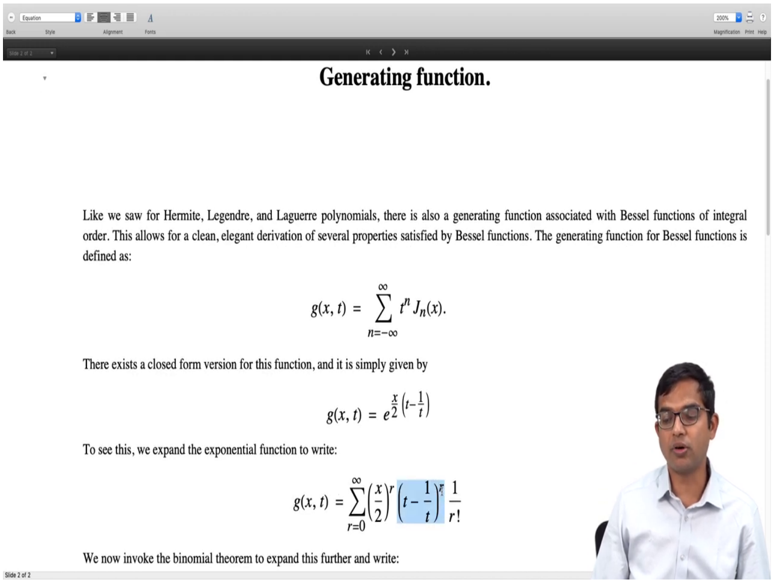
Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Bessel Functions
Lecture - 59
Bessel functions of integral order: Generating function

Ok. So, when we were studying Hermite polynomials, Legendre polynomials, Laguerre polynomials, we saw that the idea of the generating function, it is quite useful and so, it turns out that Bessel functions of integral order also have, there is a its possible to write down a generating function you know as a series expansion and in fact, there is also closed form expression available right so, which we will show in this lecture. And also, show an example of how this can be a; can be used to extract some properties of Bessel functions ok.

(Refer Slide Time: 00:57)



So, what do we do when we want to come up with the generating function? We create this series in terms of this you know variable t in the what we are calling t here, it is really a you know a dummy variable in some sense, we it get summed from minus infinity to plus infinity, all the coefficients of this series expansion are really the polynomials or you know the functions of your of interest.

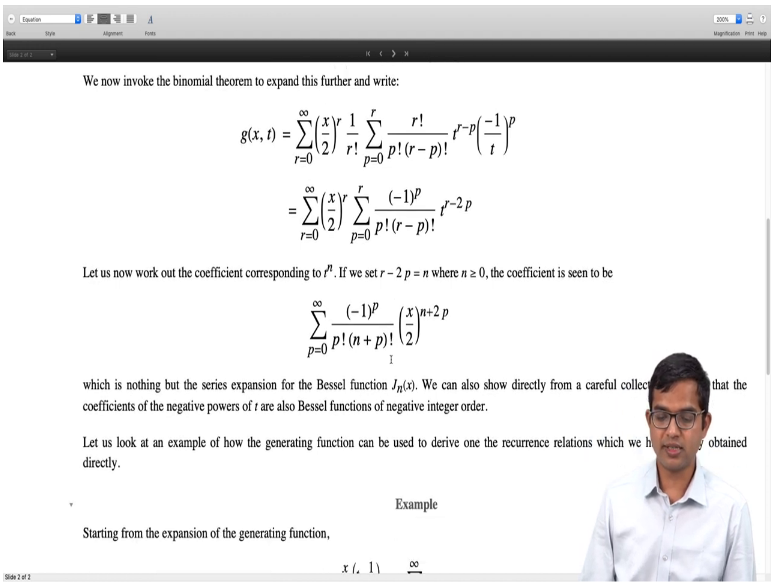
In this case, its these are not polynomials, these are you know each of these functions J n of x ms them themselves are defined in terms of an infinite series, but I mean so, if you tag them along, although J n of x in general itself you know does not have this sort of closed form expression available, it turns out quite remarkably that in fact, you can find a closed form expression for this series and that turns out to be just simply given by this expression e to the x by 2 times t minus 1 by t right. So, this is the generating function for the Bessel function of integral order.

Now, to see this right, I will sort of sketch the argument. I mean all the details are not going to be included. So, the idea is simply to expand right. So, you have an exponential of some stuff so, it is easy to go ahead and expand this.

So, we have this expansion r going from 0 to infinity, you know whatever stuff you are taking the exponential off, you are going to take the power you know of r and divide by r factorial x by 2 the whole power r times t minus 1 by t the whole power r the whole thing must be divided by r factorial and r will go from 0 to infinity. So, this is the expansion of just the exponential function.

Now, we also have this stuff involving t minus 1 by t the whole power r that itself can be expanded using the binomial expansion.

(Refer Slide Time: 02:53)



So, we have g of x comma t is equal to summation over r to going from 0 to infinity x by 2 the whole power r, we leave all the stuff as it is and then, you know this 1 by r factorial we write down immediately after this and in place of this t minus 1 over t to the whole power r, we have this expansion with p going from 0 to r, r factorial divided by p factorial times r minus p the whole factorial.

So, the first term here is t so, t to the r minus p and then, we have minus 1 by t the whole power p right. So, these are like the two terms in your binomial expansion.

And now, it is a matter of some bookkeeping right. So, the idea is ok let us first rewrite this as you know summation over r going from 0 to infinity is as it is x by 2 the whole power r as it is, r factorial cancels, then summation over p going from 0 to r minus 1 to the p divided by so, this minus 1 to the p divided by p factorial r minus p the whole p factorial times so, this t to the r minus p minus p, it will become this t to the r minus 2 p.

So, now, we see that we have an expansion in terms of you know in powers of t right. So, you can see that you know r minus 2 p can take both positive values and negative values. So, you can think of this expansion as really an expansion in powers of t where you know you can think of some coefficient you know times t to the n and n going from all the way from minus infinity to plus infinity.

So, what needs to be done is to extract the coefficient of t to the n where n can be positive, or you know 0 or positive and n can be negative. So, these are two separate cases one has to treat these two cases separately in order to do it carefully. If you do it, then in fact, we can show that the coefficient corresponding to t to the n is nothing, but J n of x, this will come out from the series expansion definition of the Bessel function itself right.

So, we have to assume that you know this swapping of these summations involved and all of this is legitimate. So, it is by no means a rigorous proof the way we are doing it, but basically you know it is a plausible argument let us say and indeed, it is true. So, you can just have sort of an argument for why this works out and then, you can play with you know the generating function so obtained to derive many properties of Bessel functions which are indeed true and which we have already obtained using other methods.

So, now, let me quickly sketch the argument for when t to the n, when you are doing this bookkeeping exercise for t to the n where n is greater than or equal to 0 and then, I will allow you to fill in the details for n is n being negative.

So, the idea is you know so, if r minus 2 p is equal to n where n is positive, then we immediately see that the coefficient corresponding to t to the n can be written in terms of you know we have make the substitution r minus 2 p equal to n so, r minus p is going to become n plus p. So, in place of r minus p, you get an n plus p, in place of p remains as it is minus 1 to the power p remains as it is.

And then of course, p will now go all the way from 0 to infinity right. So, I mean you are assuming r itself to take all values from 0 to infinity. So, it is a bookkeeping exercise so, then r x by 2 the whole power r is n plus 2 p. So, but this expansion is actually nothing, but the expansion corresponding to Bessel function of order n. So, this is immediately identified to be J n of x right.

So, when n is negative, it is we have to be a little more careful, but it is just a matter of spending maybe 5 or 10 minutes and convincing yourself that indeed this works out right. So, if n is negative, then so, in place of so, once again you will put r minus 2 p is equal to n where n is negative.

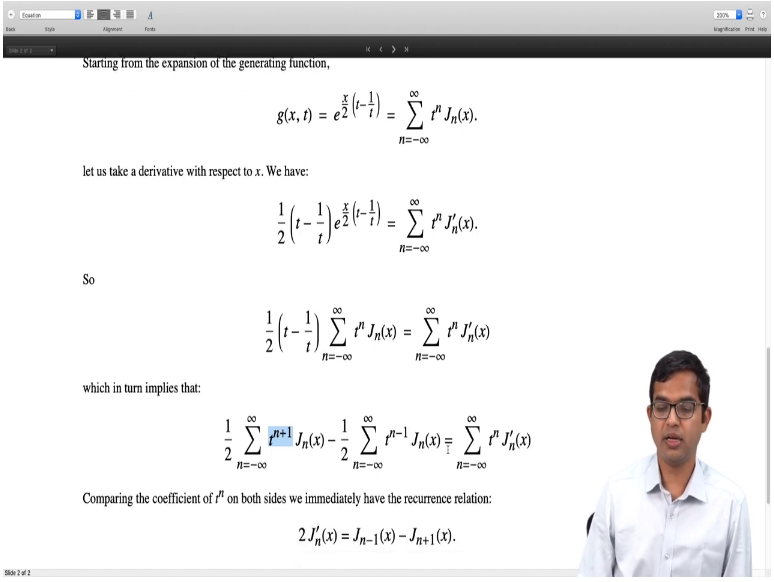
So, then we see that this denominator r minus p itself cannot become negative because if it does then, you will have 1 over so, the factorial of a negative number will give you infinity and therefore, you will have to stop r minus p from becoming negative and then, you in fact, your summation will not start from p equal to 0, but it will start from actually p equal to mod n right.

So, you know r minus 2 p is equal to n so, r minus p is equal to p plus n right. So, if r minus p must be greater than or equal to 0 so, then, you will see that n plus; n plus p so, r minus p must be greater than or equal to 0 so that n plus p must be greater than or equal to 0.

So, then, so, you will get constraints on the lower end of p which will turn out to be mod n and then, there is a way to argue that in fact, you can extend, you can add a bunch of zeros it is like a padding that you can do and then, you can carefully you know show that indeed J you know these powers will turn out to be these Bessel functions with negative integer order.

So, I will not go into all those details so, let us say that this is indeed true, it seems plausible, it definitely seems to work out for positive n and it also works out for negative n. So, let us quickly see an example of how to use this to derive a result which we have already derived using the series expansion.

(Refer Slide Time: 09:14)



So, let us start with this expansion. Assume that this is indeed the generating function so, g of x comma t is equal to e to the x by 2 times t minus 1 over t is equal to summation n going from minus infinity to plus infinity t to the n J n of x and if you take a derivative on both sides with respect to x right so, this is what generating functions can do for us right.

So, we take derivatives with respect to x sometimes, it is convenient to take derivatives with respect to t and then, you know there is this assumption of uniform convergence which is true for Bessel functions and so on. And you can do term by term differentiation.

So, therefore, here you see that in place of on the right-hand side, you have a summation over n t to the n J n prime of x and on the left-hand side, you have half times t minus 1 by t times e to the x by 2 times t minus 1 by t, but e to the x by 2 times t minus 1 by t is nothing, but this generating function itself which has this expansion.

So, we might as well rewrite the left-hand side as half times t minus 1 by t times the same Bessel function expansion is equal to now, this expansion in terms of the derivatives and now, the left-hand side is basically made-up of two terms so, this t will make it t to the n plus 1 J n of x and there is a half of course, and then, a minus a half times t to the n minus 1 J n of x and n summation from minus infinity to plus infinity.

Now, it is just a matter of comparing terms on both sides, not comparing the coefficient of t to the n on both sides if you do that, you know first of all we will multiply throughout with 2 so, the right-hand side will become 2 times t to the n J n prime of x summation over n and then, we have to just do some bookkeeping involving t to the n. So, for the first time corresponding to t to the n will appear when so, when you have J n minus 1 and here, you will get J n plus 1.

And so, immediately, we see that actually 2 times J n prime of x which is a coefficient of you know t to the n on the right-hand side after you have multiplied throughout by 2 and on the left-hand side, you get J n minus 1 of x minus J n plus 1 of x. So, which is a recurrence relation which we already worked out for you know when for any order nu, it does not have to be integer right.

So, this is an illustration of how the generating function can be used and I mean we have seen other examples involving Hermite polynomials, Laguerre polynomials where you know many other results are obtained with great ease and so, in that context that this is also its useful to also obtain the generating function for Bessel functions ok that is all for this lecture.

Thank you.