Mathematical Methods in Physics 2
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Orthogonal polynomials
Lecture - 47
More Properties of Hermite polynomials

So, we started our discussion of Orthogonal polynomials by looking at some generic from a generic perspective and looking at you know some abstract way of writing down properties. And then we also started looking at Hermite polynomials and we wrote down some of its properties. In this lecture we will continue our discussion of Hermite polynomials and look at some more properties along the lines suggested by the general approach ok.

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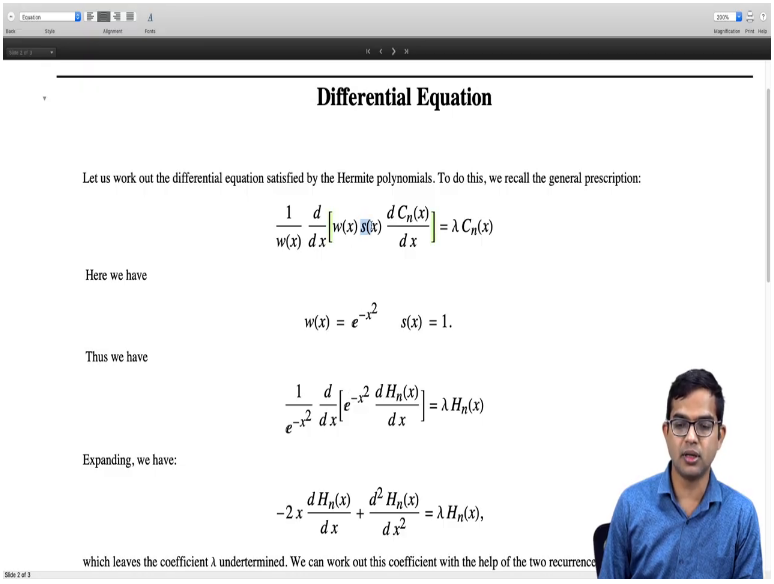

First is to work out the differential equation which is satisfied by Hermite polynomials. To do this we will use the prescription we wrote down a few lectures ago. So, the idea is to look at this quantity.

So, we managed to argue that if you take 1 over w of x and take the derivative of this complicated looking object w times s times the first derivative of this you know the polynomial of degree n. In fact, we are going to get back the same polynomial subject to some constant factor right. So, we will work out this differential equation for the Hermite polynomial and specifically we will compute this lambda right.

So, first of all we must plug in w of x is equal to e to the minus x square, that is the weight function corresponding to Hermite polynomials s of x is just 1, right. So, it goes all the way from minus infinity to plus infinity. So, s of x really does not have a role here. So, all we have to do is work out this quantity.

So, this gives us an e to the x squared times the first derivative of this quantity e to the minus x squared times the first derivative of H n of x must be equal to some lambda times H n of x, but what this lambda is we will work it out explicitly here, right.

So, expanding we have, so, if you take a derivative of this product right. So, I mean s of x does not appear. So, if this is just a derivative which is the product of 2 of these functions. It is going to give you minus 2 x times e to the minus x square this e to the minus x squared will cancel with this e to the minus x squared in the denominator.

So, you just have a minus 2 x times the first derivative of H n of x or alternatively if you just treat e to the minus x squared as a constant which cancels with this e to the minus x squared in the denominator then you have to take the second derivative of H n. So, that is the second term somewhat minus 2 x d H n by d x plus d squared H n of x by d x square must be equal to lambda times H n of x, right.

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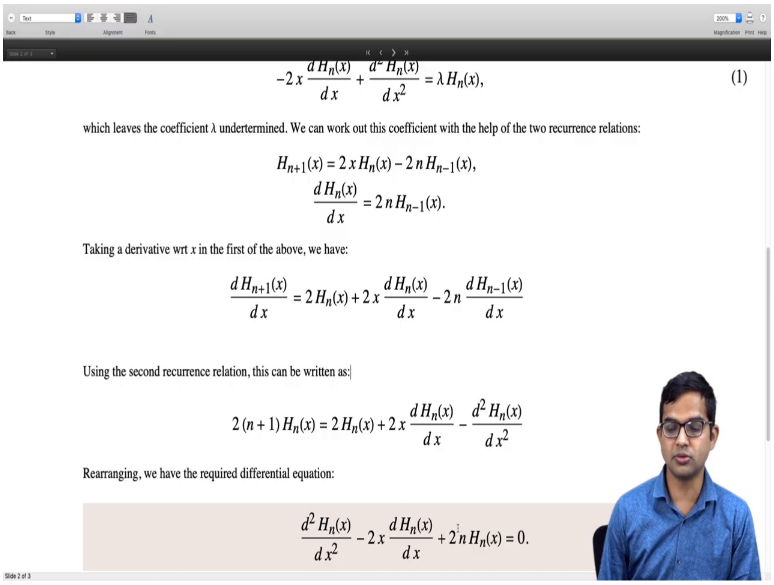

So, we will work out this coefficient lambda and it turns out that to get this we have to use these two recurrence relations that we already derived for the Hermite polynomials. And so, the way to get to equation 1 from here is to take a derivative of this first equation right; so, to start with this first equation and take a derivative with respect to x.

So, you have d by d x of H n of n plus 1 of x must be equal to 2 times H n of x plus 2 x times d H n of x by dx minus 2 n derivative of H n minus one of x d d by dx of H n minus one of x. But when we look at this equation, it is in a suggestive form. We see that ultimately we want to write every term here in terms of H n right. So, in order to do that, we will make use of the second recurrence relation.

Now, in place of d H n plus 1 of x we can write it as using this 2 times n plus 1 times H n of x, right. So, we want only one kind of polynomial in the whole equation and that is you may have different orders in the derivatives taken, but you want to just work with just H n of x.

And so, indeed H n plus 1 derivative is going to become H n according to this relation, but you have to be careful with the factor. And so, in fact, you get 2 times n plus 1 times H n of x and again on the right hand side we see that this term and this term they both involve H n of x.

So, we leave these two terms as it is, but this term has an H n minus 1 of x which we do not want. We want to write the whole equation in terms of just H n, and in order to do that once again we invoke this result, but in the other direction.

So, we see from here 2 times n times H n minus 1 of x is actually nothing but d by dx of H n of x. So, we have 2 times n times d by dx of H n minus of x which is basically like taking the derivative. If you take this equation and take its derivative of both the left hand side and right hand side with respect to x, you get this term really which is on the right hand.

So, in place of 2 n d H n minus 1 of x divided by d x we simply write down minus d squared by d x square of H n of x invoking this result. So, which is really the differential equation we are after, because if you rewrite this you can rewrite it as d square H n of x by d x squared you bring it to the left hand side minus 2 x d H n of x by d x plus. So, you have 2 times n plus 1 minus 2 which becomes just 2 to n plus 2 n H n of x.

So, you see that this equation is really the same as you know this form that we had from the general prescription was this one. So, we have you know this term is here and this term is here and indeed it is equal to lambda times H n of x, but we know what lambda is from we have computed it here. In fact, lambda is minus 2 n ok.

So, this equation is in fact, probably familiar to a lot of a lot of students you know taking this course. I guess we have seen it in a quantum mechanics course and maybe elsewhere as well.

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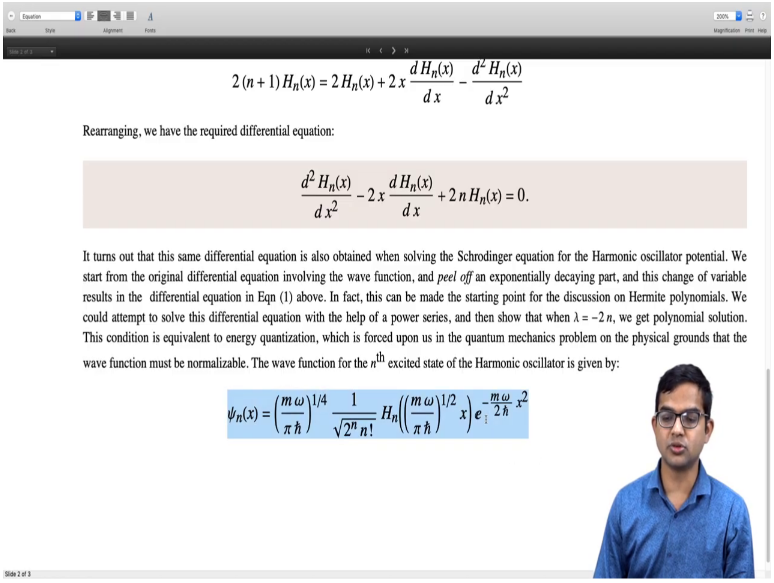

So, one familiar context in which this appears is when we are solving the Schrodinger equation for the harmonic oscillator potential. So, the way you solve the Schrodinger equation for the harmonic oscillator problem is you know there are two different ways.

One is this Schwinger's way which is a clever algebraic approach and then there is this sort of bread and butter differential equation approach which involves writing down a power series expansion, and then trying to find a power series solution.

After you have peeled out right, you start with this wave differential equation for the wave function itself. And then you argue that you know your wave function must fall off in a nice way right for large positive x and negative x and then you work with a differential equation of another quantity which basically gives you this kind of an equation right.

So, and then in fact, this 2 n that you get you know lambda equal to minus 2 n is really you know the condition that you have for ensuring that you get polynomial solutions for you know such a differential equation right. So, and that is essentially the same as the quantization condition for the harmonic oscillator problem, right.

So, the harmonic oscillator problem, well I mean in general the differential equation corresponding to the harmonic oscillator problem can be solved, right. So, from a mathematical point of view you will definitely find solutions there will be power series solutions. It is just that these power series are not going to have nice enough properties that you can give them and the interpretation of a wave function, right.

So, they would blow up at plus x and minus x. It's only when you impose this, what is really a quantization condition. So, this forces the energies to have certain allowed levels when you have these special levels of energy lambda equal to minus 2 n.

Then in fact, you are able to normalize your wave function which basically is means that you can you get polynomials, your power series solutions you know the power series truncates and you get, in fact, just a polynomial and those polynomials are nothing but these Hermite polynomials. So in fact, you could have actually also viewed this as a kind of an eigenvalue problem right. So, I guess it is even more transparent if you look at this equation.

So, you have some stuff: some operator acting upon H n of x is equal to some lambda times H n of x. So, H n of x has this interpretation of an eigenfunction for this eigenvalue problem and lambda is an allowed eigenvalue right. So, H n of x has to be if H n of x has to be a polynomial then lambda must take only these allowed values. So, when you do this in fact you can write down the general wave function for the nth excited state of the harmonic oscillator and this is probably a familiar expression.

So, you have this overall you know e to the minus x squared I follow off. I mean in suitable units you have to put in all these numbers, these constants which come from the you know mass and natural frequency and so on. And once again the Hermite polynomial is sitting here and you have this overall factor outside. ok.

So, this is often a starting point. So in fact, there are many discussions of Hermite polynomials, which start with a differential equation of this kind and then work out when do you get a polynomial. And then you say, ok, these they have a sequence of polynomials they are all orthogonal and then you work out recurrence relations and so on.

But, I mean the method we have followed is to start with in and from an abstract point of view and then now we are in fact, obtaining the differential equation as a sort of a consequence of the you know the property of these polynomials ok.

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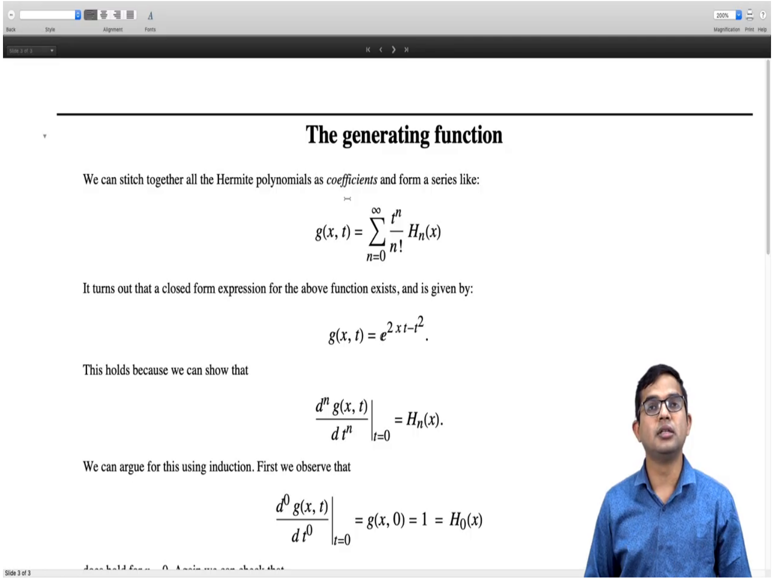

Let us look at another aspect of the story which is the so-called generating function. So, it turns out that you can actually stitch together these polynomials; Hermite polynomials along with a very you know carefully designed set of coefficients and form a series like this.

So, t to the n divided by n factorial, n going from 0 to infinity H n of x. It turns out you can actually write a closed form expression for this and that is this object e to the 2 x t minus t square. So, in order to show this we have to simply show that if you take the n-th derivative of this function with respect to t and put t equal to 0, you get H n of x right. So, if you can show this indeed then this is the generating function and this can often be used to prove many of the results.

We have already proved some of these results like the recurrence relation for example, right starting from the Rodrigues' formula and so on. But in fact, you know this generating function often provides a very clever way of proving many properties of you know various polynomials involved various orthogonal polynomials involved have different generating functions and starting from here its a method by which many properties can be extracted.

But first let us for the Hermite polynomials; let us show that indeed this is the generating function and basically which boils down to showing that the n-th derivative of g of x comma t at t equal to 0 will give you the Hermite polynomial. So, let us argue for this using induction. So, there is a way to argue for this directly from the Rodrigues' formula as well which perhaps coming you would be encouraged to try and work out on your own, but let us argue for this using induction.

So, first we observe that this is true for n equal to 0. So that means, if you do not take a derivative, but simply put t equal to 0 you get 1 right, which is definitely the first member of the Hermite polynomial sequence of polynomials H naught of x is indeed 1.

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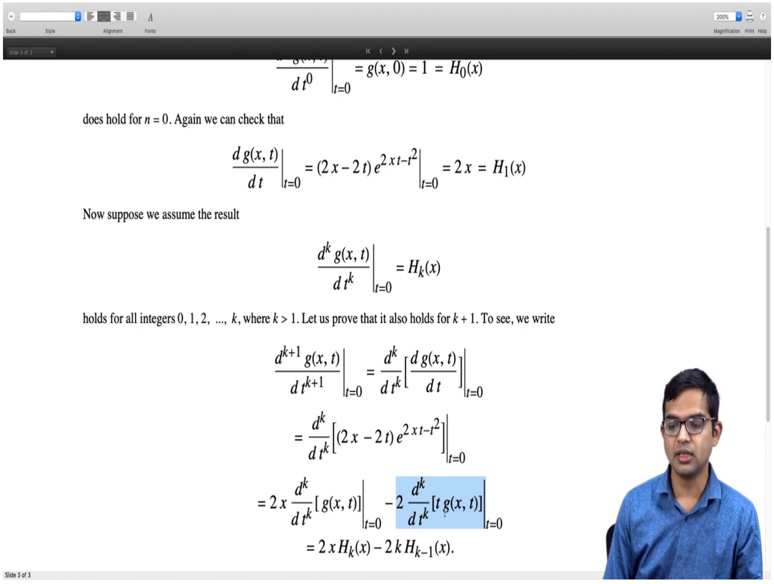

And again we can also verify for the first derivative. If you take d g by d t and put t equal to 0, you get H 1 of x because dg by d t is nothing but 2 x minus 2 t times e to the 2 xt minus t squared and now if you put equal to 0 you simply get 2 x which is indeed the next member of the Hermite polynomial class of polynomial.

So, now suppose we assume that this result holds all the way from you know 0, 1, 2 and all the way up to k, where k is greater than 1 right. So now, by induction we will show you know the result holds and in order to do this we have to show that if 0, 1, 2 three all the way up to k, where k greater than 1 is true then we will argue that it is also true for k plus 1. And therefore, it will be true for the next one and so on. So, by induction this result will hold for all these coefficients in this expansion right.

So, how do we do this? So, we consider the k plus 1th derivative with respect to t of this function g of x comma t equal to at t equals 0, which is the same as taking the k-th derivative of the first derivative of g d g by d t right. But, now what is the first derivative of g? Which is actually nothing but 2 x minus 2 t times e to the 2 xt minus t square right.

So, you can just start with this function and take one for the first order derivative and that is nothing but 2 times 2 x minus 2 t. So, it is a little bit like here, but you do not put t equal to 0, so you just leave it as it is. And then now you argue that you see as far as. So, this is a sum of two terms and this 2 x basically has nothing to do with this derivative with respect to t. So, 2 x can come out.

So, I can write this as 2 x times the k-th derivative with respect to t of just g of x comma t because after all this is nothing, but g of x comma t at t equals 0. And then minus 2 times 2 can come out, but t cannot come out t has to stay, the k-th derivative of t times g of x comma t at t equal to 0.

Now, but the first term we observe that this is actually nothing but this is nothing but at t equal to 0 it's nothing, but you know because we have this induction result it's nothing but 2 x times H k of x and once again this we I will argue now in a moment. We can show that this is nothing but 2 times k times H k minus 1 of x, right.

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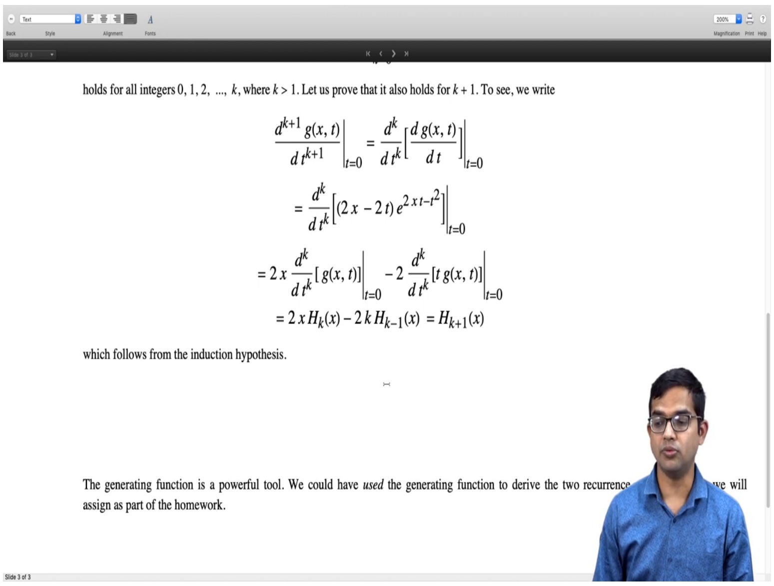

So, this is how we argue for this ok. Let us go back to this original result. So, what we have; the induction hypothesis is that g of x comma t is equal to summation over n going from 0 to infinity t to the n times some coefficient like if I call it C n.

But basically we have argued that the C n is equal to H n of x divided by n factorial for the first you know k terms, where k goes from 0 0, 1, 2 all the way up to k. So now, if I multiply this by t, then I have t to the n plus 1 divided by n factorial right.

So, I am not giving you all the steps. This is something that you should convince yourself is true. So, if I take t times g its t to n plus 1 and then I multiply the denominator and numerator by n plus 1. So, I have n plus 1 times t to the n plus 1 divided by n plus 1 the whole factorial H n, then I can do the shift.

So, I have this n plus 1 in place of n plus 1. I do not know. So, let me redefine n plus 1 as m. Then I have m times t to the m divided by m plus m minus 1 sorry H h m minus 1 and then I have a m factorial. So, basically it is going to look exactly like what you have here except that you will get an H m minus 1 and then there will be an extra factor of m sitting here.

So, that is what I am saying will boil down to this minus 2 k times H k minus 1 of x right. So, this is a consequence of the induction hypothesis where we have assumed this is true for all these integers starting from 0 to k and then we use that result in the original form of expanding this.

And then finally, we argue that this is actually nothing but H. I should explicitly write this down; H H k plus 1 H k plus 1 H k plus 1 of x, which is really a consequence of one of the recurrence relations right. So, we have this recurrence relation - we saw this earlier. So, we have H n plus 1 of x is 2 x H n of x minus 2 n H n minus 1 of x. So, this is just the property of the Hermite polynomial. So, if you take this combination of polynomials indeed this is H k plus 1 of x.

So, we have managed to show that if you have this induction hypothesis then the next term in this sequence is also the Hermite polynomial. Therefore, this result holds right. So, which therefore, by the principle of mathematical induction indeed H k of H n of x is the nth derivative of g of x comma t evaluated at t equal to 0. So, indeed the function we have given here, this equivalently this is the generating function for our Hermite polynomial.

So, I mean you could have worked this out directly from first principles starting from the Rodrigues' formula right. So, that is also something that allows you to work out this alternate way of seeing this. And so, the reason to go after such a generating function is because it can derive many properties for you. So, it is a powerful tool and in fact, we could have used the generating function to derive these two recurrence relations.

We derived it from other methods starting from the Rodrigues' formula. But in fact, if you just use a generating function it's very quick to obtain these results right, so which we will assign as part of homework ok.

So, that is all for this lecture.

Thank you.