Mathematical Methods 2
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Orthogonal Polynomials
Lecture - 54
Laguerre polynomials: differential equation

Ok, so we resume our discussion of Laguerre polynomials. So, in this lecture, we look at how the differential equation corresponding to the Laguerre polynomials can be obtained by a method similar to what we have done in the earlier two cases invoking the general prescription.

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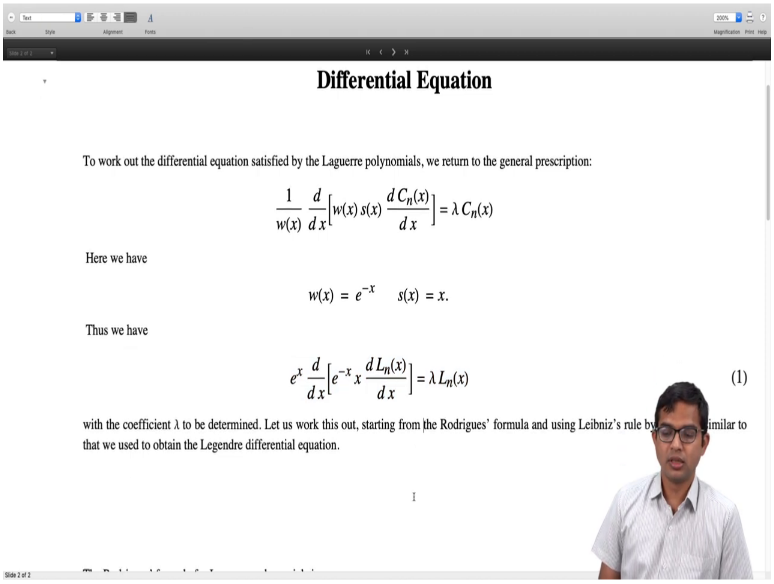

So, the idea is to look at this really what is an Eigenvalue equation right. So, there is this operator acting upon some Eigenfunction which we have shown necessarily must give back the same Eigenfunction with some Eigenvalues. So, this Eigenvalue needs to be determined right so that it is specific to the kind of polynomials that we are looking at.

But if you know take this combination of quantities and take their product and take the derivative and then divide by this weight function, then you know we have shown the theory is that this necessarily must be the same polynomial itself with some factor lambda ok.

So, let us look at how to extract this for the Laguerre polynomials. So, the Laguerre polynomials the weight function is this exponential e to the minus x, s of x is equal to x and so it has an important role at the lower end at x equal to 0. So, it is you know this ensures that the function goes to 0 at x equal to 0. And this will ensure that you know it is all good at the other end at plus infinity.

Now, so we have to consider this combination of factors e to the minus x times x times the derivative of Laguerre polynomial of the first derivative, and then you have to take another derivative with respect to all of this stuff, and then multiply by e to the x because you are doing 1 over w of x, then this must be equal to lambda times ln of x right. So, lambda is a coefficient that needs to be determined right.

So, in fact, we will verify that this holds basically right we will directly work it out from Rodrigues' formula. And in fact, in the process, lambda also will become clear right. I mean in some treatments you know you start with this sort of Eigenvalue equation and look for Eigenfunctions and Eigenvalues.

So, but in general like we have said in the past with respect to other different other polynomials you know if you look for Eigenfunctions, you know solutions to what basically this is a differential equation where lambda is not one of these Eigenvalues, then you will still get solutions but they will turn out to be not polynomial.

There will be some series solutions at which I mean after all in our case we are interested in the polynomial, then if we have constructed the polynomials, now we are working out the differential equation.

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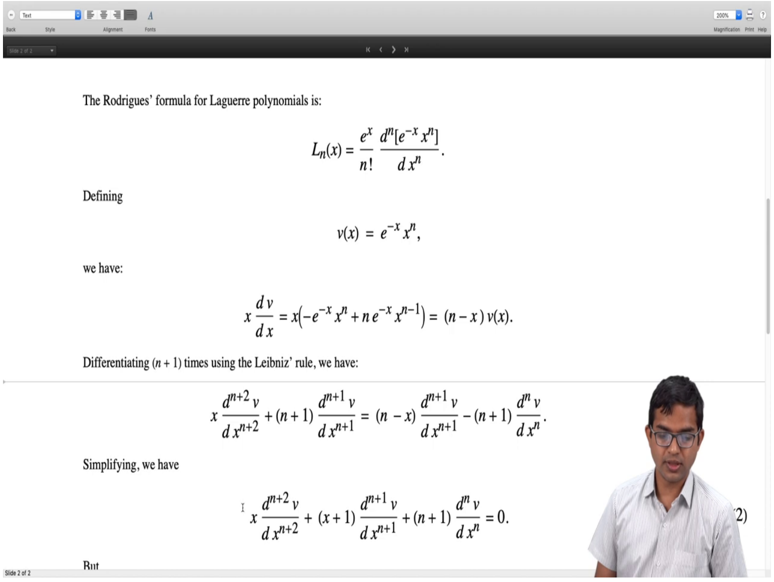

So, to do this, we will go back to the Rodrigues' formula. Rodrigues' formula is very simple for Laguerre polynomials. This is just e to the x time times the nth derivative of this product e to the minus x times x to the n, then the whole thing must be divided by n factorial, so that our normalization condition works out. And now we look at this function v of x is equal to e to the minus x times x to the n.

So, if you want to take the derivative right, so we have seen that I mean let us work this out d dv by dx is equal to e to the minus x times minus 1, so that is this term here minus e to the minus x times x to the n plus n times x to the n minus 1 times e to the minus x that is the second term. And then it is convenient to multiply throughout with x, so that is this x.

And then we observe that basically so the reason we multiply with this x is to fill in this you know gap that is created here in some sense x to the n minus 1 is you know if it is multiplied by x you can connect it back to v, so that is the idea. So, x times x to the n minus 1 will become just x times e to the minus x and then x x to the n times e to the minus x which will connect it back to this v of x.

So, then we have you know this part will be just minus x times and this stuff is again v of x, and then this stuff also will become v of x, and so effectively it is just n minus x times v of x. So, now the idea is to use a Leibniz rule right. So, like it is a very similar trick that we used in the you know the previous case right. So, now, we want to differentiate it n plus 1 times using the Leibniz rule right.

So, if you use Leibniz rule, we have um you know the first so I mean indeed this is a product of two functions, you have n minus x and then v of x right, but n minus x is a is just a linear polynomial, it is just a linear function in x. Therefore, there are going to be only two terms: either you differentiate it or you do not differentiate it right. So, you probably do not even need to know the full Leibniz rule here.

It is just a matter of getting these two terms. If you do not differentiate it with respect to this at all, in fact, you differentiate all of these with respect to v of x right. So, then you have x, so that is x times d to the n plus 2 times v divided by dx, well, I mean that is the right hand side.

So, let us do it step by step. So, the left hand side is so x times dv by x. So, if you differentiate all of these with respect to only the second one, so then you get x times the n plus 2 derivative of v. Or if you know if you have the chance to differentiate with respect to x 1s, but where you do it you have n n plus 1 different ways of choosing that?

So, you have this n plus 1. And then I mean of course the answer is one when you differentiate it. And then the other n, n times you are going to differentiate with respect to v with respect to the second of these factors which is dv by dx. So, you get d to the n plus 1 divided by dx n plus 1 v.

And then we have you know this option of either differentiating with respect to n minus x or not right, so that is. So, if you do not do anything but differentiate all of these with respect to v, then you get n minus x times d to the n plus 1 by dx to the n plus 1 times v. And then if you do it once, then you get a minus 1 and then you also have this factor n plus 1, and then you get to differentiate with respect to v only n times right.

So, that is there are two terms both on the left hand side and there are two terms both on the right hand side as well. Now, you just collect all these terms, bring all these things to the left hand side, and so the right hand side is just a 0. So, this first term is as it is x times the n plus 2th derivative divided by so the n plus 2th derivative of v remains as it is. Then we have n plus 1th derivative. So, you have an n plus 1 on the left hand side, and then we have an n minus x on the right hand side.

So, n and n will go. And so you are left with x plus 1 times n plus 1th derivative of v with respect to x. Then we have this other extra term which will just come to the left hand side as n plus 1 times the nth derivative of b with respect to x.

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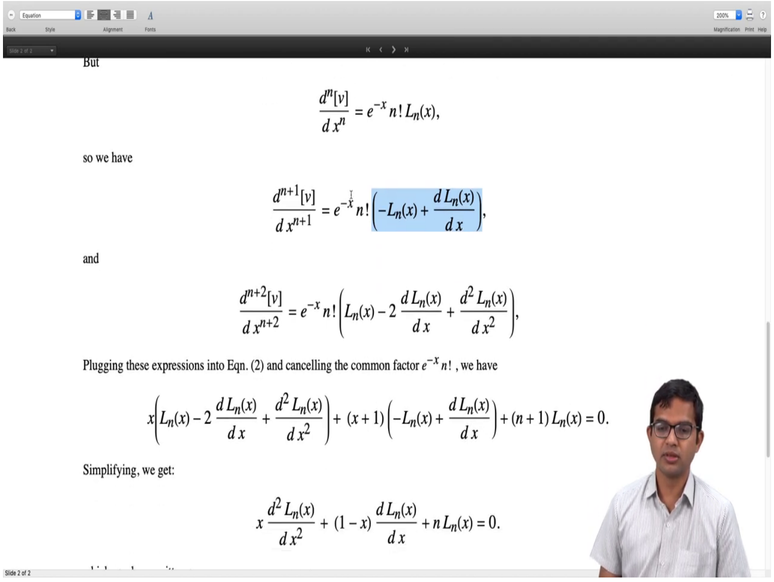

Now, we know that the nth derivative of v right after all v is this yeah you know product of these two functions which we already know corresponds to the Rodrigues' formula, it appears in the Rodrigues' formula this very product. And so the nth derivative of this is in fact nothing but the Rodrigues' Laguerre polynomial itself except for these two factors.

So, if you bring these factors back in, we can rewrite this in this manner. So, now, we have to be a bit more careful because there is also this function e to the minus x sitting here. So, the weight function is a bit more non-trivial in this case, it is not just 1. So, if it were only n factorial it will just cancel throughout, but now you have to be careful.

And so when you take a derivative of this function n plus 1, so you get e to the minus x times n factorial times minus Ln of x right, so that is when you are differentiating with respect to this, or you get another plus d Ln of x by dx.

And then if you take another derivative, it is the same type of argument e to the minus x n factorial stays as it is. But now this minus Ln of x could, well, it one term is when it becomes plus because you are differentiating with respect to this. Or, then you have you know you could either get a derivative with respect to this that will add with this and give you minus 2 d Ln of x by dx.

And then you also have the third option which is to differentiate with respect to with respect to this, then that is going to give you d squared Ln of x by dx square right. So, you can check this right. So, and convince yourself that indeed it is reasonable what we have done right.

So, now the idea is to plug these expressions back into equation 2. And then we cancel this common factor of e to the minus x times n factorial, and we are left with x times Ln of x minus 2 times 2, so that is just this, this whole stuff right. So, we have to do x times this whole stuff plus x plus 1 times just this stuff minus Ln of x plus the first derivative of Ln plus n plus 1 times just Ln of x equal to 0.

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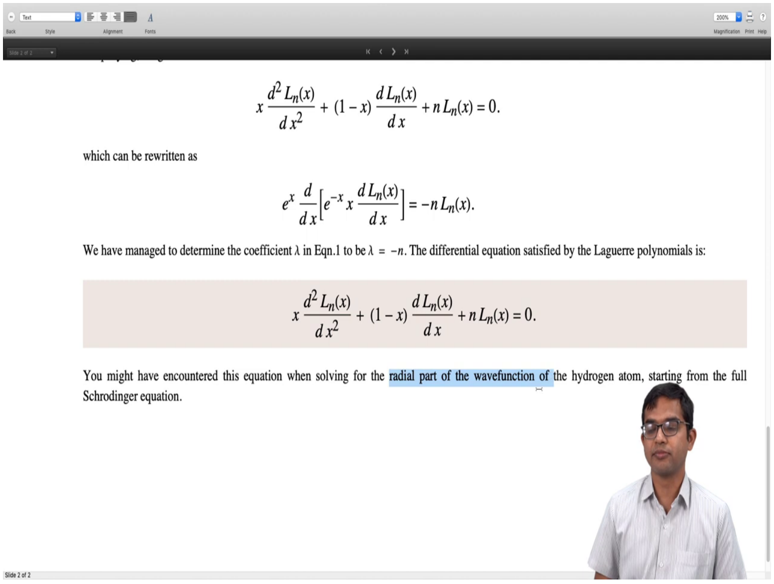

Now, it is a matter of simplifying. And so we see that the first we pick the highest order derivative which is x times d squared Ln of x divided by dx squared plus we get 1 minus x. So, how does that happen? So, we have d Ln of x by dx. So, we have a minus 2 x, but then we also have a x plus 1, so x plus 1 minus 2 x will become 1 minus x times d Ln of x by dx. And then we have this n plus 1 sitting here, but we also have a minus 1 times Ln of x. So, it is just plus n Ln of x which we of course rewrite as e to the x times d by dx.

Then we multiply e to the minus x times x times the derivative of Ln of x right. So, we are aware that it is always going to be possible to put this differential equation in this special form right because we have this general prescription. So, we just try to match it, and then we immediately see that the Eigenvalue in this particular case is actually minus n right. We have worked this out - it is a direct consequence of the Rodrigues' formula and some manipulations that we have done.

So, basically the we have managed to work out the differential equation of the Laguerre polynomials, and it simply turns out to be x times the second derivative of Ln of x plus 1 minus x times the first derivative of Ln of x plus n times Ln of x is equal to 0. So, this is a differential equation you might have already seen. And you might have seen some properties of the Laguerre polynomials you know considering this as your differential equation and trying to work out the solution because of this differential equation right.

So, the standard approach is I mean you consider a more general differential equation where in place of n you have some lambda. And then you argue that if lambda is equal to n, then you get polynomial solutions. And those polynomial solutions turn out to be the Laguerre polynomials, and then you work out these properties of Laguerre polynomials that is one approach right.

And you know the context in which Laguerre polynomials appear and are important are you know when you are studying the quantum mechanics of the hydrogen atom, and so it is the radial part of the wave function corresponds to Laguerre polynomials the angular part of course gives us the Legendre polynomials and associated Legendre polynomials.

In general, whenever there is some kind of spherical symmetry involved there are going to be Legendre polynomials that also appear in various problems in E and M right. But the context in which Laguerre polynomials appear is certainly you know this one familiar context is the radial part of the wave function of the hydrogen atom ok. That is all for this lecture.

Thank you.