Mathematical Methods 2
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Orthogonal polynomials
Lecture - 46
Hermite polynomials

Ok so, we have looked at several properties of orthogonal polynomials starting from a very general perspective and from an abstract perspective. Starting with this lecture we will start looking at some specific examples of you know these sets of polynomials. Specifically in this lecture we will look at what are called Hermite polynomials.

So, we will see how these general principles play out. Some part of it will be covered in this lecture and then there are more properties of Hermite polynomials to discuss ahead ok.

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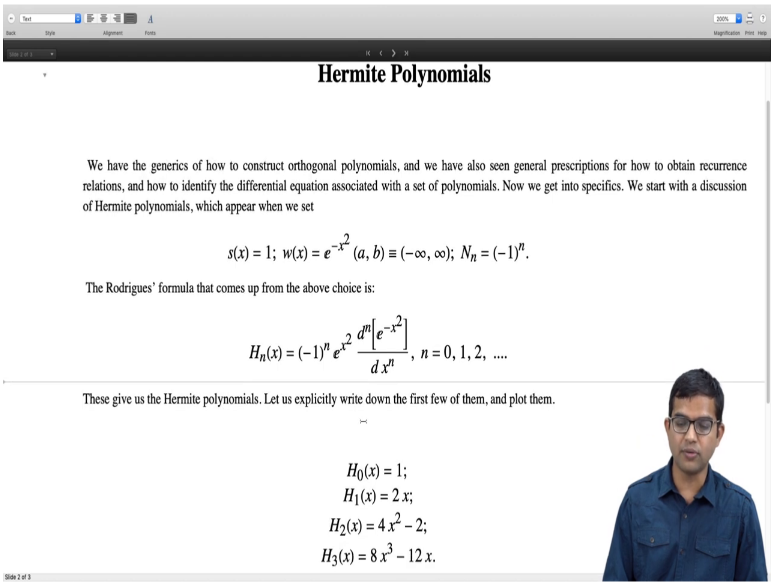

So, the Hermite polynomials according to our notation appear when we put s of x to be 1. The weight function is chosen to be e to the minus x squared and the range in which these polynomials are going to be valid they go all the way from minus infinity to plus infinity. And it is convenient and it is also a matter of convention to choose the normalization constant to be minus 1 to the n right.

So, the formula that we derived is the so-called Rodrigues formula for writing down these polynomials in terms of the nth derivative of some function. So, in this case you are going to take the nth derivative of the function e to the minus x squared that is the weight function s of x is just 1.

And then after having taken this derivative you also have to multiply by this e to the x squared right. What appeared as 1 over w of x, now becomes e to the x squared and then you have this minus 1 to the n sitting right at the beginning. So, that is the normalization factor. So, this is valid for n equal to 0, 1, 2, so on right.

So, let us look at how this formula explicitly gives out a bunch of polynomials as we start looking at small n right. You can play out this exercise yourself and check that indeed you know this formula gives us polynomials right. So, for n equal to 0 of course, you are not doing any derivative.

So, its like e to the x squared times e to the minus x squared will just give you 1 minus 1 to the power 0 is one and then when you take a derivative once you get this minus 2 x times e to the minus x squared e to the minus x squared and e to the x squared will cancel.

And so, now, you see that you have this minus sign in order to compensate for this minus sign. This overall outside minus 1 minus sign is useful here and so, then you write it down as 2 x, but if you take a derivative again you can check this. So, the polynomial that will result is 4 x squared minus 2. So, now, minus 1 the whole squared is just 1, you do not need this extra padding.

So, the key point is that whenever you take successive derivatives there is always e to the minus x squared which is going to stick around and that is going to cancel with this overall outside e to the x squared. And so, it is just a matter of bookkeeping how many times you take derivatives with respect to the you know e to the minus x squared itself.

Sometimes you may take derivatives with respect to some power of x right and then you have to distribute it appropriately and all such terms will come in right. So, you can verify that indeed H 2 of x is this H 3 of x is 8 x cube minus 12 x. And you see that this overall minus 1 to the n ensures that the highest power in all of these polynomials is necessarily positive and there is also another observation we can make just from looking at these first few polynomials, but let us first plot these polynomials.

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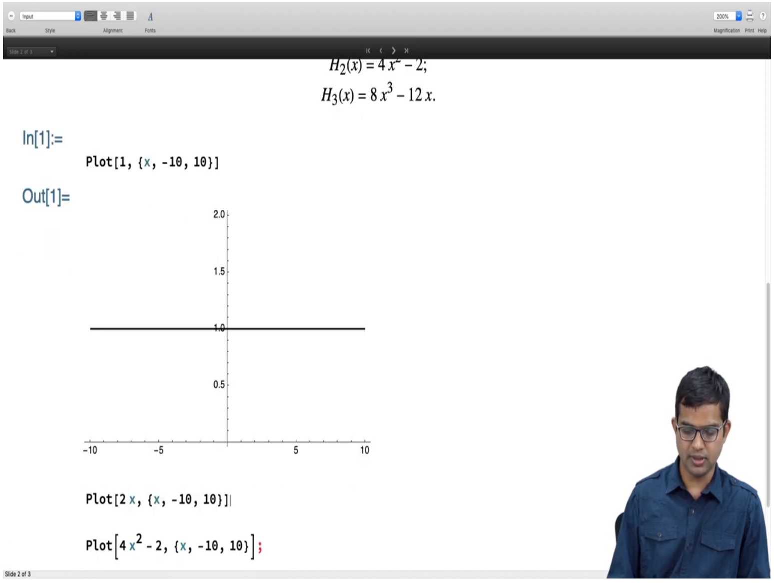

So, if I plot the first one, it is just a constant. So, it is just 1 and it goes all the way from minus infinity to plus infinity nothing changes. And whereas here 2 x is also a straightforward function this is a polynomial number 2, which is H 1 of x is just 2 x it is just a linear curve with slope 2.

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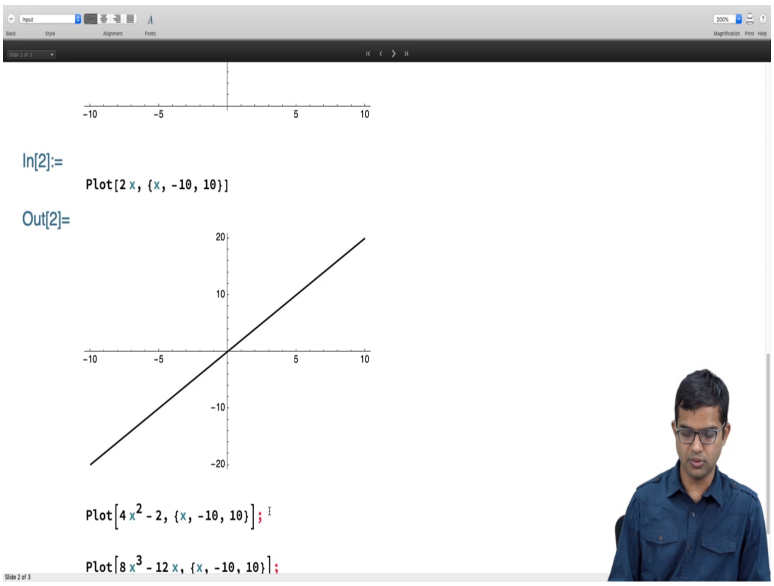

And then when you go to the next polynomial you see a quadratic function.

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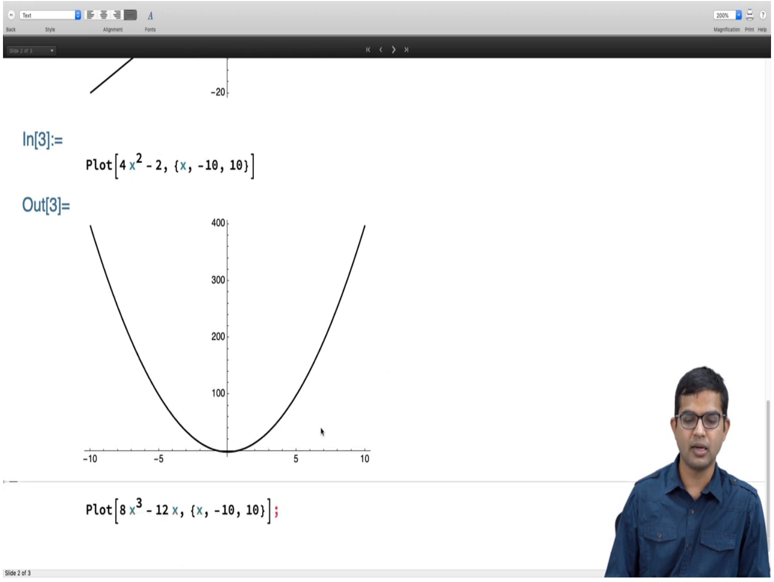

It is a parabolic curve and then if you go to the next curve you get a cubic function right.

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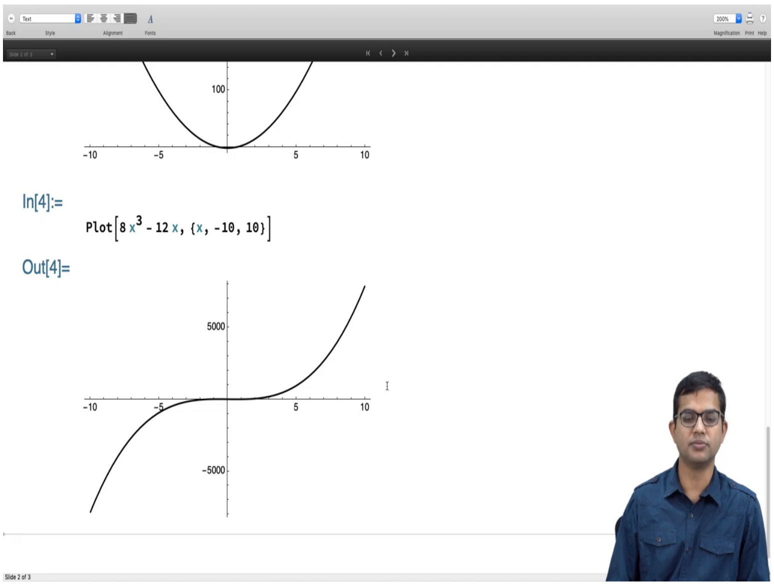

So, a closer look at these functions reveals a pattern. You see that whenever you are looking at an odd index only odd powers seem to get covered here. So, it is only x here and when H 3 of x for H d of x you see that only x cube and x will appear.

And likewise you can check that when you go to H phi of x you will have only H x to the phi x to x cube and x and again when H naught of x, if you are looking at H naught of x is just x to the power 0; H 2 of x has x squared and x to the 0 and this pattern will continue right. So, this is a property which we will exploit in a moment, right.

So, this is to get a feeling for these functions right. I mean probably we have encountered Hermite polynomials in the context of quantum mechanics, but here we are studying it from a sort of sort of you know we first wrote down the prescription and then we are working out properties of such functions. It is a slightly different route to understanding these polynomials. So, let us go ahead and look at some properties of these polynomials.

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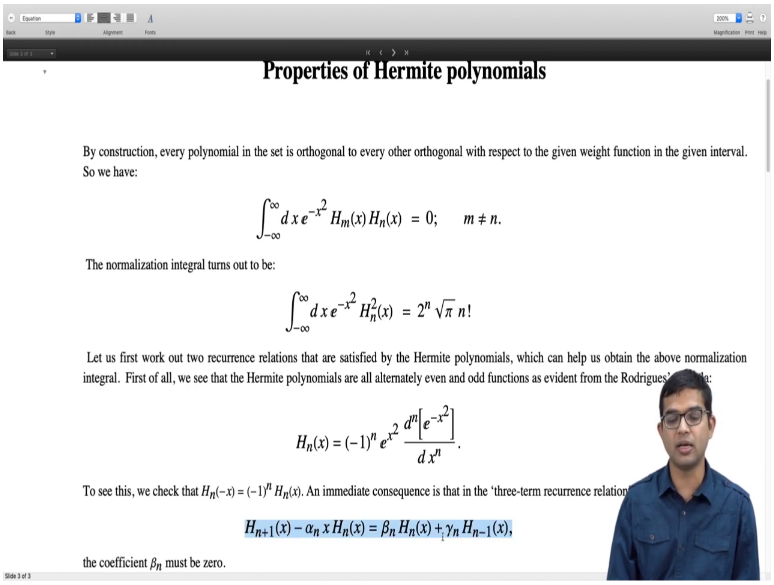

So, by construction of course, every Hermite polynomial is orthogonal to every other Hermite polynomial, so, with respect to the weight function. Of course so, specifically it means in this context integral minus infinity to plus infinity d x e to the minus x squared H m of x and H n of x, it is going to be 0 as long as m is not equal to n right.

And it turns out that I mean of course, if you put m equal to n it is not 0 and it has a very specific value and that is the that is what is called the normalization integral and in fact, it is possible to show that then this normalization integral will evaluate to 2 to the n times square root pi times n factorial right.

We will actually show this result a little bit later after we have derived some recurrence relations. So, from which this result will follow, but let us first workout some recurrence relations. So, one of these recurrence relations is actually the general three term, so-called three term recurrence relation which we showed which is valid for you know sets of orthogonal polynomials in general.

And then we will see that there is another interesting recurrence relation which is specific to Hermite polynomials and using both of these in a judicious way. In fact, we can also work out this normalization integral ok. So, first we start with the Rodrigues formula.

So, the Rodrigues formula says that H n of x is minus 1 to the n e to the x squared and the nth derivative times the nth derivative of e to the minus x squared. And then we exploit the fact that H n of x has definite parity. So, every member of this sequence is either exactly odd or exactly even right.

So, since you know an odd member of this H 0 H one for example, has only powers of x and x alone H 3 of x will have x cube and x. So, a Hermite polynomial with an odd index will have only odd powers of x and therefore, it is going to be an odd function.

So, H n of minus x specifically you can directly check this by putting in H n of minus x in this formula. And you will see that you know e to the x squared and e to the, e to the minus x the squared is the same. So, this part basically remains unchanged when you put in minus x. I likewise this part also is unchanged when you have a minus x. So, it is only here that there is going to be a change.

So, and this has an n sitting here. So, you are taking the nth derivative. So, x appears n times. So, if n is even so, you get minus 1 to the n and that is not going to change anything. But, on the other hand if x is odd, so, if you take a minus x to the whole power n so you have an extra overall minus 1 to the n which comes out. So, in fact, you can say H n of minus x is minus 1 to the n times H n of x. So, it is an odd function if n is odd it is an even function if n is even right.

An immediate consequence of this is the three term recurrence relation of this form. So, this is the general form for the three term recurrence relations right. Alpha n, beta n and gamma n are specific things to be worked out, but a consequence of this is that beta n must be 0 right. So, one way to argue for this is the following right.

So, if you look at the left hand side H n plus 1 is a polynomial of order n plus 1 and x times H n of x is also a polynomial of order n plus 1 right. So, in other words, in fact, we can say something more; H n of x is an nth order polynomial with definite parity.

So, if you take such an nth order nth degree polynomial and multiply by x, if you had all even powers x times this is going to give you all odd powers. And anyway, we know that H n plus 1 has definite parity. So, when you take the sum of these two again all of these terms are of a certain type or all the terms will be odd powers or all of them will be even power.

And likewise H n minus 1 will have the same parity as H n plus 1 and x times H n of x its only H n of x which has a different parity. So, it is incompatible for an equation like this to hold unless beta n is 0, right. So, it forces beta this simply the parity forces beta n to be 0.

So, we have this recurrent, the three term recurrence relation has actually become a two term recurrence relation for this in this case for the case of Hermite polynomials and. In fact, we can also go ahead and work out we can work out this alpha n right.

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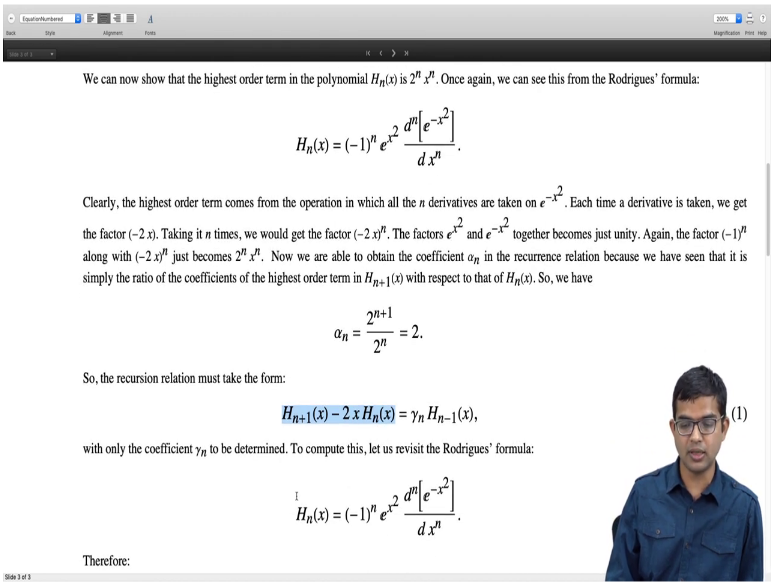

So, in order to do this first we will show that the highest order term in the polynomial H n of x is 2 to the n x to the n right. So, the way to do this is to start with the Rodrigues formula. We start with the Rodrigues formula and then the highest, how would we get the highest power right? So, we know that H n of x is a polynomial with degree n. So, how will we get x to the n?

So, when you are taking this nth derivative you know the first time if you take a derivative you are going to get minus 2 x times e to the minus x squared. Then if you take a derivative a second time you have this choice whether to first you know you will get two terms. One is when you take the derivative with respect to x, but leave e to the minus x squared as it is or you leave x as it is and then you take a derivative with respect to e to the minus x squared right.

So, you can convince yourself that the highest power will come when at every step we simply take a derivative only with respect to e to the minus x squared, which is the term which is going to keep on giving you more x's here. So, the first time you will get an x second time you will get an x squared x cube and so on. So, when you do this n times you it will give you x to the n.

That is the only term which gives you the highest power. So, it is actually straight forward to work out the coefficient corresponding to the highest power. So, that comes out to be just you know every time you get a minus 2 x. So, in fact you get all these factors of minus 2 x there are exactly n of them. So, you get a minus 2 x the whole power n.

So, that is nothing but 2 to the n times x to the n times minus 1 to the n. Minus 1 to the n will go with this minus 1 to the n, you will get a plus 1. Therefore, the highest power term in this any H n of x in general is going to be just 2 to the n times x to the n right. So, we argued that this coefficient alpha n is nothing but the ratio of this highest order coefficient of H n plus 1 divided by the corresponding coefficient for the function H n of x right.

So, this is something we argued on very general grounds in the previous discussion. So, you can go back and check that again if you do not recall this. So, from this immediately we are able to write down alpha n as 2 to the n plus 1 divided by 2 to the n and it's just equal to 2 as simple as that.

So, our recurrence relation here is of this form. It must take this form H n plus 1 of x minus 2 x H n of x is equal to gamma n times H n minus 1. This gamma n is something which we need to determine. Again, to find this let us go back to the Rodrigues formula H n of x is this.

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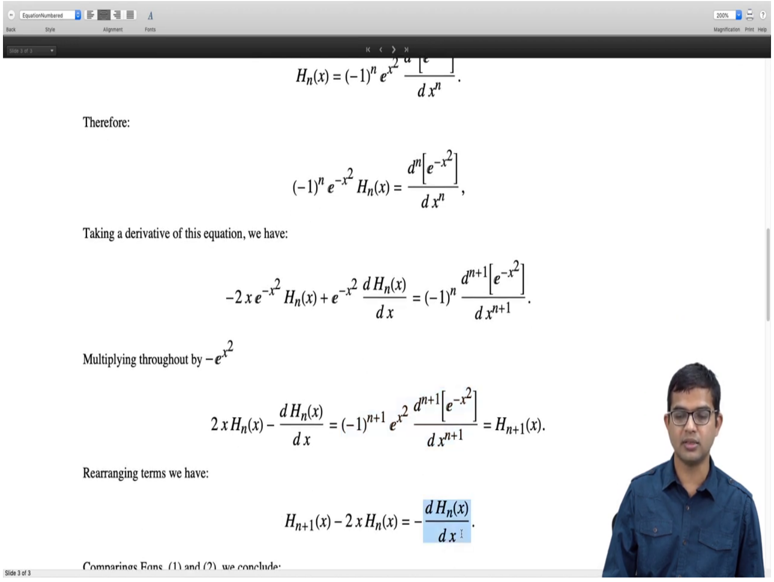

And therefore, we will pull out all this stuff. We will bring this whole stuff to the left hand side. So, we have minus 1 to the n e to the minus x squared H n of x is this and now we take a derivative of this equation on both sides. The left hand side, there are two terms. So, you get a minus 2 x times e to the minus x squared H n of x plus e to the minus x squared the derivative of H n of x is equal to minus 1 to the n the n plus 1th derivative of e to the minus x squared right.

So, what I have done is again I have brought this minus 1 to the n back to the right hand side because its convenient to keep only this minus 1 to the n on the right hand side. And so, the left hand side of course, I have just taken a derivative with respect to x and then if I multiply throughout with minus e to the minus e to the x squared right.

So, this x e to the x. So, this e to the x minus x squared will cancel, this e to the minus x squared will cancel and then I am just left with a minus sign which will become a plus sign. So, it becomes 2 x times H n of x minus the derivative of H n with respect to x is equal to minus 1. You have an extra power minus 1 to the n plus 1 then you have an e to the x squared the nth plus 1th derivative of e to the minus x squared.

But a little thought reveals that in fact, the right hand side is nothing, but the Rodrigues formula for the Hermite polynomial of order n plus 1. So, we have managed to show that H n plus 1 of x minus this is just rearranging terms minus 2 x times H n of x is minus the derivative of H n of x.

So, if we compare this equation and this earlier equation 1 and 2. So, we see that in fact, both of these look very similar. H n of x minus 2 x H n of x is the right hand side. The right hand sides are different, left hand sides are the same. So, the right hand sides must equal each other.

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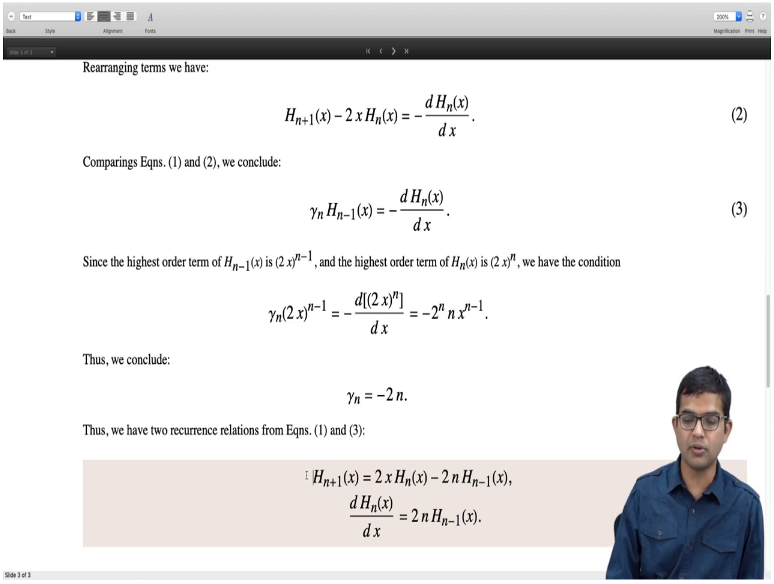

So, which immediately implies that gamma n H n minus 1 of x is actually equal to minus that derivative of H n of x right. So, this will actually lead to another recurrence relation which is also very useful. And so, there is a quick way to evaluate gamma n from this relation.

So, the thing is that you have H n minus 1 is a polynomial of degree n minus 1 and when you take this polynomial of the nth degree polynomial and take a derivative right. And so, this polynomial is going to contain many terms and this polynomial 2 is going to contain many terms in general right.

So, it suffices I mean that these two are equal term by term. Every term of the polynomial on the left hand side is equal to the corresponding term on the right hand side. So, if you just simply compare the highest order terms on both sides we can immediately get this gamma n out. Just by looking at which you already know the highest power and use that to compute gamma n.

The highest order term of H n minus 1 of x is 2 x to the n minus 1 and the highest order term of H n of x is 2 x the whole power n. So, if you take a derivative of this we will get minus 2 to the n times n times x to the n minus 1. So, we have to equate that to gamma n the whole power times 2 x to the n minus 1 and so, x to the n minus 1 and 2 to the n will cancel and we get gamma n is equal to minus 2 n.

So, we have worked out all these unknown coefficients and we have in fact, got two recurrence relations right. So, for this the labor that we have put in, we got an extra recurrence relation for free. So, H n plus one of x is equal to 2 x times H n of x minus 2 n H n minus 1 of x which is really the three term recurrence relation the general one.

And then we have this relation for the derivative of a Hermite polynomial. In fact, it gives you the next lower order Hermite polynomial except that there is this factor 2 n involved right. So, both are useful properties.

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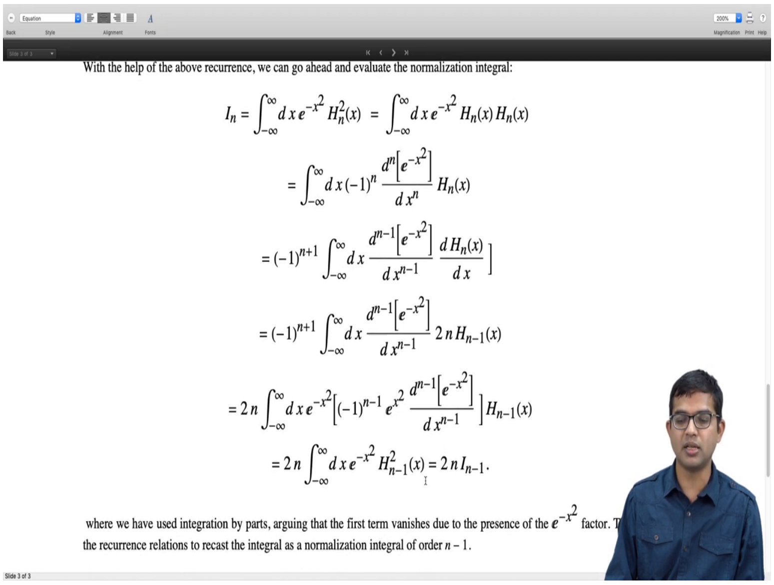

And so, we will use them to work out the normalization integral now right. So, let us look at the normalization integral. So, this is the normalization integral, I n minus infinity to plus infinity d x e to the minus x squared H n squared of x. So, we write this as you know, write this out explicitly as a product H n of x times H n of x and then bring in the Rodrigues formula for just one of them.

The first one of them let us say I am replacing this H n of x in terms of the Rodrigues formula and then I have this e to the x square which will couple with this e to the minus x squared and it cancels. So, I am left with just you know this integral to evaluate, but this integral I can evaluate with the aid of integration by parts.

So, this is like u this is like d v. So, I have some function times a perfect derivative of another function which I know how to work out the integral of. So, its u d v is u v minus v d u. So, this will be u v, but u v has to be evaluated at minus infinity and plus infinity at both these limits since you have this e to the minus x square sitting there right.

So, all these derivatives of e to the minus x square will definitely give you at least at the end there will be an e to the minus x square sitting there. So, at both ends plus infinity and minus infinity that term is going to ensure that it is going to go to 0. Therefore, you do not have to worry about the boundary term and then you have a minus sign. So, the minus sign will go with this minus 1 to the n.

So, you have a minus 1 to the n plus 1 minus infinity to plus infinity then you have to take this d n minus 1 by d x to the n minus 1 of this function, but you have to take a derivative of this. So, this is just an exercise in integration by parts. And now we invoke this result that d n by d x is actually nothing but 2 n H n minus 1 of x. So, we bring this result back in and so, we have in place of d H n of x by d x we plug in 2 n times H n minus 1 of x.

And now, it is convenient to rewrite minus 1 to the n plus 1 as minus 1 to the n minus 1 right, it does not matter. So, you can always multiply by minus 1 the whole squared which is just 1. And then we pull out this factor 2 n bring this minus 1 to the n minus 1 inside and then we also write 1 as e to the minus x squared times e to the x squared.

So, e to the x squared goes inside here. It is just to get a convenient expression right and so that we can read off this quantity as in fact nothing but the Rodrigues formula for H n minus 1 of x. So, e to the minus x squared it's useful to have because it is the weight and then there is an H n minus 1 of x which is sitting outside.

You multiply these two and basically we have managed to show that I n which is a normalization integral corresponding to the nth degree polynomials is the same as 2 n times I n minus 1 which is a normalization integral corresponding to n minus 1 degree. So, this immediately actually gives us another kind of recurrence relation, recurrence recursive relation if you wish I n is equal to 2 n times I n minus 1.

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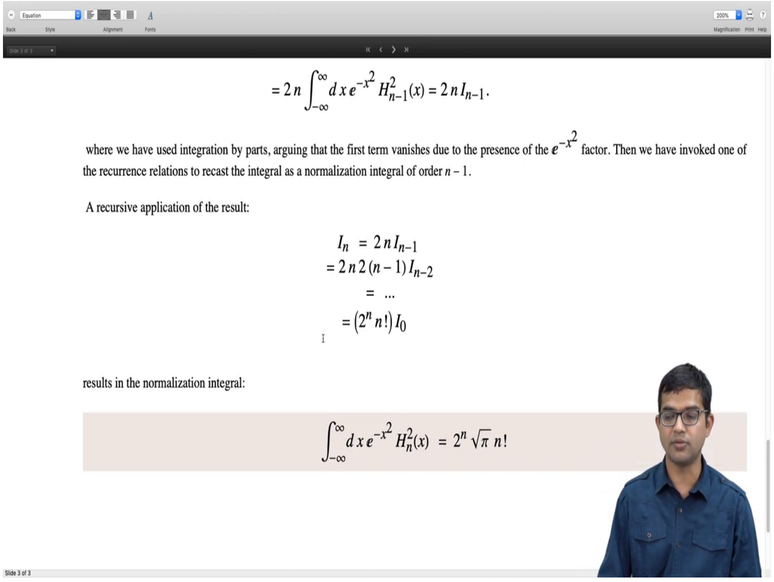

But then in place of you can use the same thing again. So, in place of n minus 1 I n minus 1 you can write it as I n minus 2. So, 2 n times 2 times n minus 1 times I n minus 2 and then you keep on doing this. You will eventually end up with n into n minus 1 into n minus 2 so and so on that will give you n factorial and then you will also have 2 to the n the whole thing multiplied by I naught.

But I naught is nothing but H naught squared is just 1, H naught of x is 1. And then integral minus infinity to plus infinity d x e to the minus x squared is just square root pi. So, we have the result. This normalization integral is just 2 to n times square root pi times n factorial ok.

So, in this lecture we managed to work out a couple of very useful recurrence relations for the Hermite polynomials and exploit these relations and some properties corresponding to Hermite polynomials. We also worked out their normalization integral.

Thank you.