Mathematical Methods 2

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Orthogonal polynomials
Lecture - 49
Legendre polynomials: recurrence relation

So, in this lecture we look at some more properties of Legendre polynomials. Specifically, we will work out the recurrence relation that is satisfied by Legendre polynomials, ok.

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So, we have seen that the general method for figuring out this recurrence relation would be to first write down these three quantities s of x, w of x in a comma b which we know for the Legendre polynomial is the way we have set it up. So, s of x is x squared minus 1, w of x is just 1 and the interval of interest is minus 1 to plus 1.

So, the Rodrigues formula is P n of x is equal to 1 over 2 to the n n by times n factorial which is the normalization we chose in such a way that P n of 1 is the same for all n. And it is so, this normalization times the nth derivative of x squared minus 1 the whole power n right. So, we have also seen how it is convenient to pull out this minus 1 to the n factorial outside ok.

We can directly verify from this Legendre from the Rodrigues formula that in fact, Legendre polynomials are somewhat like the Hermite polynomial. We have also seen this visually that they have you know definite parity. So, for a given n you know depending upon if n is even or odd you would get even or odd parity.

So, if n is equal to 2 for example, you will get P n of minus x is equal to P n of plus x as you can directly verify by plugging in minus x in the Rodrigues formula. For example, if you put it here you will see that since you know this part has only x squared in it. So, it does not care about whether you have a minus sign or a plus sign and the denominator has an x to the n. So, you are taking a derivative with respect to dx to the n. So, you will get this minus 1 to the n.

So, therefore, depending on if n is even or odd you will get a minus 1 or a plus 1 here. So, in fact, you will get exactly like Hermite polynomials. You will as you increase n you are going to alternate between even and odd polynomials. Now, this has an immediate consequence right.

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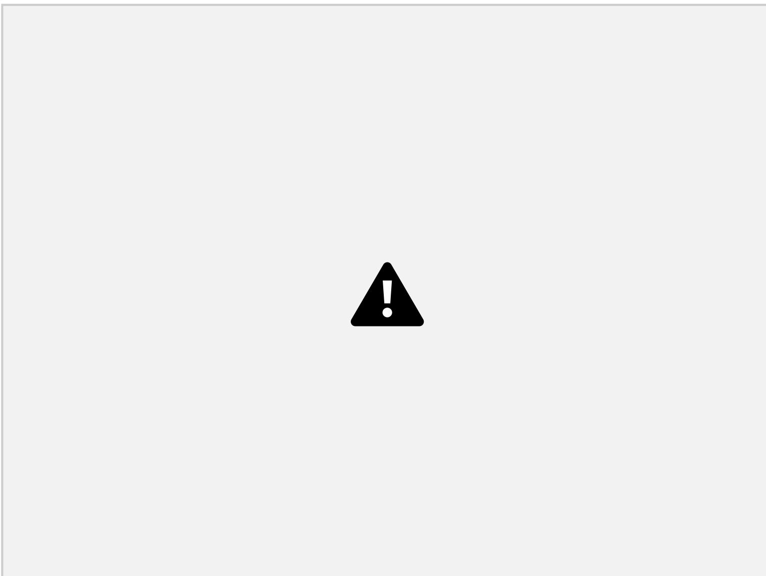

So, this is this parity and definite parity and the alternating nature of it immediately implies that the standard three term recurrence relation is actually a two term recurrence relation. So, the argument is very similar to how we did it with Hermite polynomials. So, the idea is basically, so, if you have the three term recurrence relation you have some P n n plus 1 of x is connected to x times P n and there is a P n minus 1 of x term, but there is also a term involving just P n.

Now, you know each of these three terms because there is an x times P n and a P n minus 1 and P n plus 1 of x will have the same parity. You know because parity is the same when you increase you know n minus 1 and n plus 1 of course, it is clear and P n of x has a different parity, so, but if you multiply by x that is going to become the same as the parity corresponding to n plus 1 or n minus 1.

So, P n of x has no business to be here. So, the coefficient which tags along with P n is going to go to 0. So, that is why we might as well start with this two term recurrence relation and work out these unknown coefficients. So, this alpha n needs to be worked out, gamma n needs to be worked out. And these we will again invoke the Rodrigues formula and some properties of the Legendre polynomials we have already seen to work these out ok.

Let us start with the Rodrigues formula and we have seen that there is also already an inbuilt prescription for evaluating alpha n, right. So, to find alpha n, we need to find the coefficient corresponding to the largest power of x in this polynomial. So, the polynomial P n of x has you know is a polynomial of degree n. So, you need to work out the coefficient corresponding to x to the n.

And if you want to work out the coefficient corresponding to x to the n then you need to work out the coefficient you know what happens when you do this nth derivative right and basically it is the highest order term that is the only one which is going to survive as far as you know taking n derivatives is concerned.

So, we see immediately from here that the highest order term is actually just x to the 2 n then plus lower order terms it does not matter so much because when you take the nth derivative you know it is only this term which is going to result in an x to the n, everybody else after that is going to have lower order which is not the you know the terms of interest at this point.

So, if you look at just this object then you see that taking n derivatives is going to give you 2 n times 2 n minus 1 times 2 n minus 2 all the way up to n plus 1. So, concentrating on just the first term, we see that it is going to be just this and there is a compact way of writing this.

So, the numerator you can multiply by n factorial and denominator you multiply by n factorial and you see that the numerator can then be written as 2 n the whole factorial. So, the coefficient of the highest power x to the n in P n of x is seen to be 2 n the whole factorial divided by 2 to the n times n factorial square, right.

So, from the general prescription we know that to find alpha n you must take the this kind of a coefficient in P n plus 1 of x which is basically this and then divide by this the you know highest coefficient in P n of x. So, division by this is the same as multiplying by the denominator and dividing by the numerator.

So, the first term is I have just replaced n with n plus 1. So, I have 2 times n plus 1 the whole factorial divided by 2 to the n plus 1 and then I have these two factors n plus 1 factorial times n plus 1 factorial in the denominator. And then I have to multiply by 2 to the n times n factorial times n factorial divided by 2 n the whole factorial.

So, if I work out this algebra lots of simplifications happen. 2 n to the n will cancel with 2 to the n plus 1 these are 2 and then you have 2 times n plus 1 the whole factorial. And you know so, you can expand and write it as 2 n plus 2 the whole factorial that is going to give you just these two terms.

2 n factorial will cancel with this then you have 2 n plus 1 times 2 n plus 2, 2 n plus 2 can be written as 2 times n plus 1 one of these n plus 1s will go. And then one of these n plus 1 stays and there is a 2 which also cancels and basically you are left with just 2 n plus 1 divided by n plus 1. You can convince yourself by checking this explicitly that all these simplifications indeed result in just 2 n plus 1 over n plus 1.

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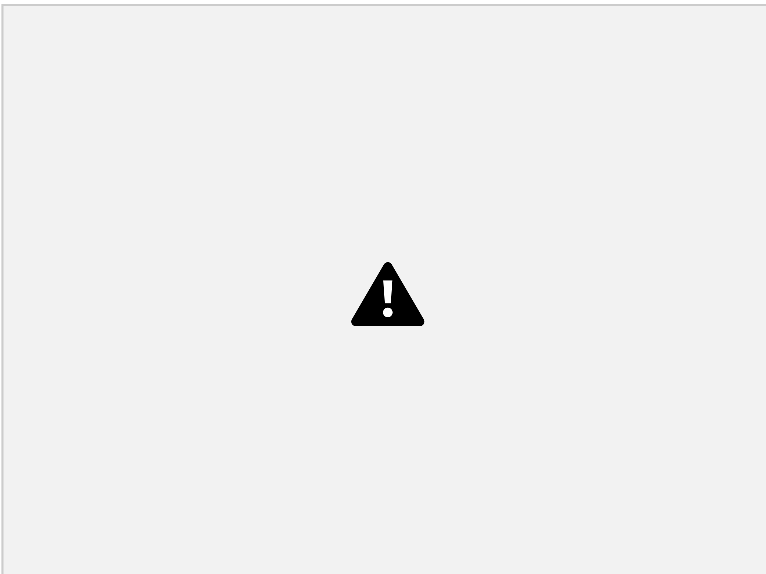

So, therefore, our recurrence relation now takes this form P n plus 1 of x must be equal to in place of alpha n we write to n plus 1 divided by plus 1 times x times P n of x plus there is this coefficient which still needs to be determined gamma n times P n 1 P n minus 1 of x.

So, in order to work this gamma n what we will do is we will exploit the normalization properties of well orthogonality and normalization properties of P n plus 1, right. So, normalization of course, comes from the specific type of coefficients we have chosen for the you know polynomial and we have already worked out the normalization integral.

So, let us exploit the facts that are already available to us and cleverly use them here. So, what we do is first we will multiply throughout with P n minus 1 of x and integrate from minus 1 to 1. So, when we do this we see that the left hand side must go to 0 because P n plus 1 is orthogonal to P n minus 1. So, this integral minus 1 to 1 P n minus 1 of x times P n plus 1 of x d x is indeed 0.

And on the right hand side we have you know this integral first of these integrals we have to work out. So, 2 n plus 1 divided by n plus 1 minus 1 to 1 dx times x times P n of x times P n minus of P n minus 1 of x we leave it as it is. But when you do this second one plus gamma n times you know it is this integral of P n minus 1 with P n minus 1 minus 1 to 1 d x which is really the normalization integral which we have already evaluated.

And so, we have to just put down 2 divided by 2 times n minus 1 plus 1 oh. So, which basically boils down to 2 divided by 2 n minus 1 right. So, it is just the normalization integral, but with the index n minus 1 right. So, now, what we have managed to show is that we can rewrite the same expression as an expression for gamma n in terms of this integral which we have to evaluate. So, we can write this as minus 2 n plus n minus 1 times 2 n plus 1 divided by 2 times n plus 1 integral minus 1 to 1 d x x P n of x times P n minus 1 of x.

So, all that we need to do is evaluate this integral and then we are done. So, it turns out that we can be clever once again. So, what we will do is we will evaluate this integral again using this equation 1. So, we want to evaluate this, but we will go back to this equation and multiply by a different quantity this time.

So, let us suppose we multiply throughout not with P n minus 1, but suppose we multiply throughout with P n plus 1. So, then we have the left hand side and then integrate from minus

1 to 1. So, the left hand side becomes the normalization integral and then P n plus 1 with P n minus 1 is going to cancel.

So, on the right hand side only the first term will survive. The second term will go because P n plus 1 and P n minus 1 are orthogonal polynomials. So, you have eliminated this term involving the unknown gamma n. So, we will get this expression. So, on the left hand side of course, you have to write 2 divided by 2 times n plus 1 plus 1.

So, that is going to give us 2 divided by 2 n plus three on the left hand side that is just the normalization integral for the polynomial n plus 1. Then we have this integral 2 n plus 1 divided by n plus 1 integral minus 1 to 1 dx x times P n of x times P n plus 1 of x and then plus 0. So, basically that this is an unknown integral, but it is not really an unknown integral because we know the left hand side.

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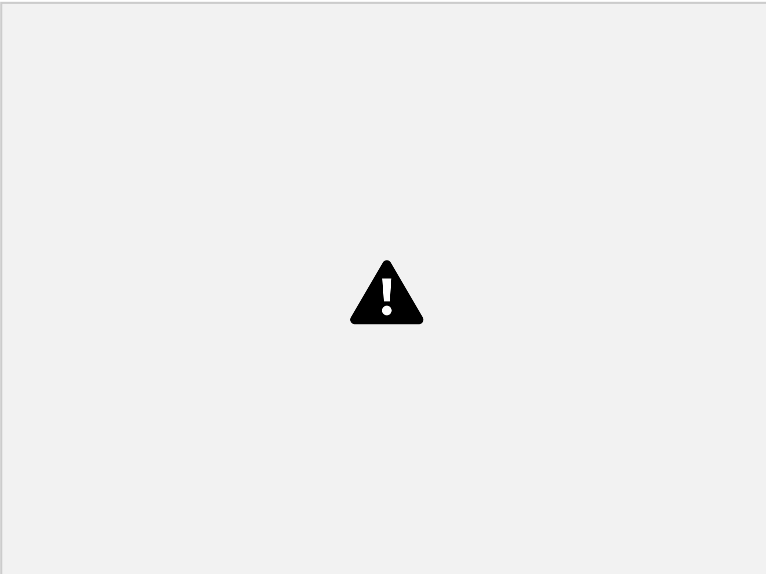

So, we can rearrange this and in fact, it is convenient to change the index n to n minus 1. Because really what we are after is this integral and in fact, this integral is also something like this except that there is a shift. You know n minus 1 has become n and n has become n plus 1 here. So, we might as well you know use.

So, we have basically worked this integral out. All you do is you know send all these factors to the left hand side and then change n to n minus 1 and so, we have the result 2 divided by 2 n plus 1. So, in place of n we put n n minus 1. So, we have 2 divided by 2 n plus 1 here and then n plus 1 will become n and then this will become 2 n minus 1.

So basically, we have managed to extract the result that we were after using this clever you know clever way of using the orthonormality properties of our Legendre polynomials. So, now all we have to do is go back and plug all this information back in.

And so, gamma n is in equation 2 is given by minus 2 n minus 1 times 2 n plus 1 divided by 2 times n plus 1 times we have you know this stuff that we have to multiply. 2 divided by 2 n plus 1 times n divided by 2 n minus 1. So, we have lots of cancellations once again and we are just left with 2 goes away. We are just left with minus n divided by n plus 1.

So, we are basically done now. We all have, we have all we have to do is to collect all the terms and then we can write. So, in place of gamma n, we write minus n divided by n plus 1. Now of course, it makes sense to multiply throughout with n plus 1 and rewrite this recurrence relation as n plus 1 times P n plus 1 of x is equal to 2 n plus 1 times x times P n of x plus minus minus n times P n minus 1 of x.

So, this is actually a very important and useful recurrence relation. So, in fact, what you can do is you can use this to find higher and higher order polynomials higher and higher order Legendre polynomials. So, we already know P 0 of x we know P 1 of x. So, using this we can work out P 2 of x. And then since you know P 1 of x and P 2 of x you plug it in here on the right hand side and get P 3 of x.

So, if you know any n minus 1 and n you can get to n plus 1 right that is what this is telling us. So, it is a very useful recurrence relation. So, this can be used to set up on a computer to, for example, work out the whole sequence of Legendre polynomials, right.

So, we will see that there is another recurrence relation as well which is satisfied by the Legendre polynomials. And then there are a bunch of results which are all closely connected and which are, you know, often extracted with great ease with the use of the generating function approach, right.

But that is coming up later. As far as this lecture is concerned we have directly used the Rodrigues formula and some basic properties of orthonormality along with the normalization integral of Legendre polynomials and worked out this recurrence relation.

Thank you.