Mathematical Methods 2
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Complex Variables
Lecture - 05
Functions of a complex variable and the notion of continuity

So we have looked at complex numbers and some of their basic properties and some applications, right. Most of the material that we have covered so far has been in the nature of recall; we have all presumably seen these ideas at some point. So, starting from this lecture, we will look at functions of a omplexvVariable, right. So, we are familiar with functions of a real variable and you know notions of continuity and differentiability and so on.

So, we will see that the extension of these ideas to functions of a complex variable result in some special constraints. They appear naturally when we are working with functions of a complex variable. So, we will discuss the notion of continuity in this lecture, ok.

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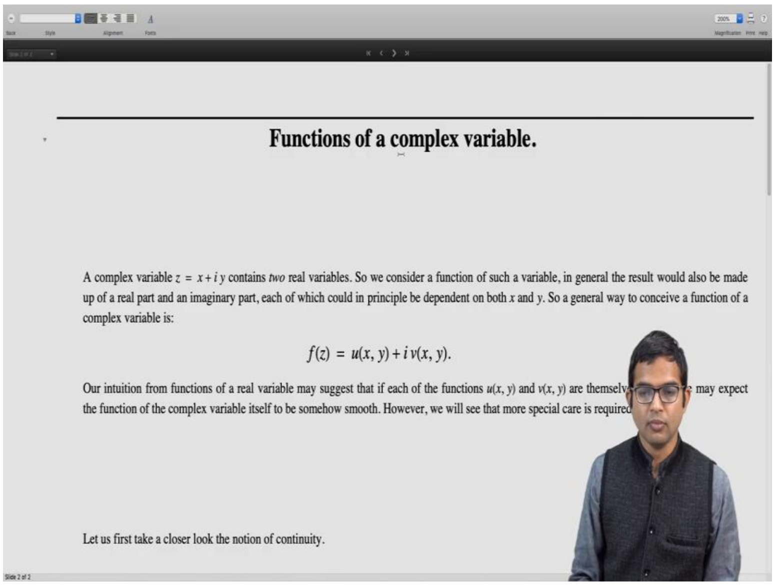

A complex variable has a real part and an imaginary part. So, it is made up of two independent real variables, right. So, if we want to make a function out of a complex variable, so in general; the resulting answer will also be a complex number. So, the the output of this operation would have a real part and an imaginary part.

So, it is customary to write it as you know f of z is equal to a real part which is itself is a function of both the real part and the imaginary part of the original complex variable and the imaginary part of the function f of z itself is a function of both x and y.

So, we write it as f of z is equal to u of x,y plus i times v of x,y, right. So, if u and v themselves are in some sense nice functions, we might expect that there is something smooth about the overall function f of z; so but we will see that special care is required. So, you cannot simply put in any arbitrary u and any arbitrary v and expect that the function f of z is going to turn out, with very nice properties, which we care about for such functions, right.

So, let us analyze this carefully. So, to do this we will first look at the notion of continuity, right. So, what is our intuition about continuity from functions of a real variable? So, it's the idea that there is nothing very jarring in the movement of the function; as you change the real variable, so the value that the function takes at nearby points is basically the same or small changes in the input will give you only small changes in the output if you wish to think of it like that, right.

So, x is your independent variable and so then y would be a dependent variable and so small change in x will result in a small change in y, right. So, that is the idea of continuity intuitively.

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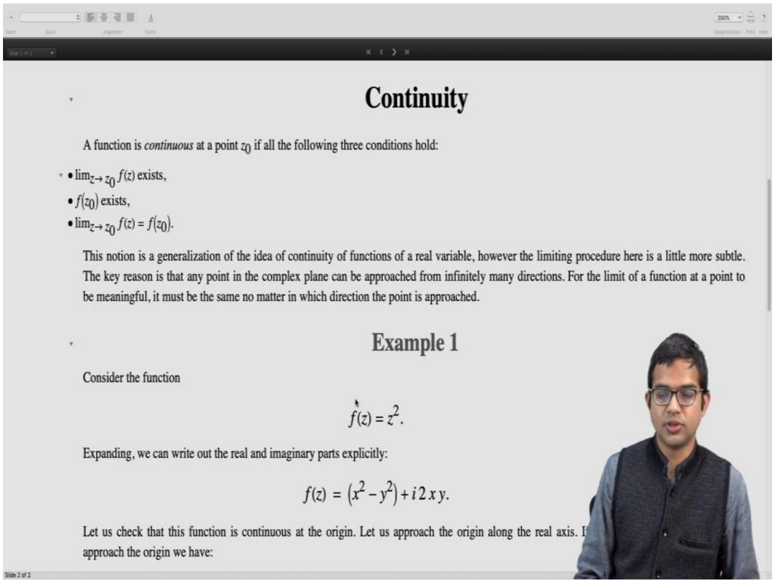

So, when you are looking at a function of a complex variable. We say that a function is continuous at a point z0 if three conditions hold. First of all the limit of this function f of z at as z approaches z0 must exist; f of z0 itself must be very defined and this limit z going to z0 f of z must be equal to f of z0, right.

So, this idea of taking the limit of a function is also something which requires some thought, when we are working with complex functions; because the limit here is something that you can approach a certain point z0 in infinitely many directions. So, if you are working with a real variable x, you can approach a point x0 either from the right or from the left.

And so, you would expect that the value of this function is the same, whether you approach from the right or from the left; but here it is a little more involved. So, the condition here is that, the limit should be the same no matter in which direction you approach this point z0 right, so if that is true, then the limit exists and not only is it enough for the limit to exist, the value of the limit must be equal to the value of the function itself at that point and then the function is continuous.

So, this seems like a very reasonable definition, requirement for continuity, right. So, let us look at a few examples. Suppose we consider a simple function like f of z is equal to z square, right. So, indeed intuitively we would expect this to be a continuous function at all points in the complex plane and indeed it is true. So, if you expand it out explicitly and write out the real and imaginary parts, right. So, you write it as x plus i y the whole squared; so the real part is going to be x squared minus y squared and the imaginary part is going to be 2 x y, so straight forward to work this out.

And so, if you check the continuity of this function at the origin; suppose you approach the origin along the real axis, right. So, you imagine taking a point like epsilon plus i 0, right. So, you are on the real axis; so it is only epsilon that is a variable, so i always has 0 along with it.

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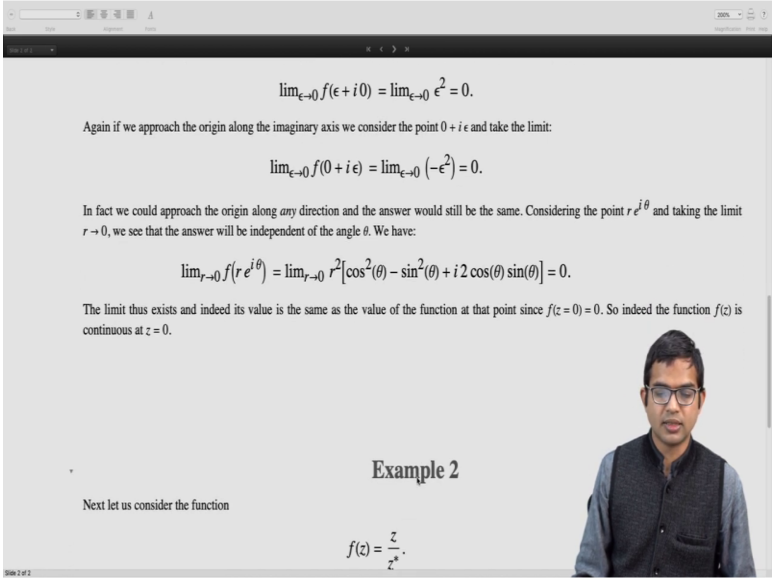

And so, the approach, as you approach the origin; so this is the limit that we must consider limit as epsilon tends to 0 of f of epsilon plus i 0, which is equal to limit epsilon tending to 0, just epsilon square, right. So, the argument is just simply epsilon and so indeed as epsilon goes to 0, this is going to go to 0.

So, again if you approach the origin, but along the imaginary axis; we would consider the point like 0 plus i epsilon, slightly away from the origin, but along the imaginary axis. And then you do the same exercise again limit epsilon tending to 0; f of 0 plus i epsilon which will now give you when you square it, you get a minus sign. But since you are taking the limit epsilon going to 0; it does not matter whether you have a positive sign or a negative sign, so the value is 0.

So, indeed you could have approached this origin in any of the other directions as well and you can convince yourself that it would still give you the same answer. So, to do that, you consider something like a displacement of r along a direction theta.

So, your z itself will be r times e to the i theta and then if you take the limit r going to 0; you will see that as you take the limit r tending to 0, this function will take this value r squared time cos squared theta minus sin squared theta plus you have i times 2 cos theta sin theta, right.

So, another way of seeing this is simply to write it as r squared e to the i to theta, right. So, that is this is just an expanded version of r squared e to the i to theta. And indeed as r goes to 0, it is going to be 0 regardless of which direction you approach it from. So, theta can be anything and you will still get the same answer. So, indeed the limit exists and it is, so the value of f of z at 0 is 0. So, we need the function f of z to be continuous at z equal to 0.

But you can verify that this function would be continuous everywhere right; by carrying out a similar exercise, but at a different point. So, let us look at another example. So, if you consider a function like f of z is equal to z divided by z star, right. So, we have seen that z star is the complex conjugate, right. So, the complex conjugate of a complex variable x plus i y, is z star equal to x minus i y.

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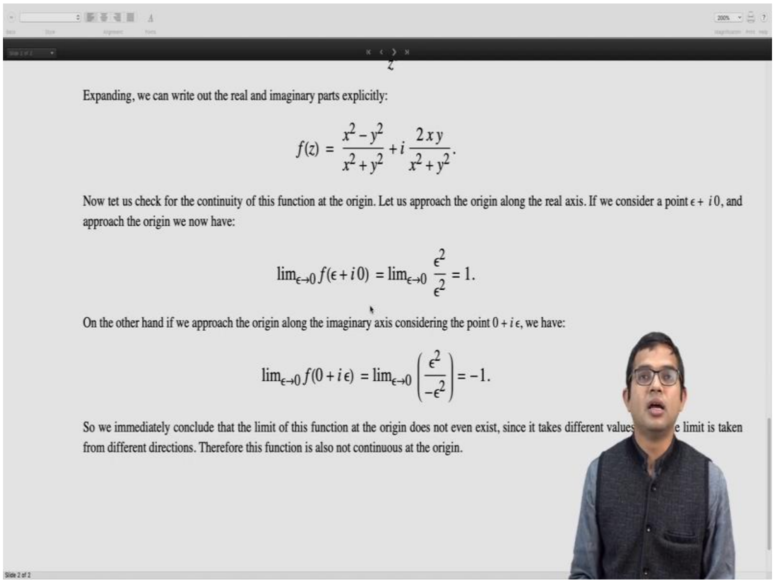

So, if you expand this out and write it out explicitly, you can write this as z times z squared divided by z times z star. So, multiply and divide the numerator and denominator, by z. So, then the denominator becomes mod z squared, which is the same as x squared plus y squared.

So, then the numerator has, is basically z squared which we just worked out so you have the numerator is x squared minus y squared plus i times 2 x y. So, you have the real part of this function is x squared minus y squared divided by x squared plus y squared and the imaginary part is 2 x y divided by x squared plus y squared.

So, now if you try to approach the origin, right. So, let us approach the origin for this function along two directions. Suppose we start on the real axis; if we consider a point like epsilon plus i 0 and we approach the origin, so this limit will be epsilon tending to 0 f of epsilon plus i 0, which will be just limit epsilon tending to 0, epsilon square divided by epsilon squared, right. So, in this case. So, the imaginary part is just 0, so if, so y is 0.

So, the imaginary part of this function itself immediately vanishes and then you are just left with epsilon square divided by epsilon square. So, as you tend epsilon to 0, it does not matter it just go to equal to 1. But on the other hand if you had approached along the imaginary axis, then you have to take this same limit epsilon going to 0; but of the function f of 0 plus i epsilon, 0 plus i epsilon means that y is you know epsilon, so x is 0.

So, you have once again the imaginary part is gone; but the real part now is going to be you know minus i epsilon squared will just give epsilon squared, but the denominator will give you minus epsilon square, right. So, and therefore, the answer overall in the limit of epsilon equal to 0 is just minus 1, right.

So, therefore, we see that, this limit of this function at the point z equal to 0, as you approach 0 from two different directions are different, right. It does not even matter, you do not have to consider some other direction; you could if you want, but you know establishing that any two directions, taking the limit gives you different values immediately tells you that, there is no continuity, the function is not continuous at that point, right.

So, this shows that we have to be careful with functions of a complex variable. So, we have looked at two examples; one a very simple benign type of a function, where indeed continuity is clear, we might intuitively expect it and indeed arguing carefully we have seen that it is continuous. And then we looked at another example where continuity is not possible at the origin and so, this we saw by looking at taking the limit carefully from two different directions and seeing that the values are different.

So, we will build on this notion of continuity and we will come up with ideas of what differentiability is and what conditions differentiability will impose upon the function and so on. So, that is coming ahead in future lectures, that is all for this lecture.

Thank you.