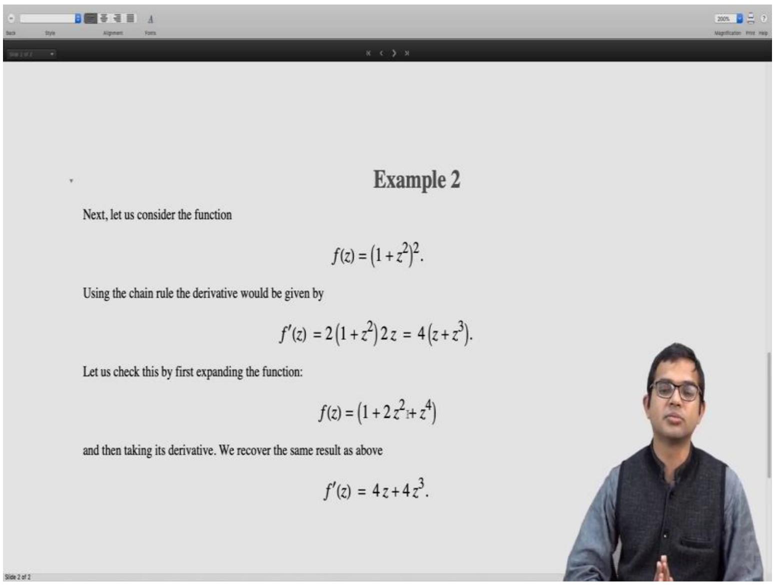

So, I can think of this as 2 times w; we have seen that the derivative of z squared is just 2 z. So, here it will be w square. So, 2 w, but w is 1 plus z squared and then I have to take another derivative w itself d w by d z. So, that will give me just 2 z. So, if I expand it out, I get 4 times z plus z cube, right.

So, this is something that we can check, by working out the same result from a different approach; we first expand this function 1 plus z squared the whole squared is 1 plus 2 z squared plus z to the 4. Now, if I take the derivative, it is similar to example 1; it is the sum of it is a polynomial, right. So, you can take the derivative of every term in this sum separately and then you get 4 z cube plus 4 z right, which is the same as the answer we got earlier using the chain rule, right.

So, it is just very simple stuff, all things which we are familiar with the same rules carry through; provided you know differentiability of different parts, bits of your function are assured, which is definitely true in this kind of a case, because we are working so far with polynomials.

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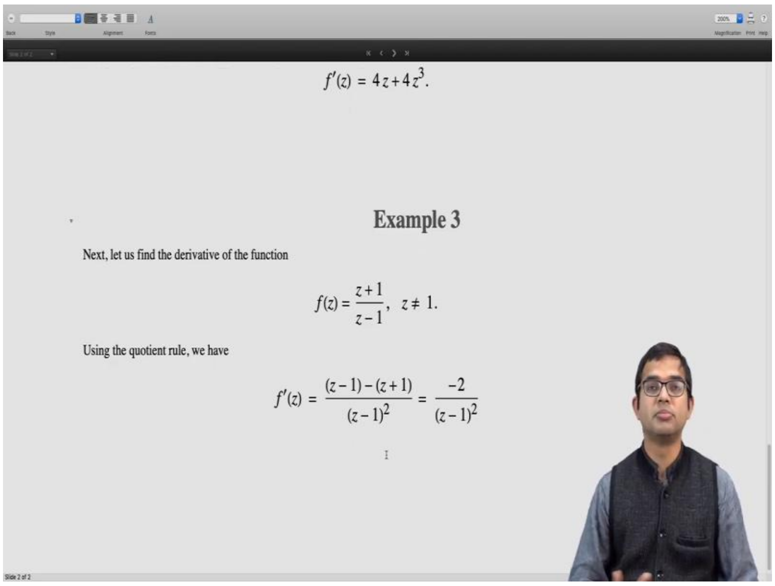

Now, let us look at a rational function like this. Suppose I want to find the derivative of a function like z plus 1 divided by z minus 1; of course I have to demand that z is not equal to 1 right; if z equal to 1, then it is not even defined.

So, z is not equal to 1, then we can use the quotient rule and the quotient rule tells me that; I take the denominator times the derivative the numerator is just 1 minus numerator times the denominator derivative of the denominator which is again just 1. So, it is z minus 1 minus z plus 1, the whole thing must be divided by z minus 1 the whole square. So, the numerator simply becomes minus 2 and then I am left with z minus 1 the whole square in the denominator right, straightforward, ok.

So, you can cook up your own functions, simple functions, rational functions, polynomials, more complicated polynomials or some you know combination of these kinds of terms to create your own interesting function; then check whether rules which we are familiar with for functions of a real variable, when we apply them how they work out to evaluate derivatives of quite a wide variety of functions, right. So, that is all for this lecture.

Thank you.