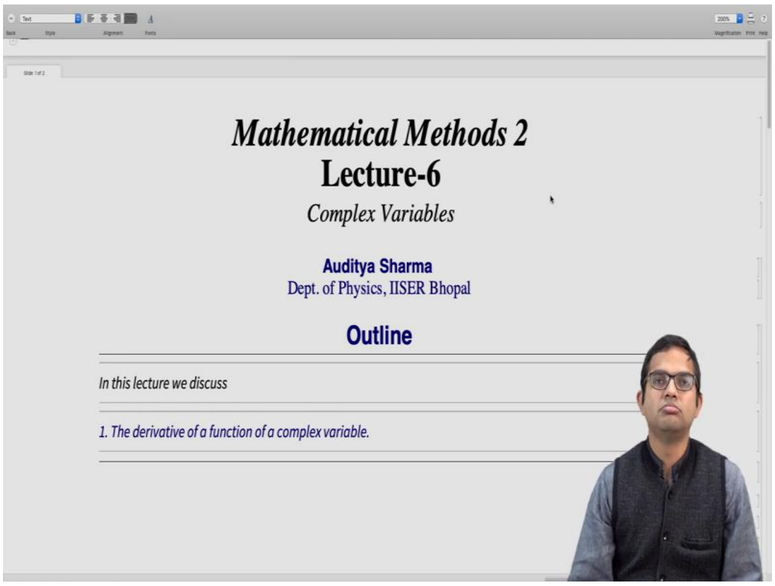
Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

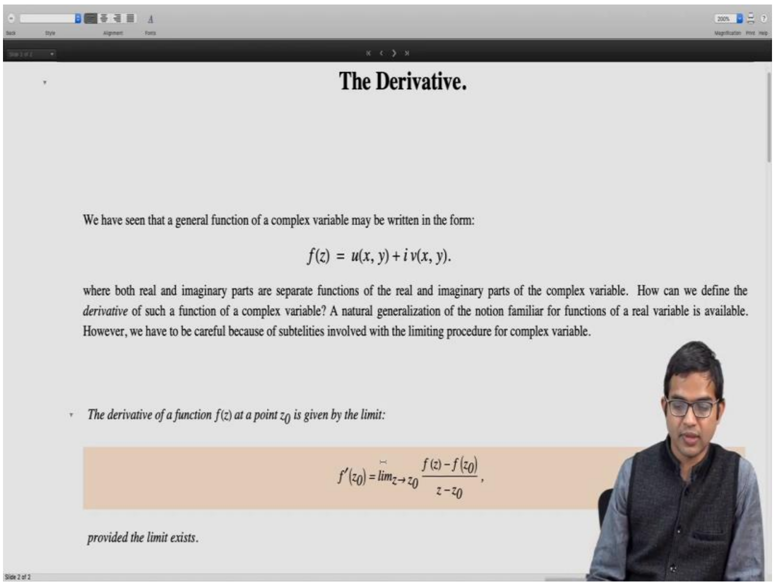
Complex Variables
Lecture - 06
Derivative of a complex function

(Refer Slide Time: 00:21)



So we have seen how to think of continuity when we are working with functions of a complex variable. So, in this lecture, we will build on these ideas and look at how to think of the derivative of a function of a complex variable, ok.

(Refer Slide Time: 00:41)



So, we have seen that, the general form for a function of a complex variable is like here f of z is equal to some real part which we can explicitly write out as u of x,y plus then there is an imaginary part for this function, which you can write as plus i times v of x,y, right.

So, in general both of these are independent functions; but there are constraints which come in if a meaningful notion of a derivative can be ascribed to such a function. So, let us look at what some of these conditions are; starting with the idea of what a derivative would be, right. So, it is like a natural intuitive generalization of the idea of a derivative, you are familiar with from a function of a real variable.

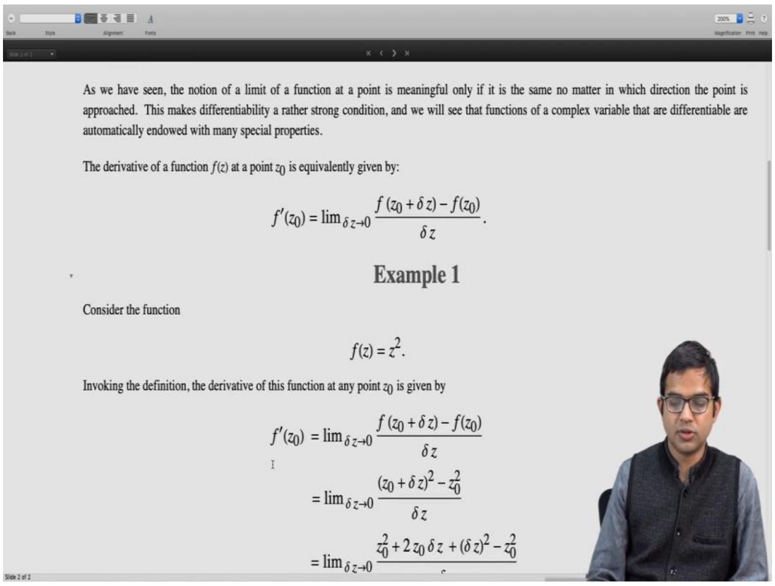
So, the derivative here is defined as a limit, right. So, suppose you are interested in finding the derivative of a function f of z at a point z0; so then we define the derivative f prime of z0 to be the limit of z tending to z0 of f of z minus f of z0 divided by z minus z0.

So, the idea is of course, how much does the function change, if you change the independent variable by a small amount. So, f of z minus f of z0 and divided by z minus z0 in the limit of z tending to z0.

So, this is exactly like the idea of a derivative with a real variable. So, the key difference of course is that, because the notion of the limit itself is different when we are working with a function of a complex variable, because the limit can be approached in infinitely different directions.

So, this limit whether it exists or not itself is something a little more subtle here, right. So, provided the limit exists; so this is the definition of the derivative of a function at some point z0.

(Refer Slide Time: 02:46)



Now so it is not always the case that for arbitrary functions u and v; even if they are very nice looking functions of x and y, the idea of a derivative may be meaningful, right. So, we will go into this carefully ahead. But here first we will just try and work with this definition itself and see how much we can extract from just this definition and look at a few examples.

So, an equivalent way of thinking of the derivative is of course to define it like this; f prime of z0 is some limit of some small delta z, as the small delta z goes to 0 you slightly perturb your z0 right, you measure the value of your function slightly away from z0 and then subtract it from the value at that point itself and then divide by the small perturbation if you wish.

So, f of z0 plus delta z minus f of z0 divided by delta z and I mean it is like the slope of a function if you wish, that is the intuition we have from a function of a real variable. But here it is better to think of it in this abstract way; how much does the function change, if you change the independent variable by a small amount and then you have to divide by the small amount it has varied and we are also take the limit of that small amount going to 0.

So, let us use this definition itself and work out some derivatives from first principles. Suppose we consider the function f of z is equal to z square. So, we already seen that this function is continuous everywhere. So, if you want to take the derivative of this function at some point z0, it's a nice smooth function everywhere, right.

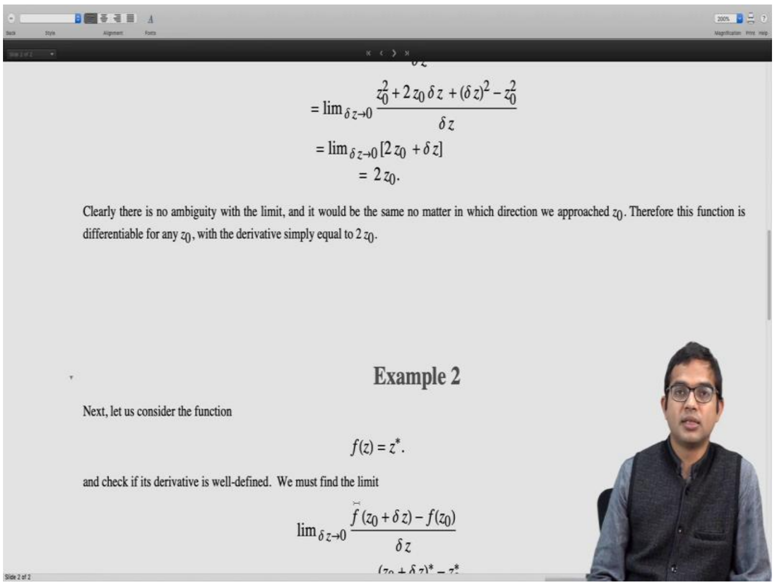
So, it is continuous everywhere and if you take the derivative of this function at some point z0; so the first principle definition will tell us that we should work out this limit of delta z tending to 0 f of z0 plus delta z minus f of z0 divided by delta z. But this is the same thing as taking the limit delta z going to 0 of this quantity, z0 plus delta z the whole square minus z0 square the whole thing divided by delta z.

So, if you expand out this term so you get z0 squared plus 2 times z0 times delta z plus delta z the whole square minus z0 square and then the whole thing needs to be divided by delta z. So, then we see that these z0 squared will cancel out and then we are left with just 2 z0 plus delta z.

So, you must take the limit delta z equal to 0 with this function and then indeed the answer is just simply 2 z0. So, in other words the derivative of the function z squared is just simply 2 z at any point z.

And so, here we have considered the particular point z0, therefore we get 2 z0; but in general, the derivative of f of z is equal to z squared is 2 z. So, very similar to the result we have for a similar function of a real variable, right.

(Refer Slide Time: 06:05)

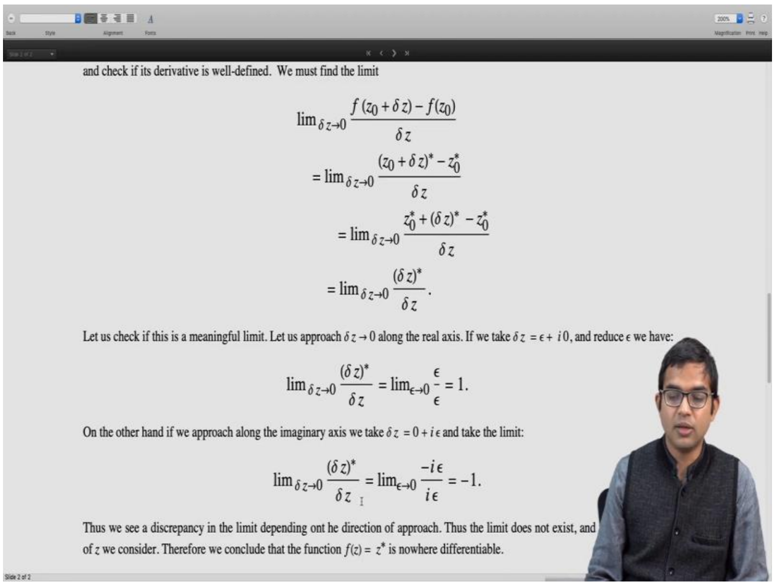


So, there is no ambiguity with the limit and it would be the same, no matter in which direction you approach z0. So, you are free to check this if you so desire, right. So, there is no ambiguity and the derivative is well defined for this function.

Now, let us look at another example; suppose we consider this function f of z is equal to z star, right. So, z star looks like a fairly simple function. So, z is equal to x plus i y, z star will be x minus i y and let us see if its derivative is well defined.

So, we must find this limit, limit delta tending to 0 f of z0 plus delta z minus f of z0 divide whole thing divided by delta z; f of z0 plus delta z is the same as z0 plus delta z the whole star minus z0 star and then whole thing divided by delta z, right.

(Refer Slide Time: 06:58)



So, we know from the properties of complex numbers that the complex conjugate of a sum is the sum of the complex conjugate. So, you get z0 star plus delta z the whole star minus z0 star z0 star cancels. And then finally, you are left with just this limit, limit delta tending, z is tending to 0 delta z star over delta z.

So, now, you see that, this is a bit of a problem right; because if you take this limit from you know delta z, let us try to take this limit along the real axis. So, in other words we look for a delta z of the form epsilon plus i 0 and keep on shrinking epsilon and take the limit epsilon going to 0.

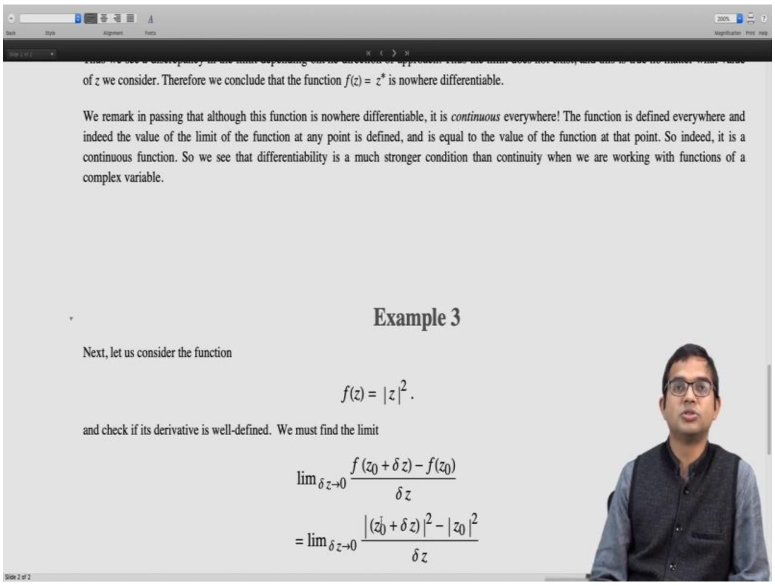
So, this limit is the same as limit delta z going to 0 of this quantity is the same as limit epsilon going to 0 of; well delta z the whole star is the same as epsilon, because there is nothing which goes along with that. So, you just get epsilon over epsilon which is just 1, right.

So, if you approach along the x axis for the limiting procedure, then you are supposed to get 1 for an answer here. But on the other hand, if you approach along the imaginary axis; so you are taking delta z to be 0 plus i epsilon and then take the limit epsilon going to 0, so now, we are going to find, you know it is same limit that needs to be carried out, but delta z the whole star is minus i epsilon.

And then you have to divide by i epsilon. So, epsilons will cancel and then you are left with minus 1. So, depending upon the direction, you know from which you are taking this limit, you may get different answers; even if there are two different answers, that immediately tells us that this limit is not well defined, right. So, if the limit is not well defined, then there is no derivative for this function at that point. So, in fact there is no derivative for this function z star at any point. So, you can check this.

So, it does not matter whether you are looking at the origin or you look at any other point, right. So, its, so in this case we have taken some arbitrary z0, right. So, it is only delta z that we are taking it to 0. So, we have already shown that this function does not have a derivative at any point.

(Refer Slide Time: 09:26)



But on the other hand, this is actually a nice function, in the sense that it is continuous everywhere, right. So, it is just x minus i y. So, if you go to any point, it has a well-defined value and so the limit of the function at that point is also well defined and it is equal to the value of the function at that point.

So, indeed the function is defined everywhere and it is also continuous everywhere; but it is nowhere differentiable, right. So, this is. So, we see that in fact differentiability is actually a much stronger condition than continuity, where we are working with functions of a complex variable, right. So, all of this of course, is connected to the fact that this limiting procedure is somewhat more non trivial.

We have all these different directions to consider, only if the limiting procedure the limit is the same, no matter which direction you approach, the point from only there is a limit defined. So, let us look at another example. So, suppose we consider the function mod z squared right and when we carry out the same procedure; we start from first principles, we must find this limit f of z0 plus delta z minus f of z0 divided by delta z with delta z going to 0.

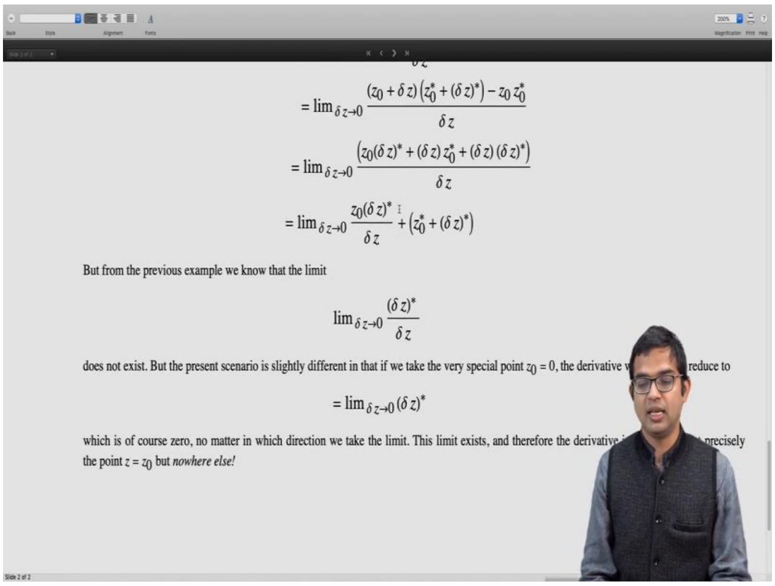
Then we find that in place of f the function, we just put in z0 plus delta z mod of this squared minus mod z0 squared whole thing divided by delta z; but mod of a complex number squared is the same as the complex number times its complex conjugate.

So, we can write this as z0 plus delta z times z0 star plus delta z the whole star minus z0 z0 star; the whole thing of course is to be divided by delta z, we have we expand it out and then of course, the z0 times z0 star will cancel.

And then we will be left with just z0 times z0 z0 times delta z the whole star plus delta z, you know times you know the z0 star plus delta z the whole star. Now, the whole thing has to be divided by delta z; these second two terms will both have this delta z with them, so they will cancel with delta z.

And so, the final answer or well I mean the limit that we need to evaluate is, this limit delta is equal to 0 of this function z0 times delta z star divided by delta z plus z0 star plus delta z star, alright.

(Refer Slide Time: 12:04)



So, but from the previous example we know that, this kind of a limiting procedure involving delta z star divided by delta z is messy business; because as you shrink you know in one direction or if you approach the limit in one direction, you get one kind of answer; if you approach in the other direction, you get another kind of an answer.

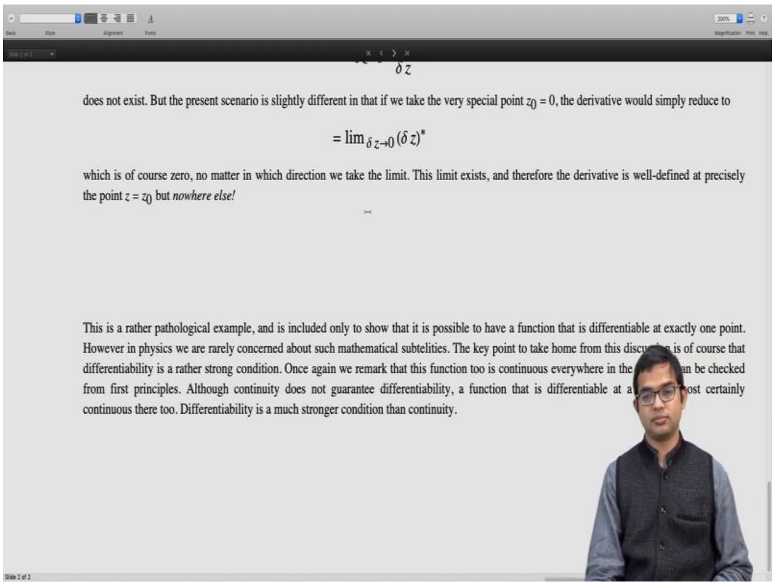
So, we would think that maybe you will probably get into the same kind of difficulties; which is true except that there is one point, where you know this limit becomes you know trivial in some sense, which is when you put z0 equal to 0. If z0 were to be 0, then the derivative would simply reduce to delta 0 going to 0, delta z of z star; because this the first term will just go to 0 right, because delta z the whole star is some small number and delta z is also a small number, z0 is exactly 0.

So, this first term is gone and then again z0 star is gone and then you are left with delta z the whole star, which also is of course 0; because if your delta z is going to 0, then so is delta z the whole star. So, there is no problem with this limit, this also is going to go to 0.

So, we see that there is this one very very special point z0 equal to 0, where this limit exists and is well defined. So, in fact this function has a derivative at exactly this point z equal to z naught, but nowhere else. So, it is kind of a weird function, right. So, it is you know it is differentiable; but only at this one very very special point, everywhere else it is continuous.

So, it has you know nice properties, except that it is not differentiable anywhere else, right.

(Refer Slide Time: 13:56)



So, this is also somewhat of a pathological example you can say; but the key point here we are trying to make is that, differentiability is a rather strong condition for functions of a complex variable. So, you can have a scenario where a function is continuous, but not differentiable like we have seen; but I mean differentiability is a stronger condition.

So, if a function is differentiable at a point, then for sure it better be continuous there, right. So, it is a stronger condition. So, exactly how strong it is and what consequences differentiability has right; what kind of inherent structure this function must obey, if it is differentiable somewhere, you know these are topics that we will discuss ahead in lectures to come, that is all for this lecture.

Thank you.