Mathematical Methods 2

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Bessel Functions
Lecture - 56
Bessel functions: series definition

Ok starting with this lecture, we start a new topic. Although it is quite connected to the previous topic, it is a slightly different topic. So, we have been looking at orthogonal polynomials, we looked at several properties. In this set of lectures which are starting from this one, we will look at Bessel functions which have properties sort of similar to orthogonal polynomials except that they are not polynomials right.

So, we will define Bessel functions with the help of a series expansion, and look at various properties right. So, like with orthogonal polynomials, we will work out the differential equation as we go along rather than you know start with the differential equation, and then work out properties ok.

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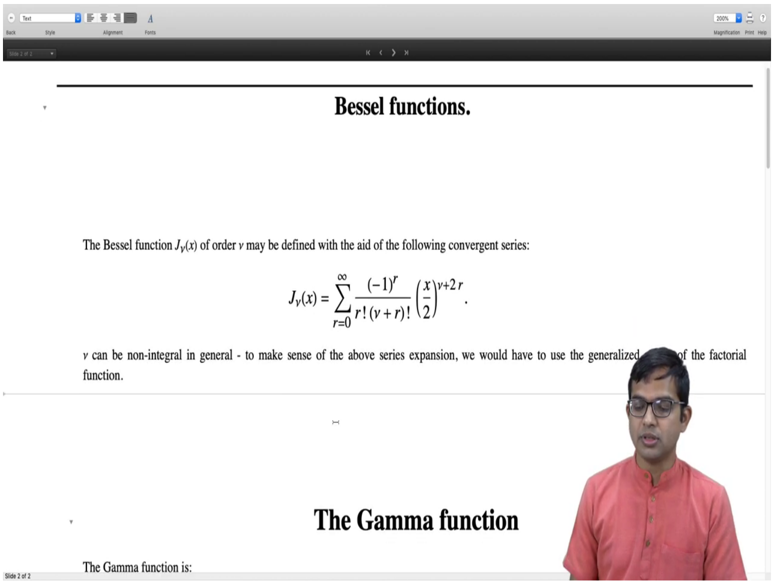

So, the Bessel function is you know can be defined as with the aid of this convergent series right. So, a Bessel function of order nu is defined like here. So, it is this infinite series r going all the way from 0 to infinity minus 1 to the r divided by r factorial nu plus r the whole factorial times x by 2 to the whole power nu plus 2 r right so.

So, now, nu can be non-integral in general right. So, for our purposes, we will take x to be a real variable all right. So, generalizations in which complex variables are allowed are also possible. And many of these properties can be you know worked out with great you know there is a lot of beauty associated with this. But we will restrict ourselves to real variables, and we will work out some properties of Bessel functions.

Now, this series definition allows for nu to be non-integral values. And so we have to make sense of this factorial right. So, we have a factorial sitting here r factorial is all right there is no issue, but if nu plus r the whole factorial must be a you know sensible idea. We must make use of the generalized notion of the factorial function which perhaps some of you are familiar with, but let us sort of use this opportunity to discuss the gamma function.

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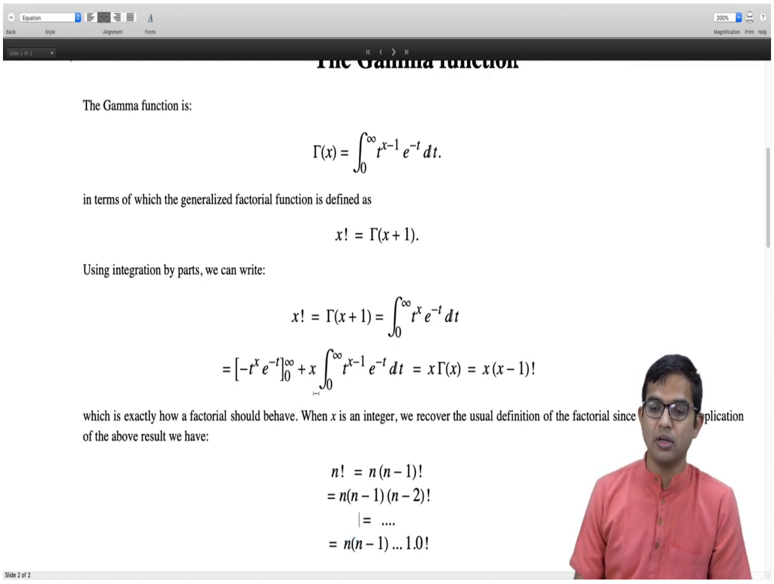

So, the gamma function is defined as this integral right so which again you know one can bring in a complex variable treatment of it you know define gamma in terms of as a function of a complex variable. And so there is a lot of very interesting mathematics you know to work out the properties of the complex function. But for our purposes we keep it simple.

So, let us think of the gamma function as this integral where x is some real variable. And so this integral goes from 0 to infinity, t is this dummy variable t to the x minus 1 times e to the minus t dt. So, why are we talking about the gamma function when we want to come up with an idea to generalize this factorial function right. So, we will see that in fact the gamma function is a kind of a factorial function.

So, in fact, we could define x factorial where x is an arbitrary real number. In fact this can be extended to also complex numbers, but x factorial can be defined as gamma of x plus 1 right. So, the way to see this is you know first of all we can argue that whenever x is an integer, positive integer, it will reduce to our familiar notion of a factorial. And also we will see how you know a key property that the factorial function satisfies holds even when this extension is made.

So, to see this, we will just integrate by part. So, x factorial is defined as gamma of x plus 1 which is the same as this integral from 0 to infinity t to the x times e to the minus t dt right. So, we are looking at x plus 1, so this has become t to the x.

Now, if you integrate by parts, so e to the minus t is the function whose integral we know it is just minus e to the minus t, so that comes out first. So, minus t to the x e to the minus t from 0 to infinity plus because there is this minus sign so that becomes a plus when you differentiate t to the x you get x times t to the x minus 1. So, 0 to infinity t to the x minus 1 e to the minus t dt.

And then we argue that you know at the lower end, so this is going to vanish; and at the higher end, this quantity is going to vanish. So, basically this boundary term is 0 at both ends. So, basically this x factorial is equal to gamma of x plus 1. And then we read off from here that you know this is really nothing but gamma of x. So, x factorial is seen to be x times gamma of x.

But gamma of x according to this definition is x minus 1 factorial. So, in fact, we have this you know result that x factorial is equal to x times x minus factorial when it is defined in this manner. And so this is basically how factorials behave right. So, therefore, it is a very reasonable definition.

And we can also quickly check that in fact whenever x is an integer a positive integer you know n factorial according to this rule will become n into n minus 1 factorial which in turn can be written as n into n minus 1 into n minus 2 factorial so on all the way up to 1 and then 1 itself 1 factorial itself can be written as 1 times 0 factorial.

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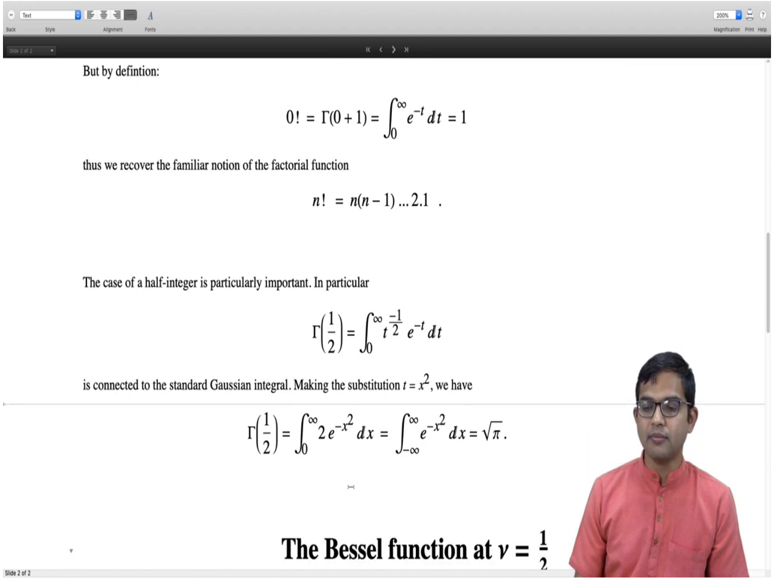

Now, 0 factorial, if you plug in into this definition is nothing but gamma of 0 plus 1 which is gamma 1 which is just this integral 0 to infinity e to the minus t dt which is seen to be 1. So, 0 factorial is 1. So, we recover exactly what we have for the factorial of a positive integer which is just n factorial. It is just n into n minus 1 into n minus 2, all the way down to 1 ok that is all good.

And so there is this particularly important case of a half integer. When, whenever you have a half integer I mean if you have gamma of n by 2, where n is some positive integer that you can write it as in terms of you know gamma of n minus 1 by 2 which in turn you can write it as in terms of gamma, so n by 2 minus 1, then n by 2 minus 2 and so on.

So, eventually you will come down to gamma of a half. So, we will work out this very special integral which is gamma of half which is connected to Gaussian integrals. So, it is important to you know it is a result that one should remember gamma of half, we will work it out now, integral from 0 to infinity t to the minus a half e to the minus t dt is connected to a standard Gaussian integral which can be seen by making this substitution in place of t you put t equal x square, and gamma of half is 0 to infinity.

So, t to the minus a half will become x squared to the minus a half which is x to the minus 1. But then when you do dt, you get 2 x dx. So, the x will cancel. And then you are just left with 2 times e to the minus x squared dx. But 2 to the times e to the minus x square d x - you might as well write it as from minus infinity to plus infinity this function is an even function.

So, you can write this as you know in place of 2 times this integral from 0 to infinity is the same as going from minus infinity to plus infinity, this Gaussian integral we know this result to be just square root of pi right. So, therefore, gamma of half is the square root of pi. And so this sort of detour into a discussion of the gamma function allows us to treat the Bessel function with index nu equal to half which is also a special case, and which we can actually work it out in terms of a familiar function right.

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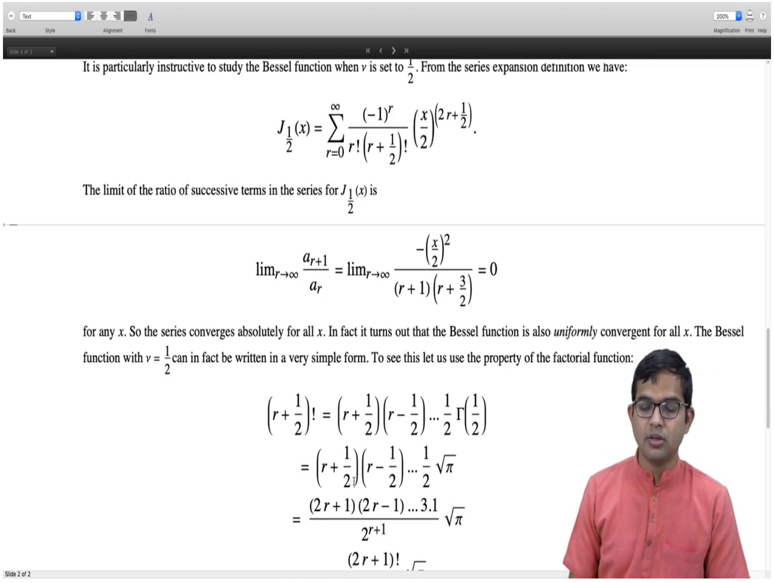

So, let us look at what is j half of x. So, by plugging in this series expansion, we have the series. And so in place of nu, I have put half so that appears here, and this appears here. And so now the first thing to observe here is you know there is this ratio test. So, if you take the ratio of successive terms, you get this result. And take the limit of r going to infinity, so this goes to 0.

So, basically this tells us that this is an absolutely convergent series right so which is I mean which is a fact that I already sort of mentioned. But here for this particular case you can check by using the ratio test. And this can also be used to check the convergence for other nu’s as well

So, in fact, it is not only absolutely convergent, but in fact the series is in fact uniformly convergent. What does it mean? I will not get into the details of how the argument works. Basically what it means is you know in a neighborhood x you know j j nu of x is a function of x.

So, if you if it is convergent at a point, it is going to be convergent in a whole neighborhood around that point in basically the same way right in the sense of you know you can get as close to the value that you want the function is going to converge to you know by truncating it at a higher level. And so that truncation can be done in a sort of a uniform way if you are in some neighborhood hood around that point.

So, essentially what it implies is that you can take derivatives of such a function, and then do term by term differentiation you can do term by term integration and so on. So, this is true in general for any nu not just nu equal to half, but we sort of check this ratio test with for nu equal to half. And then let us see you know I said that j half of x is special.

So, let us look at this. So, we have this result r plus half the whole factor from our earlier discussion of the gamma function is just r plus half times r minus half times all the way down to gamma of half so which is nothing but gamma of half is square root of pi.

So, and then we pull out these 2s and then we rewrite the numerator as 2 r plus 1 times 2 r minus 1 so on all the way up to 1. So, all the odd numbers from 1 to 2 r plus 1 are covered. And the denominator is 2 to the r plus 1 as you can check there is also square root of pi hanging around

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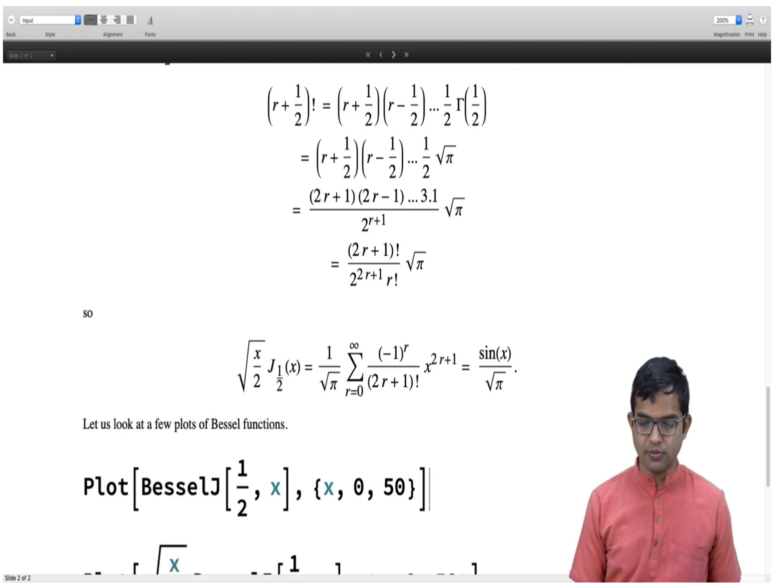

So, what we can do is you know, fill in these gaps in the numerator. So, you can put a 2r plus 2, then you can put a 2r, 2r minus 2, 2r, there is no 2r plus 2, but 2 2r 2r, 2 r minus 2, so on all the even numbers all the way down to 2. And then divide by the same stuff in the denominator which you can check you can pull out all those you know 2s in the denominator.

And so you will be left with just this r factorial as far as the denominator is concerned. But you also have a 2 to the r right. So, every one of those r numbers will give you a factor of 2. So, you have a 2 to the r which you can combine to write it as 2 to the 2r plus 1 in the denominator.

So, basically r plus a half the whole factorial is the same as this whole stuff times square root of pi right. And so in our expression for you know we have this expression. So, if we multiply throughout with square root of x by 2. You will see in a moment why we are doing this.

So, if we do this, and then we can rewrite this r plus half factorial in terms of this whole stuff, so you get this 2r plus 1 whole factorial. And then there are all these cancellations, you get a 1 over square root pi. And then because you multiplied the square root of x by 2, you get you know, so this half will become 1. So, then you get an x to the 2r plus 1.

And then you can check that all of these cancellations allow you to rewrite the series in this form which is actually a familiar series minus 1 to the r divided by 2 r plus 1 the whole factorial, r going from 0 to infinity times I mean there is also this x 2 the 2r plus 1. So, which is a familiar series, it is nothing but the series expansion for sin of x.

So, what this tells us is that the square root of x by 2 times j half of x is actually nothing but sin of x divided by square root pi. So, this particular function is very special. So, let us just quickly look at some plots of Bessel functions.

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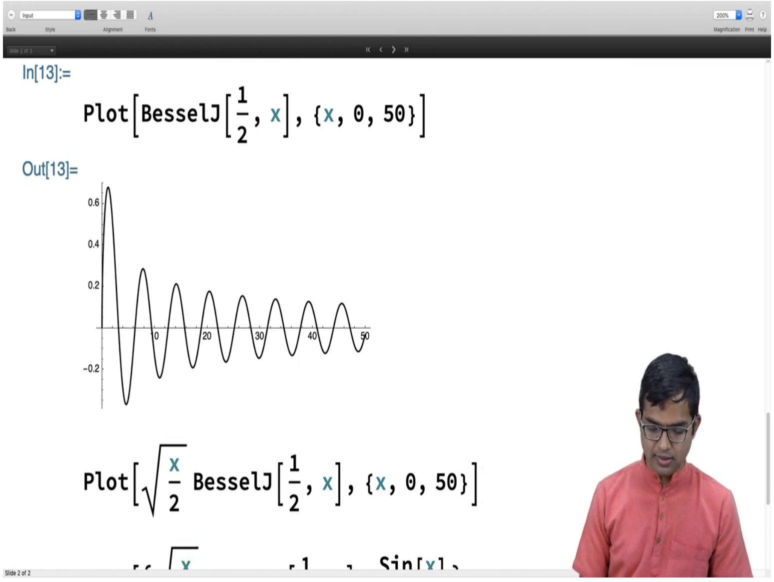

So, you know in particular if I look at this Bessel function plot for when nu is equal to half, you see all these oscillations and there is a decay right. So, often Bessel functions show up whenever there is oscillatory behavior, but also there is some kind of decaying behavior. They appear in all kinds of context, very familiarly encountered in physical problems. So, it is definitely worth being familiar with some of these properties of Bessel functions.

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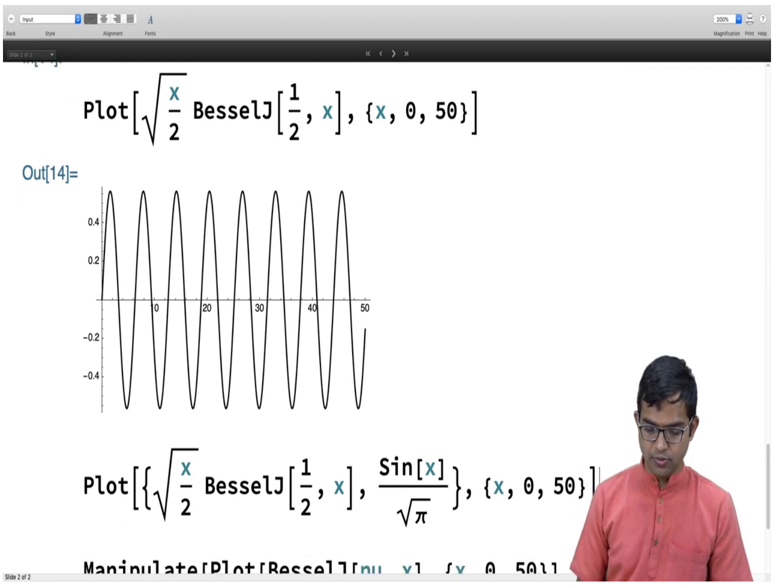

So, we can check that if you multiply by square root of x by 2 and then look at this plot it looks perfectly periodic and there is no decay.

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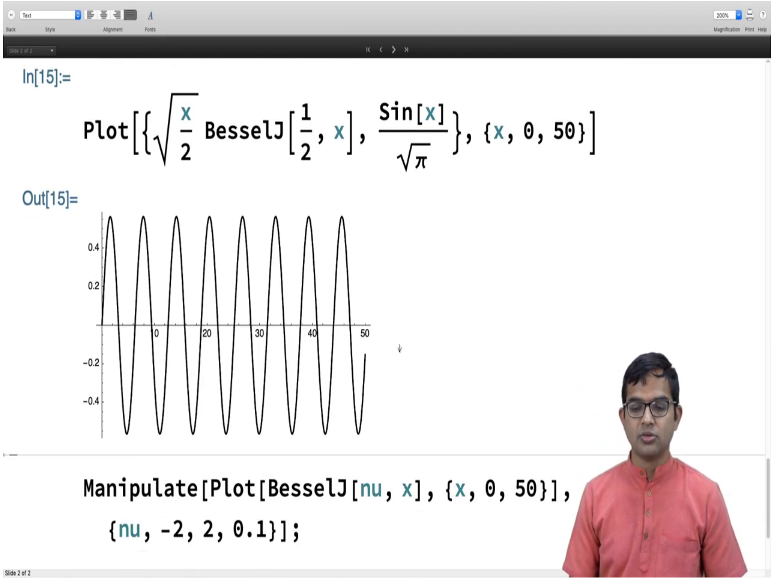

And in fact if we super or if we plot simultaneously on the same graph also sin of x divided by square root of pi, we see that the two curves overlap. You do not see any distinction between the two. So, indeed as we have derived you know how these two are the same. So, we can plot them, and we check that indeed they are the same.

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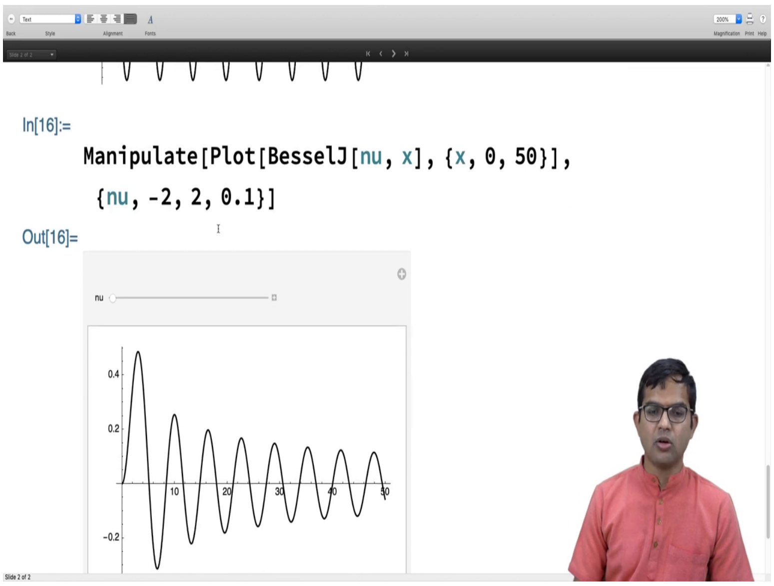

So, we also have this you know way of looking at Bessel functions. So, you can also take negative values for nu.

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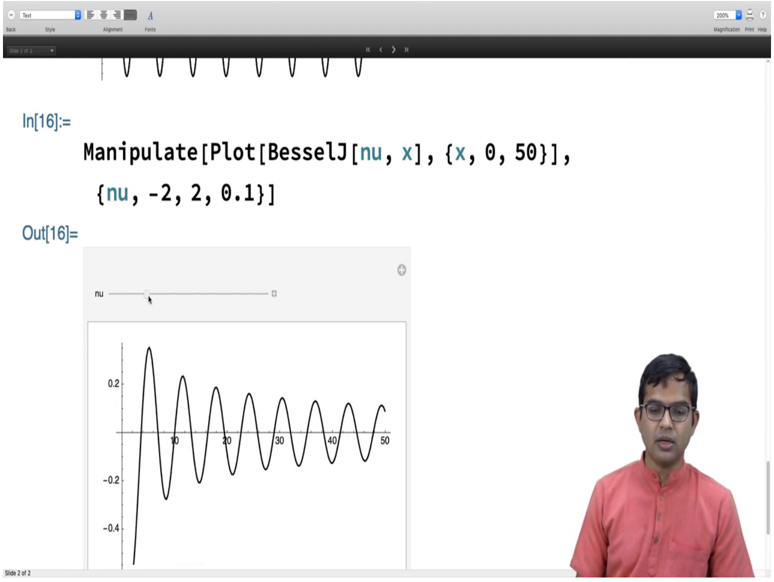

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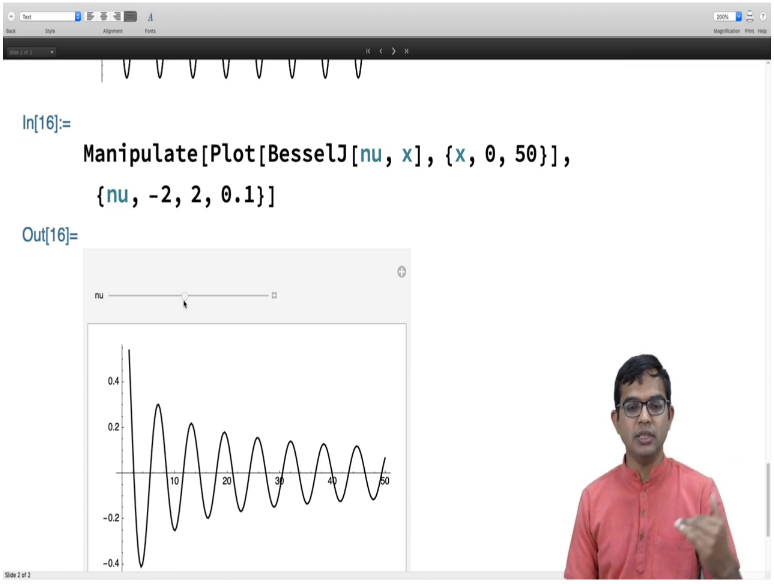

So, you see that as you change nu the Bessel function you know tends to remain basically a periodic kind of function, there is oscillatory behavior, but also with decay right.

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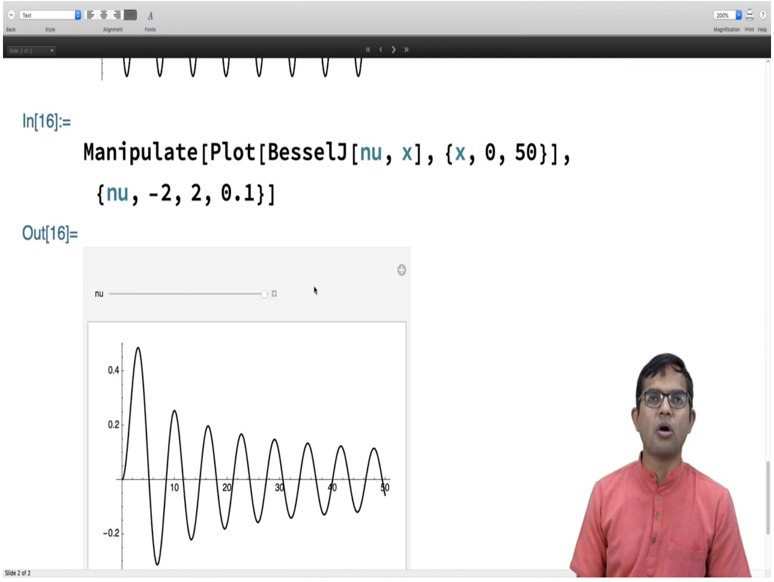

So, you see you know as you change nu there is this kind of structural change. So, this is just a sort of a broad you know look at what Bessel functions are. We will look at more properties in the following lectures.

Thank you.